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FINITE TRIFACTORISED GROUPS AND π-DECOMPOSABILITY

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Abstract

We derive some structural properties of a trifactorised finite group $G = AB = AC = BC$, where A, B, and C are subgroups of G, provided that $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups, for a set of primes π.

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1. Introduction

Throughout this paper all groups considered are finite. Within the study of factorised groups one has frequently to consider *trifactorised* groups, that is, groups of the form $G = AB = AC = BC$, where A, B and C are subgroups of G. This occurs, for instance, when aiming to get information on a normal subgroup N of a factorised group $G = AB$, with A, B subgroups of G. In this case, an important tool is to analyse the structure of the so-called *factoriser* of N, denoted by $X(N)$, which is the intersection of all factorised subgroups containing N. (A subgroup S of $G = AB$ is *factorised* if $S = (A \cap S)(B \cap S)$ and $A \cap B \leq S$.) The factoriser subgroup $X(N)$ turns out to be a trifactorised group; more precisely, $X(N) = N(A \cap BN) = N(B \cap AN) = (A \cap BN)(B \cap AN)$ (see [1]).

One of the classical results in the literature on finite trifactorised groups is due to Kegel [13]. He proved that a finite group $G = AB = AC = BC$, which is the product of two nilpotent subgroups A and B, is nilpotent (supersoluble), provided that C is likewise nilpotent (supersoluble). A corresponding statement holds when A and B are nilpotent and C belongs to a saturated formation containing all nilpotent groups

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(Peterson [1, Theorem 2.5.10]). It is worth emphasising that such a group is soluble, by the celebrated theorem of Kegel and Wielandt on the solubility of a product of two nilpotent groups.

Some criteria for the π-separability of a trifactorised group, for a set of primes π, under assumptions of existence, conjugacy and dominance of Hall π-subgroups, were obtained by Pennington in [14] (see Theorem 3.1 and Corollary 3.5 below). A much deeper result in the universe of all finite groups was proved by Kazarin in [7] using the classification of finite simple groups: if the group $G = AB = AC = BC$ is the product of three soluble subgroups A, B and C, then G is soluble. Some related results were obtained in [3], again in the universe of soluble groups, by considering some well-known families of subgroup-closed saturated formations of so-called nilpotent type (see [5] for an account of such classes of groups).

In this paper we go further with the research on trifactorised groups, dealing with π-decomposable groups. A group X is said to be *π-decomposable* for a set of primes π, if $X = X_\pi \times X_{\pi'}$ is the direct product of a π-subgroup X_π and a π'-subgroup $X_{\pi'}$, where π' stands for the complement of π in the set of all prime numbers. For any group X and any set of primes σ, we use X_σ to denote a Hall σ-subgroup of X. In particular, X_p will denote a Sylow p-subgroup of X, for a prime p.

For our purposes the following result is crucial.

Theorem 1.1 ([12], Main Theorem). *Let π be a set of odd primes. Let the group $G = AB$ be the product of two π-decomposable subgroups $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$. Then $A_\pi B_\pi = B_\pi A_\pi$, and this is a Hall π-subgroup of G.*

This theorem, whose proof uses the classification of finite simple groups, is part of a development carried out in [8, 9, 11, 12] and motivated by the search for extensions of the theorem of Kegel and Wielandt mentioned above (see also [10]). We apply Theorem 1.1 to obtain new results on trifactorised groups within the general universe of finite groups.

The notation is standard and is taken mainly from [6], and we refer to this book for the basic terminology and results about classes of groups. We refer to [16] for the elementary facts regarding π-separable groups for a set of primes π. In particular, we denote by $l_\pi(G)$ the π-length of a π-separable group G. If X, Y are subgroups of a group G, we set $X^Y = \langle x^y \mid x \in X, y \in Y\rangle$; in particular, X^G is the normal closure of X in G.

2. Preliminary results

We will frequently use the following well-known result, whose proof is straightforward.

Lemma 2.1. *Let the group $G = AB$ be the product of the subgroups A and B. Assume that $D \subseteq A \cap B$ and that D is a normal subgroup of B. Then $D^G \leq A$.*

The next lemma is a reformulation of a result of Kegel, later improved by Wielandt, which appears in [1, Lemma 2.5.1] (see also [9, Lemma 2]).

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Lemma 2.2. *Let the group $G = AB$ be the product of the subgroups A and B and let A_0 and B_0 be normal subgroups of A and B, respectively. If $A_0B_0 = B_0A_0$, then $A_0^gB_0 = B_0A_0^g$ for all $g \in G$.*

Moreover, if A_0 and B_0 are π-groups for a set of primes π, and $O_\pi(G) = 1$, then $[A_0^G, B_0^G] = 1$.

For a set of primes π, we recall that a π-separable group is a *D_π-group*, that is, every π-subgroup is contained in a Hall π-subgroup, and any two Hall π-subgroups are conjugate in the group. We will use, without further reference, the following fact on Hall subgroups of factorised groups, which is applicable to π-separable groups (see [1, Lemma 1.3.2]).

Lemma 2.3. *Let $G = AB$ be the product of the subgroups A and B. Assume that A and B have Hall π-subgroups and that G is a D_π-group for a set of primes π. Then there exist Hall π-subgroups A_π of A and B_π of B such that $A_\pi B_\pi$ is a Hall π-subgroup of G.*

We need specifically the following result, whose proof uses the classification of finite simple groups.

Lemma 2.4 ([15], Theorem 7.7). *Let G be a finite group, $A \trianglelefteq G$, and π a set of primes. Then G is a D_π-group if and only if A and G/A are D_π-groups.*

3. Main results

Our first results on trifactorised groups, Theorem 3.2 and Corollaries 3.3 and 3.4, provide an alternative approach to that of Pennington [14] concerning the π-separability of trifactorised groups. The main goal is to avoid hypotheses of existence, conjugacy and dominance of Hall π-subgroups (D_π-properties), in contrast to Pennington's results. This will follow as consequence of Theorem 3.2, which provides the D_π-property of a trifactorised group, as a first application of Theorem 1.1.

We gather first the above-mentioned results of [14]. We recall that a group G is *π-closed* for a set of primes π if the π-elements of G generate a normal π-subgroup.

Theorem 3.1 ([14], Theorem, Corollary 2). *Let $G = AB = AC = BC$ be a D_π-group where A and B are π-closed subgroups and C is a π-separable subgroup, for a set of primes π. Then:*

(1) *G is π-separable and $O_\pi(C) \subseteq O_\pi(G)$ and $O_{\pi'}(C) \subseteq O_{\pi,\pi'}(G)$;*
(2) *$l_\pi(G) \le l_\pi(C) + 1$ and $l_{\pi'}(G) \le l_{\pi'}(C) + 1$;*
(3) *if A and B are also π'-closed (that is, A and B are π-decomposable), then $l_\pi(G) = l_\pi(C)$ and $l_{\pi'}(G) = l_{\pi'}(C)$ (and also $O_{\pi'}(C) \subseteq O_{\pi'}(G)$).*

Theorem 3.2. *Let π be a set of odd primes. Let the group $G = AB = AC = BC$ be the product of three subgroups A, B and C, where $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups and C is a D_π-group. Then G is a D_π-group.*

Proof. Note first that $A_\pi B_\pi$ is a Hall π-subgroup of G by Theorem 1.1.

We argue by induction on $|G|$. The hypotheses of the result hold for factor groups. Hence whenever N is a nontrivial normal subgroup of G, the inductive hypothesis implies that G/N is a D_π-group. If in addition N is a D_π-group, then the result follows by Lemma 2.4. In particular, we may assume that $O_\pi(G) = O_{\pi'}(G) = 1$. By Lemma 2.2 it follows that $[A_\pi^G, B_\pi^G] = 1$.

We consider now the case where $A_\pi \neq 1$ and $B_\pi \neq 1$. We claim that $A_\pi^G \cap B_\pi^G = 1$. Otherwise, if N is a minimal normal subgroup of G contained in $A_\pi^G \cap B_\pi^G$, then $[N, N] = 1$, that is, N is abelian and then either $N \leq O_\pi(G) = 1$ or $N \leq O_{\pi'}(G) = 1$, a contradiction.

On the other hand,

$$A_\pi^G = A_\pi^{A_\pi A_{\pi'} B_\pi B_{\pi'}} = A_\pi^{B_{\pi'}} = A_\pi[A_\pi, B_{\pi'}] \neq 1$$

and

$$B_\pi^G = B_\pi^{B_\pi B_{\pi'} A_\pi A_{\pi'}} = B_\pi^{A_{\pi'}} = B_\pi[B_\pi, A_{\pi'}] \neq 1.$$

Let H be a π-subgroup of G. We aim to prove that $H \leq (A_\pi B_\pi)^g$ for some $g \in G$.

We apply induction on the factor groups G/A_π^G and G/B_π^G, and may assume that

$$H \leq B_\pi A_\pi[A_\pi, B_{\pi'}]$$

and

$$H \leq (A_\pi B_\pi[B_\pi, A_{\pi'}])^g = (A_\pi B_\pi[B_\pi, A_{\pi'}])^b$$

for some $g = ab$ with $a \in A$, $b \in B$, since $B_\pi[B_\pi, A_{\pi'}]$ is normal in G and A_π is normal in A. Consequently,

$$\begin{aligned} H \leq (B_\pi A_\pi[A_\pi, B_{\pi'}]) \cap (A_\pi B_\pi[B_\pi, A_{\pi'}])^b &= ((A_\pi B_\pi[A_\pi, B_{\pi'}]) \cap (A_\pi B_\pi[B_\pi, A_{\pi'}]))^b \\ &= (A_\pi B_\pi([A_\pi, B_{\pi'}] \cap (A_\pi B_\pi[B_\pi, A_{\pi'}])))^b, \end{aligned}$$

since $A_\pi[A_\pi, B_{\pi'}]$ is normal in G and B_π is normal in B.

We claim that $[A_\pi, B_{\pi'}] \cap (A_\pi B_\pi[B_\pi, A_{\pi'}])$ is a π-group. Since $A_\pi B_\pi$ is a Hall π-subgroup of G, this will imply that $H \leq (A_\pi B_\pi([A_\pi, B_{\pi'}] \cap (A_\pi B_\pi[B_\pi, A_{\pi'}])))^b = (A_\pi B_\pi)^b$, as we aimed to prove.

Let $c \in [A_\pi, B_{\pi'}] \cap (A_\pi B_\pi[B_\pi, A_{\pi'}])$. Then $c = td$ with $t \in A_\pi B_\pi$ and $d \in [B_\pi, A_{\pi'}]$. Hence, $t = cd^{-1}$. But $[A_\pi, B_{\pi'}] \cap [B_\pi, A_{\pi'}] = 1$ and $[[A_\pi, B_{\pi'}], [B_\pi, A_{\pi'}]] = 1$, because $A_\pi^G \cap B_\pi^G = 1$ and $[A_\pi^G, B_\pi^G] = 1$. Consequently, it follows in particular that the order of c divides the order of t, which is a π-number. This proves the claim and the result in the case under consideration.

In the case where $A_\pi = 1$ and $B_{\pi'} = 1$, the group G has Hall π-subgroups and Hall π'-subgroups, which implies that G is a D_π-group (see [2]).

Hence, we may assume without loss of generality that $A_\pi = 1$, $A_{\pi'} \neq 1$, $B_\pi \neq 1$ and $B_{\pi'} \neq 1$. Since $G = AB = AC = BC$ and $A_\pi = 1$, it is easy to deduce by order arguments that $B_\pi \leq C$. Hence, the facts that $B_\pi \lhd B$ and $G = BC$ imply, by Lemma 2.1, that $B_\pi^G \leq C$. Set $N = B_\pi^G$. Since C is a D_π-group, it follows that N and so also G are D_π-groups, by Lemma 2.4, which concludes the proof. □

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COROLLARY 3.3. *Let π be a set of primes. Let the group $G = AB = AC = BC$ be the product of three subgroups A, B and C, where $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups and C is π-separable. Then G is π-separable.*

Moreover, $O_\pi(C) \subseteq O_\pi(G)$ and $l_\pi(G) = l_\pi(C)$ (and also $O_{\pi'}(C) \subseteq O_{\pi'}(G)$ and $l_{\pi'}(G) = l_{\pi'}(C)$).

PROOF. We may assume that $\pi \neq \emptyset$ and $\pi' \neq \emptyset$. Let $\sigma \in \{\pi, \pi'\}$ such that $2 \notin \sigma$. Then C is σ-separable and so C is a D_σ-group. By Theorem 3.2, G is a D_σ-group and the result follows by Theorem 3.1. □

The following result is easily deduced.

COROLLARY 3.4. *Let π be a set of primes. Let the group $G = AB = AC = BC$ be the product of three subgroups A, B and C. If A, B and C are π-decomposable groups, then G is π-decomposable.*

It may be of interest to compare Corollary 3.4 with the following result, which appears in [14] as a corollary of Theorem 3.1(1). Indeed, Corollary 3.4 may also be seen as consequence of the following result together with Theorem 3.2.

COROLLARY 3.5 ([13, Satz 1], [14, Corollary 1]). *Let G be a D_π-group. Then G is π-closed if and only if there are subgroups A, B and C of G, all π-closed and satisfying $G = AB = AC = BC$.*

The following example shows a trifactorised group $G = AB = AC = BC$ with subgroups A, B and C such that A and B are π-decomposable but G and C are not π-separable.

EXAMPLE 3.6. Consider $X = \mathrm{Alt}(5)$ the alternating group of degree 5 and let $G = X \times X$. Let $Y, Z \leq X$ with $Y \cong \mathrm{Alt}(4)$ and $Z \cong C_5$ the cyclic group of order 5. Let $A = Y \times Z$, $B = Z \times Y$, and let $C = D(X) = \{(x, x) \mid x \in X\} \cong A_5$ be the diagonal subgroup. Set $\pi = \{5\}$, so $\pi' = \{2, 3\}$. Then $G = AB = AC = BC$ and A and B are π-decomposable groups, but G and C are not π-separable.

We show next that under the hypotheses of Corollary 3.3 the π-length of the group G can be arbitrarily large.

EXAMPLE 3.7. Consider P a nontrivial π-group and Q a nontrivial π'-group, for a set of primes π. For every $i \geq 1$, we define inductively a group X_i as follows:

$$X_1 = P, \quad X_2 = P \sim Q$$
$$X_i = X_{i-1} \sim P, \quad X_{i+1} = X_i \sim Q \quad \text{when } i \geq 3,\ i \text{ odd},$$

where $R \sim S$ denotes the regular wreath product of R with S, for any pair of groups R and S.

Consider $X = X_n$ for any positive integer n. Write $X^{(1)} = X^{(2)} = X$ and set $G = X^{(1)} \times X^{(2)}$. Take $X^{(i)}_\sigma$ a σ-Hall subgroup of X, for each $\sigma \in \{\pi, \pi'\}$ and $i = 1, 2$. Now let $A = X^{(1)}_\pi \times X^{(2)}_{\pi'}$, $B = X^{(1)}_{\pi'} \times X^{(2)}_\pi$ and $C = D(X) = \{(x, x) \mid x \in X\} \cong X$ the diagonal subgroup. Then $G = AB = AC = BC$, A and B are π-decomposable groups, C and G are π-separable and $l_\pi(G) = l_\pi(C)$ is either $n/2$ or $(n+1)/2$, depending on whether n is even or odd.

It is well known that the Fitting subgroup of a product of two nilpotent groups is factorised (see [1, Lemma 2.5.7]). As an application of Corollary 3.4, we obtain the following generalisation of that result for π-decomposable groups. A particular case in the universe of finite soluble groups was obtained in [4, Theorem 2].

Proposition 3.8. *Let $\mathcal{F}$ be the class of all π-decomposable groups, for a set of primes π. If $G = AB$ is a π-separable group and A and B are $\mathcal{F}$-groups, then the $\mathcal{F}$-radical $G_{\mathcal{F}}$ of G is a factorised subgroup, that is, $G_{\mathcal{F}} = (G_{\mathcal{F}} \cap A)(G_{\mathcal{F}} \cap B)$ and $A \cap B$ is contained in $G_{\mathcal{F}}$. (Recall that $G_{\mathcal{F}} = O_\pi(G) \times O_{\pi'}(G)$.)*

Proof. Assume that the result is not true and let G be a counterexample of minimal order. Since G is π-separable, $G_{\mathcal{F}} = O_\pi(G) \times O_{\pi'}(G) \neq 1$, and the choice of G implies that the $\mathcal{F}$-radical $L/G_{\mathcal{F}}$ of the factor group $G/G_{\mathcal{F}} = (AG_{\mathcal{F}}/G_{\mathcal{F}})(BG_{\mathcal{F}}/G_{\mathcal{F}})$ is factorised; in particular,

$$(AG_{\mathcal{F}}/G_{\mathcal{F}}) \cap (BG_{\mathcal{F}}/G_{\mathcal{F}}) \leq L/G_{\mathcal{F}}.$$

Set $X = X(G_{\mathcal{F}})$, the factoriser of $G_{\mathcal{F}}$ in $G = AB$. Then $G_{\mathcal{F}} < X = AG_{\mathcal{F}} \cap BG_{\mathcal{F}} \leq L$ and

$$\begin{aligned} L &= (L \cap AG_{\mathcal{F}})(L \cap BG_{\mathcal{F}}) = (L \cap A)G_{\mathcal{F}}(L \cap B) \subseteq (L \cap A)X(L \cap B) \\ &= (L \cap A)(X \cap A)(X \cap B)(L \cap B) = (L \cap A)(L \cap B) \subseteq L, \end{aligned}$$

that is, $L = (L \cap A)(L \cap B)$.

If L were a proper subgroup of G, then by the minimal choice of G the $\mathcal{F}$-radical of L would be factorised with respect to the factorisation $L = (L \cap A)(L \cap B)$. But $A \cap B \leq X \leq L$, and so $A \cap B = (L \cap A) \cap (L \cap B) \leq L_{\mathcal{F}}$. Then $G_{\mathcal{F}} = L_{\mathcal{F}}$ would also be factorised with respect to $G = AB$, a contradiction.

Consequently, $L = G$ and $G/G_{\mathcal{F}}$ is an $\mathcal{F}$-group, that is,

$$G/G_{\mathcal{F}} = O_\pi(G/G_{\mathcal{F}}) \times O_{\pi'}(G/G_{\mathcal{F}}).$$

Since $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ and G is π-separable, we deduce by Lemma 2.3 that $A_\pi B_\pi$ is a Hall π-subgroup of G, and $A_{\pi'} B_{\pi'}$ is a Hall π'-subgroup of G. It follows that $A_\pi B_\pi G_{\mathcal{F}} = A_\pi B_\pi O_{\pi'}(G)$ and $A_{\pi'} B_{\pi'} G_{\mathcal{F}} = A_{\pi'} B_{\pi'} O_\pi(G)$ are normal subgroups in G. Now, applying Corollary 3.4,

$$X = (A \cap BG_{\mathcal{F}})G_{\mathcal{F}} = (B \cap AG_{\mathcal{F}})G_{\mathcal{F}} = (A \cap BG_{\mathcal{F}})(B \cap AG_{\mathcal{F}})$$

is an $\mathcal{F}$-group, that is, $X = X_\pi \times X_{\pi'}$.

Let $\sigma \in \{\pi, \pi'\}$. Since $G_{\mathcal{F}} = O_\pi(G) \times O_{\pi'}(G) \leq X$, we deduce in particular that $[X_\sigma, O_{\sigma'}(G)] = 1$. Since G is π-separable, X_σ is contained in some Hall σ-subgroup of G. But every Hall σ-subgroup of G has the form $(A_\sigma B_\sigma)^t$ for some $t \in O_{\sigma'}(G)$, as $A_\sigma B_\sigma O_{\sigma'}(G) \trianglelefteq G$, so it contains X_σ. Hence $X_\sigma \leq O_\sigma(G)$. Consequently, $X = G_{\mathcal{F}}$, the final contradiction. $\square$

The next example shows that the above result is not true if G is not a π-separable group.

EXAMPLE 3.9. Let $N = L_2(2^6)$ and let ϕ be the Frobenius automorphism of N and $\psi = \phi^2$, which is an automorphism of N of order 3. Consider $G = [N]\langle\psi\rangle$ the natural semidirect product of N with $\langle\psi\rangle$. We note that $|G| = 2^6 \cdot 3^4 \cdot 5 \cdot 7 \cdot 13$ and also that $C_G(\psi) \cong L_2(2^2)$. Set $\pi = \{2, 3, 7, 13\}$.

The group G can be factorised as $G = AB$, where $A = N_G(G_2)$ is a π-group, $B = N_G(G_{13}) = B_\pi \times B_{\pi'} \cong ([C_{13}]C_3) \times C_5$ is π-decomposable and $|A \cap B| = 3$. Hence, if $\mathcal{F}$ is the class of all π-decomposable groups, the $\mathcal{F}$-radical of G is $G_\mathcal{F} = 1$, and it is not factorised.

Theorem 3.11 below provides a stronger version of Corollary 3.4 for a trifactorised group where two of the factors are π-decomposable and the third factor is a subnormal subgroup. We will need the following preliminary result. For any formation $\mathcal{F}$ and any group X, we denote by $X^\mathcal{F}$ the $\mathcal{F}$-residual of X.

LEMMA 3.10. *Let $\mathcal{F}$ be a Fitting formation. If the group $G = HK$ is the product of two subnormal subgroups H and K, then $G^\mathcal{F} = H^\mathcal{F} K^\mathcal{F}$.*

PROOF. We argue by induction on $d_H + d_K$, where d_X denotes the subnormal defect of X in G for each $X \in \{H, K\}$, that is, the smallest nonnegative integer d_X such that there exists a series $X = X_0 \lhd X_1 \lhd \cdots \lhd X_{d_X} = G$ of subgroups of G. If H and K are normal subgroups of G ($d_H + d_K \le 2$), the result follows by [6, II, Lemma 2.12]. Without loss of generality assume that H is not normal in G and let $H < \hat{H} \lhd G$. We observe that $\hat{H} = H(\hat{H} \cap K)$, so, by the inductive hypothesis, $G^\mathcal{F} = \hat{H}^\mathcal{F} K^\mathcal{F}$ and $\hat{H}^\mathcal{F} = H^\mathcal{F}(\hat{H} \cap K)^\mathcal{F}$. Since $\mathcal{F}$ is closed under taking subnormal subgroups, it follows that $(\hat{H} \cap K)^\mathcal{F} \le K^\mathcal{F}$, and so $G^\mathcal{F} = H^\mathcal{F} K^\mathcal{F}$. □

THEOREM 3.11. *Let π be a set of primes. Let the group $G = AB = AC = BC$ be the product of three subgroups A, B and C, where $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups and C is a subnormal subgroup of G. If $\mathcal{F}$ is the class of all π-decomposable groups, then $G^\mathcal{F} = C^\mathcal{F}$.*

PROOF. We may assume that π is a set of odd primes.

First notice that the class $\mathcal{F}$ of all π-decomposable groups is a Fitting formation. Suppose the result is not true and let G be a group of minimal order among the groups X having two π-decomposable subgroups H and K and a subnormal subgroup L such that $G = HK = HL = KL$ and $G^\mathcal{F} \ne L^\mathcal{F}$.

Then there exist two π-decomposable subgroups A and B of G and a subnormal subgroup C of G such that $G = AB = AC = BC$ and $G^\mathcal{F} \ne C^\mathcal{F}$. We choose C with $|C|$ maximal. We split the proof into the following steps.

Step 1. $G^\mathcal{F} = C^\mathcal{F} N$ for every minimal normal subgroup N of G, $C^\mathcal{F} \lhd G^\mathcal{F}$ and $\mathrm{Core}_G(C^\mathcal{F}) = 1$.

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Let N be a minimal normal subgroup of G. Since $(G/N)^{\mathcal{F}} = G^{\mathcal{F}}N/N$, the minimal choice of G implies that $G^{\mathcal{F}}N = C^{\mathcal{F}}N$. Moreover, $C^{\mathcal{F}} \leq G^{\mathcal{F}}$ which implies that $G^{\mathcal{F}} = C^{\mathcal{F}}(G^{\mathcal{F}} \cap N)$. Since $G^{\mathcal{F}} \neq C^{\mathcal{F}}$, it follows that $G^{\mathcal{F}} \cap N = N$ and so $N \leq G^{\mathcal{F}}$. Then $G^{\mathcal{F}} = C^{\mathcal{F}}N$ and also $\mathrm{Core}_G(C^{\mathcal{F}}) = 1$. Moreover, since $C^{\mathcal{F}}$ is a subnormal subgroup of G, N normalises $C^{\mathcal{F}}$ (see [6, A, Lemma 14.3]), which implies that $C^{\mathcal{F}} \trianglelefteq G^{\mathcal{F}}$.

Step 2. If there are two different minimal normal subgroups, then they are abelian.

Assume that N_1, N_2 are minimal normal subgroups, $N_1 \neq N_2$. By Step 1, $G^{\mathcal{F}} = C^{\mathcal{F}}N_1 = C^{\mathcal{F}}N_2$. Since $[N_1, N_2] = 1$, we deduce that $N_i' \leq C^{\mathcal{F}}$ for $i = 1, 2$. Since $\mathrm{Core}_G(C^{\mathcal{F}}) = 1$ it follows that N_1 and N_2 are abelian.

Step 3. $G_{\mathcal{F}} = O_{\pi}(G) \times O_{\pi'}(G) \leq C$.

Suppose C is a proper subgroup of $CG_{\mathcal{F}}$. Since $G = AB = A(CG_{\mathcal{F}}) = B(CG_{\mathcal{F}})$, $CG_{\mathcal{F}}$ is a subnormal subgroup of G and $|C| < |CG_{\mathcal{F}}|$, it follows by the maximality of C that $G^{\mathcal{F}} = (CG_{\mathcal{F}})^{\mathcal{F}}$. By Lemma 3.10, $G^{\mathcal{F}} = C^{\mathcal{F}}$, a contradiction. Therefore $C = CG_{\mathcal{F}}$ and so $G_{\mathcal{F}} \leq C$.

Step 4. $G_{\mathcal{F}} = O_{\pi}(G) \times O_{\pi'}(G) \neq 1$. Moreover, if $\sigma \in \{\pi, \pi'\}$ such that $O_{\sigma}(G) \neq 1$, then $O_{\sigma'}(G) = 1$.

Assume that $O_{\pi}(G) = 1$ and $O_{\pi'}(G) = 1$. We know that $A_{\pi}B_{\pi}$ is a subgroup of G by Theorem 1.1. Then Lemma 2.2 implies that $[A_{\pi}^G, B_{\pi}^G] = 1$. Consequently, from this fact together with Step 2, we deduce that, if $A_{\pi} \neq 1$ and $B_{\pi} \neq 1$, then there is an abelian minimal normal subgroup, and so a normal p-subgroup, for a prime p, which is a contradiction. Therefore, we may assume without loss of generality that $A_{\pi} = 1$ and $B_{\pi} \neq 1$. Since $G = AB = AC = BC$, by order arguments it follows that $B_{\pi} \leq C$. Moreover, $B_{\pi} \trianglelefteq B$ and $G = BC$, which implies, by Lemma 2.1, that $B_{\pi}^G \leq C$. Then there is a minimal normal subgroup N of G contained in C, and $G^{\mathcal{F}} = C^{\mathcal{F}}N \leq C$. Hence $G^{\mathcal{F}}/C^{\mathcal{F}} \cong N/(N \cap C^{\mathcal{F}}) \in \mathcal{F}$. Now N is a nonabelian minimal normal subgroup, and so it is a direct product of copies of a nonabelian simple group. But $N \cap C^{\mathcal{F}}$ is a direct product of simple components of N, because it is a normal subgroup of N. It follows that N is a π'-group, and so $N \leq O_{\pi'}(G) = 1$, a contradiction. Therefore, $G_{\mathcal{F}} = O_{\pi}(G) \times O_{\pi'}(G) \neq 1$.

The last statement follows because $G^{\mathcal{F}}/C^{\mathcal{F}} \cong N/(N \cap C^{\mathcal{F}})$ for every minimal normal subgroup N of G.

Step 5. $G^{\mathcal{F}} \leq O_{\sigma}(G)$. If there is a minimal normal subgroup which is an elementary abelian p-group for a prime p, then $G^{\mathcal{F}}$ has the same properties. Moreover, G is σ-separable (and σ'-separable).

Let N be a minimal normal subgroup of G, $N \leq O_{\sigma}(G)$. Since $G^{\mathcal{F}} = C^{\mathcal{F}}N$, we see that $G^{\mathcal{F}}/C^{\mathcal{F}} \cong N/(N \cap C^{\mathcal{F}})$. Then $O^{\sigma}(G^{\mathcal{F}}) \leq \mathrm{Core}_G(C^{\mathcal{F}}) = 1$, which implies that $G^{\mathcal{F}}$ is a σ-group. If there is a minimal normal subgroup which is an elementary abelian p-group for a prime p, analogous arguments prove that $G^{\mathcal{F}}$ has the same properties. Moreover, it follows now that G is σ-separable, as is $G/G^{\mathcal{F}}$.

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Step 6. $G = G_\sigma G_{\sigma'}$, $G^{\mathcal{F}} \le G_\sigma \trianglelefteq G$, $G^{\mathcal{F}} G_{\sigma'} \trianglelefteq G$, $G^{\mathcal{F}} = (G^{\mathcal{F}} G_{\sigma'})^{\mathcal{F}}$.

This follows by Step 5 and Lemma 3.10.

Step 7. The final contradiction.

If $A_{\sigma'} = 1$, then we may take $G_{\sigma'} = B_{\sigma'} \le C$, which implies that $G^{\mathcal{F}} G_{\sigma'} \le C$ and so $G^{\mathcal{F}} = (G^{\mathcal{F}} G_{\sigma'})^{\mathcal{F}} \le C^{\mathcal{F}}$, a contradiction. Analogously $B_{\sigma'} = 1$ is not possible. Consequently $A_{\sigma'} \neq 1$, $B_{\sigma'} \neq 1$ and $O_{\sigma'}(G) = 1$. Again, by Lemma 2.2, $[A_{\sigma'}^G, B_{\sigma'}^G] = 1$, and together with Step 2, we can consider a minimal normal subgroup $N \le B_{\sigma'}^G$ which is abelian. In particular, $[A_{\sigma'}^G, N] = 1$.

By Lemma 2.3 there exists a Hall σ'-subgroup of C, say $C_{\sigma'}$, such that $A_{\sigma'} C_{\sigma'}$ is a σ'-Hall subgroup of G. Since N is an elementary abelian group and $C_{\sigma'}$ acts coprimely on N, we can apply Maschke's theorem (see [6, A, Theorem 11.5]) to deduce that the $C_{\sigma'}$-invariant subgroup $C^{\mathcal{F}} \cap N$ has a $C_{\sigma'}$-invariant complement in N, say H. Moreover, since $\mathrm{Core}_G(C^{\mathcal{F}}) = 1$, it follows that $H \neq 1$. So $G^{\mathcal{F}} = C^{\mathcal{F}} N = C^{\mathcal{F}} H$ with $C^{\mathcal{F}} \cap H = 1$.

Now $C_{\sigma'} G^{\mathcal{F}} / C^{\mathcal{F}} \le C / C^{\mathcal{F}}$ is an $\mathcal{F}$-group. But $C_{\sigma'} G^{\mathcal{F}} / C^{\mathcal{F}} = C_{\sigma'} H C^{\mathcal{F}} / C^{\mathcal{F}} \cong C_{\sigma'} H$, because $C^{\mathcal{F}} \cap C_{\sigma'} H = 1$. This means that $C_{\sigma'} H$ is an $\mathcal{F}$-group, and so H centralises $C_{\sigma'}$. Since $[N, A_{\sigma'}] = 1$, it follows that H centralises $G_{\sigma'} = A_{\sigma'} C_{\sigma'}$, which is a Hall σ'-subgroup of G. In particular, $H \times G_{\sigma'} \in \mathcal{F}$.

Since $G^{\mathcal{F}} = C^{\mathcal{F}} H$ is an elementary abelian subgroup by Step 5, again by Maschke's theorem, there exists a complement of H in $G^{\mathcal{F}}$, say T, which is $G_{\sigma'}$-invariant. But then, by Step 6, we see that $G^{\mathcal{F}} = (G^{\mathcal{F}} G_{\sigma'})^{\mathcal{F}} = (THG_{\sigma'})^{\mathcal{F}} \le T$, which is a proper subgroup of $G^{\mathcal{F}}$, the final contradiction. □

Remark 3.12. Example 3.6 shows that the statement in Theorem 3.11 does not remain true if the subgroup C fails to be subnormal.

As a particular case of Theorem 3.11 we recover the following extension of Kegel's result quoted in the introduction, which appears in [3].

Corollary 3.13. *Let the finite group $G = AB = AN = BN$ be the product of three subgroups A, B and N, where N is subnormal in G. If A and B are nilpotent, then the nilpotent residual of G coincides with the nilpotent residual of N. In particular, the nilpotent residual of N is normal in G.*

One might expect that the result of Peterson [1, Theorem 2.5.10] mentioned in the introduction should generalise to a corresponding positive result by replacing the class of nilpotent groups by a class of π-decomposable groups for a set of primes π. The following example shows that this is not the case, even if the factor C is assumed to be a π-separable normal subgroup and the saturated formation to contain all π-decomposable groups.

Example 3.14. Let π be a set of primes. Assume that the group $G = AB = AC = BC$ is the product of three subgroups A, B and C, where $A = A_\pi \times A_{\pi'}$ and $B = B_\pi \times B_{\pi'}$ are π-decomposable groups, and C is a π-closed normal subgroup of G. If $\mathcal{F}$ is a saturated

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formation containing the class of all π-decomposable groups, the next example shows that it is not true in general that $G \in \mathcal{F}$ whenever $C \in \mathcal{F}$.

Choose the groups $T = \langle t \rangle \cong C_7$, $Y = \langle y \rangle \cong C_3$, $X = \langle x \rangle \cong C_2$ and consider the natural action of $Y \times X \cong \mathrm{Aut}(T)$ on T as automorphism group; more precisely, $t^y = t^2$, $t^x = t^{-1}$. Let TYX be the corresponding semidirect product. We consider now an irreducible and faithful TYX-module V over the field of five elements (see [6, B, Theorem 10.3]) and form $G = VTYX$ the corresponding semidirect product.

Take π to be the set of all odd primes, so $\pi' = \{2\}$, $A = VTY$ which is a π-group, $B = YX$ which is a π-decomposable group, and $C = VTX$ which is a π-closed normal subgroup of G. We note that $G = AB = AC = BC$. By [6, IV, Proposition 1.3)], the class of groups

$$\mathcal{H} = (G \mid \mathrm{Aut}_G(S) \in (C_2, \mathcal{E}_{2'}) \text{ for all 7-chief factors } S \text{ of } G)$$

is a formation, where $(C_2, \mathcal{E}_{2'})$ denotes the class of groups which either are isomorphic to C_2 or belong to $\mathcal{E}_{2'}$, the class of groups of odd order.

We now consider $\mathcal{F} = \mathrm{LF}(f)$ the saturated formation locally defined by the formation function f given in the following way:

$$\begin{aligned} f(p) &= \mathcal{H} && \text{for every prime } p \neq 2, \\ f(2) &= \mathcal{E}_2 && \text{the class of 2-groups.} \end{aligned}$$

It is easy to see that the class of all π-decomposable groups is contained in $\mathcal{F}$. Moreover, $C \in \mathcal{F}$ but $G \notin \mathcal{F}$.

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