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ON ZERO-SUM TURAN PROBLEMS OF BIALOSTOCKI AND DIERKER

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Abstract

Assume G is a graph with m edges. By $T(n, G)$ we denote the classical *Turan number*, namely, the maximum possible number of edges in a graph H on n vertices without a copy of G. Similarly if G is a family of graphs then H does not have a copy of any member of the family. A *Z_k-colouring* of a graph G is a colouring of the edges of G by Z_k, the additive group of integers modulo k, avoiding a copy of a given graph H, for which the sum of the values on its edges is $0 \pmod k$. By the *Zero-Sum Turan number*, denoted $T(n, G, Z_k)$, $k \mid m$, we mean the maximum number of edges in a Z_k-colouring of a graph on n vertices that contains no zero-sum $\pmod k$ copy of G. Here we mainly solve two problems of Bialostocki and Dierker [6].

PROBLEM 1. Determine $T(n, tK_2, Z_k)$ for $k \mid t$. In particular, is it true that $T(n, tK_2, Z_k) = T(n, (t+k-1)K_2)$?

PROBLEM 2. Does there exist a constant $c(t, k)$ such that $T(n, F_t, Z_k) \le c(t, k)n$, where F_t is the family of cycles of length at least t?

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1. Introduction

In 1961, Erdos, Ginzburg and Ziv [14] proved the following theorem:

THEOREM A. *Let $\{a_1, a_2, \dots, a_{(m+1)k-1}\}$ be a collection of integers. Then there exists a subset $I \subset \{1, 2, \dots, (m+1)k-1\}$, $|I| = mk$, such that $\sum_{i\in I} a_i \equiv 0 \pmod k$.*

This theorem was the starting point of the seminal paper of Bialostocki and Dierker [2], in which they introduced the concept of zero-sum colouring.

Graphs in this paper are finite and have neither multiple edges nor loops. By $R(G; k)$ we denote the least positive integer r such that in any k colouring of the edges of the complete graph K_r, there is a monochromatic copy of G. From now on we assume that G is a graph with m edges. By the *Zero-Sum Ramsey number*, denoted $R(G; Z_k)$, $k \mid m$, we mean the least positive integer r such that in any colouring of the edges of the complete graph K_r by Z_k, the additive group of integers modulo k, there is a copy of G such that the sum of the values on its edges is $0 \pmod k$. The existence of $R(G; Z_k)$ follows from the existence of the classical Ramsey number $R(G; k)$, since, following the definitions, we have

$$R(G; Z_k) \leq R(G; k). \tag{1}$$

By $T(n, G)$ we denote the classical *Turan number*, namely, the maximum possible number of edges in a graph H on n vertices without a copy of G. Similarly if G is a family of graphs, then H does not contain a copy of any member of the family. By the *Zero-Sum Turan number*, denoted $T(n, G, Z_k)$, $k \mid m$, we mean the maximum number of edges in a Z_k-colouring of a graph on n vertices that contains no zero-sum $\pmod k$ copy of G.

There is a rapidly growing literature on zero-sum problems as can be indicated from the list of references (which is by no means complete) [1–5, 8–13, 15, and 18].

Bialostocki and Dierker [2–6] raised several problems whose essence is summarized in the following

PROBLEM 1. Determine $T(n, tK_2, Z_k)$ for $k \mid t$. In particular, is it true that $T(n, tK_2, Z_k) = T(n, (t+k-1)K_2)$?

PROBLEM 2. Does there exist a constant $c(t, k)$ such that $T(n, F_t, Z_k) \leq c(t, k)n$?

The connection between these problems and the above theorem is quite obvious, and fortunately we are able to solve them completely.

Our notation is standard and follows [7, 16]. In particular, $e(G)$ denotes the number of edges of G.

A graph H is said to be *a topological graph* of a given graph G if H is obtained from G by replacing some edges of G by paths, an operation also called *sub-division*. The family of all topological graphs of a given graph G will be denoted by TG.

Let $H \in TG$. Observe that to every edge e of G there corresponds a path P_e in H. This defines a natural one-to-one mapping $\alpha\colon E(G) \to \{P_e \in$

H : P_e is a path obtained by subdividing $e \in E(G)\}$.

We are ready now for our results.

2. Results and proofs

We start with a result from [12] whose simple proof is given in order to keep this paper self-contained.

THEOREM 1 [12]. *Let $t \geq k \geq 2$ be integers such that $k \mid t$. Then for every graph G $T(n, tG, Z_k) \leq T(n, (t+k-1)G)$.*

PROOF. Let H be a graph on n vertices with $T(n, (t+k-1)G)+1$ edges. Let $c: E(H) \to Z_k$ be a Z_k-colouring of the edges of H. By the definition of the classical Turan numbers, H must contain $t+k-1$ vertex-disjoint copies of G, denoted by G_i. Put $a_i = \sum_{e \in E(G_i)} c(e)$. Then by Theorem A, as $t = km$, there is a subset $I \subset \{1, 2, \ldots, (t+k-1)\}$, $|I| = t$, such that $\sum_{i \in I} a_i \equiv 0 \pmod{k}$. Hence $\bigcup_{i \in I} G_i = tG$ has the required property.

We are able now to solve Problem 1.

THEOREM 2. *Let $t \geq k \geq 2$ be integers such that $k \mid t$. Then for $n \geq 5t$,*

$$T(n, tK_2, Z_k) = T(n, (T+k-1)K_2) = \binom{t+k-2}{2} + (t+k-2)(n-t-k+2).$$

PROOF. By Theorem 1, $T(n, tK_2, Z_k) \leq T(n, (t+k-1)K_2)$. The Turan numbers for matching were determined, 30 years ago, by Erdos and Gallai (see [7]) who proved that for $n \geq 5t$,

$$T(n, (t+k-1)K_2) = \binom{t+k-2}{2} + (t+k-2)(n-t-k+2).$$

Consider now the following construction. Color the edges of the complete bipartite graph $K_{t-1, n-t+1}$ and the complete graph K_{t-1} by 0. In the vertex class of cardinality $n-t+1$, colour by 1 the graph $K_{k-1} \cup K_{k-1, n-t-k+2}$.

It is easy to see that in the resulting graph H on n vertices there is no zero-sum $\pmod{k}$ copy of tK_2, and the number of edges in this graph is

$$\binom{t-1}{2} + \binom{k-1}{2} + (t-1)(n-t+1) + (k-1)(n-t-k+2)$$
$$= \binom{t+k-2}{2} + (t+k-2)(n-t-k+2)$$

as required.

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REMARK. Using the Erdos-Gallai Theorem, and their characterization of the extremal graphs for every n, one can show with a little more effort that, also for $n < 5t$, $T(n, tK_2, Z_k) = T(n, (t+k-1)K_2)$. We omit the proof and the construction.

The next result gives a solution to a much more general problem than Problem 2, namely,

THEOREM 3. *Let G be a non-empty graph. Then there exists a positive constant $c(G, k)$ such that $T(n, TG, Z_k) \leq c(G, k)n$.*

PROOF. Suppose $e(G) \equiv r \pmod{k}$. If $r = 0$ set $G^* = G$. Otherwise, subdivide edges of G to obtain a graph $G^* \in TG$ such that $k \mid e(G^*)$. As $k \mid e(G^*)$ it is clear that $R(G^*; Z_k)$ is well defined.

CLAIM. $T(n, TG, Z_k) \leq T(n, TK_{R(G^*;Z_k)})$.

Indeed, suppose H is a graph on n vertices and $T(n, TK_{R(G^*;Z_k)}) + 1$ edges, and let $c\colon E(H) \to Z_k$ be a Z_k-colouring. By the definition of Turan numbers, there exists a copy of a graph $F \in TK_{R(G^*;Z_k)}$ in H. Now, $c\colon E(F) \to Z_k$ induces another colouring $f\colon E(K_{R(G^*;Z_k)}) \to Z_k$ by $f(e) = c(P_e)$ where e and P_e are an edge of $K_{R(G^*;Z_k)}$ and the corresponding path in F, and $c(P_{e_i}) = \sum_{e\in E(P_{e_j})} c(e)$, where addition is performed modulo k. By the definition of $R(G^*, Z_k)$ there exists a zero-sum copy $\pmod{k}$ M of G^* (with respect to f) in $K_{R(G^*;Z_k)}$.

Clearly the corresponding graph of M in F, namely, the graph induced by the paths $\bigcup_{e\in M} P_\alpha(e)$, where α is the natural mapping between $K_{R(G^*;Z_k)}$ and F, is a zero-sum $\pmod{k}$ topological graph of G^* (with respect to c). Hence, $T(n, TG, Z_k) \leq T(n, TK_{R(G^*;Z_k)})$, proving the claim.

Recall now the deep theorem of Mader [17] which, in a weak form, states that $T(n, TK_m) \leq 3n2^{m-3}$.

Now, $T(n, TG, Z_k) \leq T(n, TK_{R(G^*;Z_k)}) \leq 3n2^{R(G^*;Z_k)-3}$, which proves the theorem with $c(G, k) = 3 \cdot 2^{R(G^*;Z_k)-3}$.

Let F_t be the family of cycles of length at least t. Then we have the following corollary.

COROLLARY. $T(n, F_t, Z_k) \leq c(t, k)n$.

PROOF. Set $q = k\lceil t/k\rceil$. Then from Theorem 3 we find that $T(n, F_t, Z_k) \leq T(n, TC_q, Z_k) \leq c(C_q, k)n = c(t, k)n$.

In the case of $T(n, P_m, Z_k)$, where P_m is a path on m edges, one can improve considerably the upper bound of Theorem 3, namely,

THEOREM 4. $T(n, TP_m, Z_k) \leq T(n, P_{km}) \leq (km-1)n/2$.

PROOF. Let H be a graph on n vertices and $T(n, P_{km})+1$ edges. Then H contains a path on km edges. Order the edges of that path by $e_1, e_2, \dots, e_{km}$. Suppose $c\colon E(H) \to Z_k$ is given. Define $a_i = \sum_{j=1}^{i} c(e_j)$. Then we have a set $A = \{a_1, a_2, \dots, a_{km}\}$. Consider the subset

$$B \subset A, \quad B = \{a_m, a_{2m}, \dots, a_{km}\}.$$

If $a_{jm} \equiv 0 \pmod{k}$, then $e_1, \dots, e_{jm}$ is a suitable path. If no $a_{jm} \equiv 0 \pmod{k}$, then there must exist $0 \leq i < j \leq k$ such that $a_{im} \equiv a_{jm} \pmod{k}$, so that the path starting with the edge e_{im+1} and ending with the edge e_{jm} is zero-sum $\pmod{k}$ on $jm - im = (j-i)m$ edges. Hence, $T(n, TP_m, Z_k) \leq T(n, P_{km}) \leq (km-1)n/2$, by the well-known Erdos-Gallai upper bound for $T(n, P_m)$ (see [19]).

Theorem 3 suggests the following strong form of Mader's Theorem.

THEOREM 5. *Let G be a non-empty graph and let $k \geq 2$ be an integer. Then there exists a constant $c(G, k) > 0$ such that every graph on n vertices and $c(G, k)n + 1$ edges contains a topological graph $H \in TG$, such that $k \mid e(H)$.*

PROOF. Let F be an arbitrary graph on n vertices and $c(G, k)n + 1$ edges, where $c(G, k)$ is the constant from Theorem 3. Let $c\colon E(F) \to Z_k$ be defined by $c(e) \equiv 1$. By Theorem 3 there exists a zero-sum $\pmod{k}$ topological graph H of G. As $c(e) \equiv 1$, $e(H) = \sum_{e \in E(H)} c(e) \equiv 0 \pmod{k}$. Hence $k \mid e(H)$ as claimed.

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