

Limit theorems for semi-Markov processes

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A new construction of regeneration times is exploited to prove ergodic and renewal theorems for semi-Markov processes on general state spaces. This work extends results of the authors in *Ann. Probability* (6 (1978), 788-797).

1. Introduction

This note is a continuation of [2] and [3], where we introduced a construction of regeneration times to show that recurrent Markov chains on general state spaces act as if they had a single recurrence point which is visited infinitely often. This device was used to give renewal theoretic proofs of ergodic and renewal theorems for Markov and semi-Markov chains. The semi-Markov results were proved under a strong aperiodicity condition on the underlying Markov chain, and a further strong restriction on the distributions of sojourn times in a state. Our objective here is to remove some of these restrictions.

Consider a space S with a σ -algebra of subsets \mathcal{S} . Let $\{X_n; n = 0, 1, \dots\}$ be a Markov chain on (S, \mathcal{S}) with homogeneous transition function $P(s, E)$, $s \in S$, $E \in \mathcal{S}$, and let $\{G_{xy}(\cdot); x, y \in S\}$ be a family of distribution functions on $R^+ = [0, \infty)$. Given a realization $\{X_n = x_n; n = 0, 1, \dots\}$ of the chain, generate independent random variables $\{L_n; n = 0, 1, \dots\}$ such that

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$$(1.1) \quad P\{L_i \leq t \mid X_n = x_n, n = 0, 1, \dots\} = G_{x_i, x_{i+1}}(t), \quad t \geq 0.$$

Set

$$W(t) = (Z(t), A(t)) = \begin{cases} (X_0, t) & \text{when } 0 \leq t < L_0 \\ (X_1, t-L_0) & \text{when } L_0 \leq t < L_0+L_1 \\ \vdots & \vdots \\ (X_k, t - \sum_0^{k-1} L_i) & \text{when } \sum_0^{k-1} L_i \leq t < \sum_0^k L_i \\ \vdots & \vdots \end{cases}$$

The process $\{W(t); 0 \leq t\}$ is a Markov process. $\{Z(t); 0 \leq t\}$ is called a *semi-Markov process*, and $\{A(t); 0 \leq t\}$ is its associated *age process*. Also $\{X_n\}$ is referred to as the state process, $\{L_n\}$ as its "sojourn" times, and $\{X_n, L_{n-1}\}$ as a *semi-Markov chain*. (Note that this pair chain is actually a Markov chain; the joint distribution of $\{X_{n+1}, L_n\}$ depends only on X_n .) The objective is to determine the limiting behaviour of $W(t)$; more specifically of functions of the form $Ef(W(t))$ and

$$E\left[\sum_{n=0}^{\infty} g\left(X_n, t - \sum_{i=0}^n L_i\right)\right], \text{ for reasonable classes of functions } f \text{ and } g.$$

The key hypothesis is designed to guarantee the required recurrence structure.

HYPOTHESIS H_k . (i) *There exists a set $A \in S$ such that for some integer k ,*

$$(1.2) \quad P_x\{X_{nk} \in A \text{ for some } n \geq 1\} = 1, \quad x \in S.$$

(ii) *There exists a probability measure ϕ on $(S \cap A, S \cap A)$, a family of probability measures $\mu(x, \cdot)$ on $B(R^+)$ for $x \in A$, and a number $\lambda \in (0, 1)$, such that*

$$(1.3) \quad P_x\left\{X_k \in E, \sum_{i=0}^{k-1} L_i \in D\right\} \geq \lambda\phi(E \cap A)\mu(x, D)$$

for all $x \in A, D \in B(R^+)$.

We then have the following

THEOREM. Assume H_k . Then

(a) there exists a σ -finite, invariant measure ν for P which is unique up to multiplicative constants;

(b) if furthermore

$$(i) \quad m \equiv \int m(x)\nu(dx) < \infty, \text{ where } m(x) = E_x L_0,$$

$$(ii) \quad P \left[\sum_0^{k-1} L_i \leq u \right] = F_k(u) \text{ is non-lattice, and}$$

(iii) $f : S \times R^+ \rightarrow R^+$ is bounded, measurable, and $\nu\{x : f(x, t) \text{ is discontinuous for some } t\} = 0,$

then

$$(1.4) \quad \lim_{t \rightarrow \infty} E f(W(t)) \\ = \left[\int_S \left[\int_0^\infty f(x, u) P_x(L_0 > u) du \right] \nu(dx) \right] / \left[\int_S \left[\int_0^\infty P_x(L_0 > u) du \right] \nu(dx) \right];$$

(c) if (b) (i) and (ii) hold, and $g(x, t) : S \times R^+ \rightarrow R$ is bounded, measurable, continuous in t , and satisfies

$$(1.5) \quad g(x, t) \rightarrow 0 \text{ almost surely } (\nu) \text{ as } t \rightarrow \infty,$$

$$(1.6) \quad \int_S \sum_{n=0}^\infty \sup_{nh \leq t < (n+1)h} g(x, t) \nu(dx) < \infty,$$

and

$$(1.7) \quad \int \left\{ \sup_t |g(x, t)| \right\} \nu(dx) < \infty,$$

then as $t \rightarrow \infty,$

$$(1.8) \quad E_\varphi \sum_{n=0}^\infty g \left(X_n, t - \sum_{i=0}^{n-1} L_i \right) \rightarrow \frac{1}{m} \int_0^\infty \int_S g(x, t) \nu(dx) dt.$$

REMARKS. 1. The theorem extends results of [2] in two ways. First, the distributions $G_{xy}(\cdot)$ were allowed to depend only on x (not y) in the earlier work; that is, there the sojourn times depend only on the

state the chain is "coming from". Secondly, only the case $k = 1$ is admitted in [2].

2. The awkward conditions (1.5)-(1.7) are designed to assure the direct Riemann integrability of certain functions related to g . These conditions can be eliminated at the expense of strengthening the smoothness hypothesis on F_k in (b) (ii). (see the related discussion in Arjas, Nummelin, and Tweedie [1], and Athreya and Ney [4].)

3. Nummelin [7], [8], has proved many results related to this subject, including one like the above theorem under apparently slightly stronger hypotheses. He also uses regeneration methods, but treats the semi-Markov process as a Markov chain on the enlarged state space $S \times R$, and applies discrete renewal theory to this chain. We work instead with the continuous time process $(Z(t), A(t))$ and use renewal theory on R to draw our conclusions. This approach seems to us to yield somewhat more transparent proofs.

4. Nummelin has also shown (see [7]) that a condition very close to H_k is always satisfied (for some k), provided the semi-Markov chain satisfies a weak irreducibility condition.

5. That (1.8) is in fact a "renewal theorem", can be seen by taking $g(x, t)$ of the form $\chi_A(x)\chi_I(t)$ for $A \in S$, and $I \subset R^+$ an interval. As usual in such results, if $F_k(\cdot)$ is lattice, then there is a lattice version of (1.8).

6. Similarly (1.4) can be seen to imply the convergence of $P\{Z(t) \in E\}$, $E \in S$.

7. Since (a) asserts the uniqueness of the invariant measure, one can use any measure $\pi(\cdot)$ that is invariant with respect to P in place of ν .

8. If $k = 1$ and G_{xy} depends only on x , then H_1 automatically holds.

9. The result (c) of the theorem carries over without difficulty to the "two sided" case when the $G_{xy}(\cdot)$ are distributions on $(-\infty, \infty)$, along the lines of Theorem 4.1 of [2].

10. Renewal theorems of the type in (c) of the theorem under different hypotheses have been proved by Jacod [5] and Kesten [6].

2. Proof of the theorem

Hypothesis H_k (ii) applied to $D = R^+$ implies that the Markov chain $\{X_{nk}; n = 0, 1, \dots\}$ is $(A, \lambda, \varphi, 1)$ -recurrent in the sense of Definition (2.2) of [3]. Hence, by Theorem (6.1) of that paper, there exists a unique (up to multiplicative constants) invariant measure, say ν_k , for P . It is now easy to verify that

$$(2.1) \quad \nu = \nu_k + \nu_k P + \dots + \nu_k P^{k-1}$$

is a (necessarily unique) invariant measure for P .

As in the earlier work [2], [3], the basic idea behind the proof is an appropriate

REGENERATION LEMMA. *If H_k holds for some $k \geq 1$, then there exists a random time N such that $P_x(N < \infty) = 1$ for all $x \in S$, and*

$$(2.2) \quad P_x \left\{ X_{Nk} \in E, \sum_{(N-1)k}^{Nk-1} L_i \in D \mid X_{jk}, \sum_{i=0}^{jk} L_i, j = 0, 1, \dots, N-1 \right\} \\ = \varphi(E \cap A) \mu(X_{(N-1)k}, D),$$

almost surely.

Proof. Let $U_j = L_{jk} + \dots + L_{(j+1)k-1}$, and consider the "skeleton" semi-Markov chain $\{(X_{jk}, U_j), j = 1, 2, \dots\}$ (k is fixed throughout). Whenever $X_{jk} \in A$ for some j (say $X_{jk} = x \in A$), randomize the next transition of the chain as follows:

(i) with probability p ($0 < p < \lambda =$ the constant in (1.3)) distribute $(X_{(j+1)k}, U_j)$ over $A \times R^+$ independently with distributions $\varphi(\cdot)$ and $\mu(x, \cdot)$ respectively;

(ii) with probability $(1-p)$ distribute $(X_{(j+1)k}, U_j)$ over the entire state space $S \times R^+$ according to a transition function $Q(x, \cdot)$, chosen so that the overall transition probabilities of the chain

$\{(X_{jk}, U_{j-1})\}$ remain unchanged. This is accomplished by defining Q so that

$$P^k(x, E \times D) = p\varphi(E \cap B)\mu(x, D) + (1-p)Q(x, E \times D) .$$

That this is in fact possible, follows from (1.3). Now since A is visited by $\{X_{nk}\}$ infinitely often, and each time, with probability $p > 0$, the next transition is distributed independently according to (φ, μ) , this event will ultimately occur at some time $N < \infty$ almost surely. This proves the lemma. (The reader wishing to see a more detailed argument is referred to the proof of (3.1) in [3], which contains a careful proof of a special case of the above lemma.)

COROLLARY 1. *There exists a sequence of random times N_1, N_2, \dots for which (2.1) holds.*

COROLLARY 2. *Let $T = \sum_0^{Nk-1} L_i$ and $Z = X_{Nk}$. Then Z and T are independent random variables.*

Proof. Let f and g be bounded, measurable functions on S and R^+ , respectively. Then

$$\begin{aligned} Ef(Z)g(T) &= E\{E[f(Z)g(T) \mid (X_{jk}, U_{j-1}), j = 0, \dots, N-1]\} \\ &= E\left\{E\left[f(X_{Nk})g\left(\sum_0^{N-2} U_i + U_{N-1}\right) \mid (X_{jk}, U_{j-1}), j = 0, \dots, N-1\right]\right\} \\ &= E\left\{\int_A f(y)\varphi(dy) \int_R g\left(\sum_0^{N-2} U_i + u\right)\mu(X_{(N-1)k}, du)\right\} . \end{aligned}$$

Taking the first integral outside the expectation, this equals $Ef(Z)Eg(T)$, proving the corollary. Before turning to the main part of the proof of the theorem, we prepare one more

PROPOSITION. *Let $h : S \times S \rightarrow R$ be bounded and measurable, and $\tilde{h}(x) = E_x h(X_0, X_1)$. Then*

$$(2.3) \quad E_\varphi \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) = \int_S \tilde{h}(x)\nu(dx)$$

where $\nu(\cdot)$ is the stationary measure whose existence was asserted in part

(a) of the theorem.

Proof.

$$\begin{aligned} E_{\varphi} \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) &= E_{\varphi} \sum_{j=0}^{N-1} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1}) \\ &= E_{\varphi} \sum_{j=0}^{\infty} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1}) \chi_{(N>j)} . \end{aligned}$$

Now extend the chain $\{X_n\}$ to a chain $\{X_n, \delta_n\}$, $n \geq 0$, where $\{\delta_n; n \geq 0\}$ is a sequence of independent "coin tossing" variables with $P(\text{Head}) = P(\delta_n = 1) = 1 - P(\delta_0 = 0) = p$. The evolution of $\{X_n\}$ is independent of $\{\delta_n\}$ except when $X_{nk} \in A$, at which time the randomization described in the proof of the regeneration lemma takes place. Thus the random variable $\chi_{(N>j)}$ is completely determined by the history $\{(X_i, \delta_i), i = 1, \dots, jk\}$, and hence conditioned on this history (say F_{jk}) $(X_{jk}, \dots, X_{(j+1)k})$, and $\chi_{(N>j)}$ are trivially independent. Thus for any bounded, measurable $\psi : S^k \rightarrow R$, we have

$$\begin{aligned} E\psi(X_{jk}, \dots, X_{(j+1)k}) \chi_{(N>j)} &= E\{E[\psi \chi | F_{jk}]\} = E\{\chi E[\psi | F_{jk}]\} \\ &= E\{\chi_{N>j} E[\psi(X_{jk}, \dots, X_{(j+1)k}) | X_{jk}]\} \\ &\equiv E\{\tilde{\psi}(X_{jk}) \chi_{(N>j)}\} , \end{aligned}$$

where we have let

$$\tilde{\psi}(x) = E\{\psi(X_0, \dots, X_k) \mid X_0 = x\} .$$

Also let

$$\begin{aligned} H(x) = E_x \sum_{n=0}^{k-1} h(X_n, X_{n+1}) &= \sum_{n=0}^{k-1} E_x \tilde{h}(X_n) \\ (E_x h(X_n, X_{n+1})) &= E_x E_{X_n} h(X_n, X_{n+1}) = E_x (\tilde{h}(X_n)) . \end{aligned}$$

Then

$$\begin{aligned}
 (2.4) \quad E_{\varphi} \sum_{n=0}^{Nk-1} h(X_n, X_{n+1}) &= E_{\varphi} \sum_{j=0}^{N-1} \sum_{n=jk}^{(j+1)k-1} h(X_n, X_{n+1}) \\
 &= E_{\varphi} \sum_{j=0}^{N-1} H(X_{jk}) = E_{\varphi} \sum_{j=0}^{\infty} H(X_{jk}) \chi_{N>j} .
 \end{aligned}$$

By Theorem (6.1) of [3] we know that an invariant measure for P^k is given

by $\nu_k^{(E)} = E_{\varphi} \sum_{j=0}^{N-1} \chi_E(X_{jk})$, and hence (2.4) equals

$$\begin{aligned}
 \int_S H(x) \nu_k(dx) &= \int_S \tilde{h}(y) \sum_{n=0}^{k-1} \int_S P^n(x, dy) \nu_k(dy) \\
 &= \int_S \tilde{h}(y) \nu(dy) \text{ by (2.1).}
 \end{aligned}$$

This, with (2.4), is (2.3), proving the proposition.

We now turn to the main part of the proof of part (b) of the theorem. Since $P_x(T < \infty) = 1$ for all $x \in S$, it is sufficient to prove this result for the case when the initial state X_0 is distributed according to φ . Let $m(t) = Ef(W(t))$, and $a(t) = E\{f(W(t)); T > t\}$, where f satisfies the conditions of part (b) of the theorem, and T is as in Corollary 2. Due to the independence assertion of that corollary, T is a "regeneration time", at which the chain $\{X_n\}$ undergoes a transition and is distributed over A according to φ , independent of the history of the process up to that time. Hence $m(\cdot)$ satisfies the one-dimensional renewal equation

$$m(t) = a(t) + \int_0^t m(t-u) dF(u),$$

where

$$F(u) = P_{\varphi}(T \leq u).$$

The direct Riemann integrability of $a(\cdot)$ under the hypotheses on $f(\cdot)$ now follows exactly as in the proof of Theorem (3.1) of [2]. Hence, since $F(\cdot)$ is non-lattice (note that this is a little weaker than the hypothesis F_k non-lattice), we have by the renewal theorem that

$$(2.5) \quad m(t) \rightarrow (E_{\phi}^T)^{-1} \int_0^{\infty} a(y) dy, \text{ as } t \rightarrow \infty.$$

It thus remains only to identify the limit. To this end we write

$$(2.6) \quad \begin{aligned} \int_0^{\infty} a(t) dt &= E_{\phi} \int_0^T f(W(u)) du \\ &= E_{\phi} \sum_{n=0}^{N_{k-1}} \int_0^{L_n} f(X_n, u) du \\ &= E_{\phi} \left\{ E_{\phi} \left[\sum_{n=0}^{N_{k-1}} \int_0^L f(X_n, u) du \mid X_0, \dots, X_{N-1}, N \right] \right\} \\ &= E_{\phi} \sum_{n=0}^{N_{k-1}} h(X_n, X_{n-1}), \end{aligned}$$

where

$$\begin{aligned} h(x, y) &= E \left[\int_0^{L_0} f(X_0, u) du \mid X_0 = x, X_1 = y \right] \\ &= \int_0^{\infty} f(x, u) [1 - G_{xy}(u)] du. \end{aligned}$$

Applying (2.3) of the proposition, we see that

$$\int_0^{\infty} a(t) dt = \int_S \tilde{h}(x) \nu(dx),$$

where

$$\begin{aligned} \tilde{h}(x) &= E_x E h(X_0, X_1) \\ &= \int_S \int_0^{\infty} f(x, u) [1 - G_{xy}(u)] du P(x, dy) \\ &= \int_0^{\infty} f(x, u) P_x(L_0 > u) du. \end{aligned}$$

Thus $\int a(t) dt$ equals the numerator in (1.4). For the denominator, we

observe from (2.6) that $E_{\phi}^T = \int_0^{\infty} a_1(t) dt$, where $a_1(\cdot)$ is $a(\cdot)$ for the

special case when $f(x, t) \equiv 1$. This implies part (b) of the theorem.

Finally, turning to part (c), we again observe that we need only consider the case when X_0 has distribution φ . Let

$$K(t) \equiv E_\varphi \sum_{n=0}^{Nk-1} g\left(X_n, t - \sum_{i=0}^{n-1} L_i\right) \text{ and } T_j = \sum_{N_j k}^{N_{j+1} k-1} L_i, \text{ where } \{N_j; j \geq 0\}$$

are the successive regeneration times for $\{X_{nk}\}$. Then $\{T_j; j \geq 0\}$ are independent identically distributed as T in Corollary 2, and are regeneration times, in the previously described sense, for the continuous time process. Hence

$$\begin{aligned} E_\varphi \sum_{n=N_j k}^{N_{j+1} k-1} g\left(X_n, t - \sum_{i=0}^{n-1} L_i\right) &= E_\varphi \sum g\left(X_n, t - \sum_{i=0}^{j-1} T_i - \sum_{i=N_j k}^{n-1} L_i\right) \\ &= E_\varphi K\left(t - \sum_{i=0}^{j-1} T_i\right). \end{aligned}$$

Thus

$$\begin{aligned} M(t) \equiv E_\varphi \sum_{n=0}^{\infty} g\left(X_n, t - \sum_{i=0}^{n-1} L_i\right) &= E_\varphi \sum_{j=0}^{\infty} \sum_{n=N_j k}^{N_{j+1} k-1} g\left(X_n, t - \sum_{i=0}^n L_i\right) \\ &= E_\varphi \sum_{j=0}^{\infty} g\left(X_n, t - \sum_{i=0}^{j-1} T_i\right). \end{aligned}$$

But now we are ready to apply the one dimensional renewal theorem (exactly as in Section 4 of [2]), to conclude that if $K(\cdot)$ is direct Riemann integrable, then

$$(2.7) \quad M(t) \rightarrow (E_\varphi T)^{-1} \int_0^\infty K(u) du \text{ as } t \rightarrow \infty.$$

The hypotheses on g are exactly as those in [2], and are designed to assure this direct Riemann integrability.

To identify the limit in (1.8) we write

$$\begin{aligned} \int_0^{\infty} K(t)dt &= E_{\varphi} \sum_{n=0}^{Nk-1} \int_0^{\infty} g \left(X_k, t - \sum_{i=0}^{n-1} L_i \right) dt \\ &= E_{\varphi} \sum_{n=0}^{Nk-1} \int_0^{\infty} g(X_k, t) dt, \end{aligned}$$

(we may define $g(x, t) = 0$ for $t < 0$)

$$= \int_S \int_0^{\infty} g(x, t) \nu(dx) dt.$$

We have already seen that $E_{\varphi}(T) = m$ (b) (i)). This completes the proof of the theorem.

We again ask the reader to observe that the above proof of (c) in no way depends on the non-negativity of the L_i 's.

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