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Topological K-theory of complex noncommutative spaces

Anthony Blanc

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ABSTRACT

The purpose of this work is to give a definition of a topological K-theory for dg-categories over \mathbb{C} and to prove that the Chern character map from algebraic K-theory to periodic cyclic homology descends naturally to this new invariant. This topological Chern map provides a natural candidate for the existence of a rational structure on the periodic cyclic homology of a smooth proper dg-algebra, within the theory of noncommutative Hodge structures. The definition of topological K-theory consists in two steps: taking the topological realization of algebraic K-theory and inverting the Bott element. The topological realization is the left Kan extension of the functor ‘space of complex points’ to all simplicial presheaves over complex algebraic varieties. Our first main result states that the topological K-theory of the unit dg-category is the spectrum **BU**. For this we are led to prove a homotopical generalization of Deligne’s cohomological proper descent, using Lurie’s proper descent. The fact that the Chern character descends to topological K-theory is established by using Kassel’s Künneth formula for periodic cyclic homology and the proper descent. In the case of a dg-category of perfect complexes on a separated scheme of finite type, we show that we recover the usual topological K-theory of complex points. We show as well that the Chern map tensorized with \mathbb{C} is an equivalence in the case of a finite-dimensional associative algebra – providing a formula for the periodic homology groups in terms of the stack of finite-dimensional modules.

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1. Introduction

The idea of associating algebraic invariants to geometrical objects culminated with the study of derived categories in algebraic geometry. Kontsevich's noncommutative geometry goes further in really defining a noncommutative space as being the category of functions defined on this space. Therefore by a noncommutative space we mean a differential \mathbb{Z} -graded category (or dg-category for short). This categorical point of view on geometry is a very powerful one, allowing one to treat on an equal footing objects having *a priori* different origin: for example, in algebraic geometry the derived category of perfect complexes of quasi-coherent sheaves [BO02, Orl03]; in symplectic geometry the Fukaya category of a symplectic variety [FOOO00, Sei08]; in representation theory the derived category of complexes of representations of a quiver [Kel08, Kra08]; in singularity theory the category of matrix factorizations of an isolated hypersurface singularity [Orl05, Dyc11, Efi12]; in algebraic analysis the derived category of deformation-quantization modules [Kon01, Sch08]. The first two examples are particularly important in the mathematical formulation of mirror symmetry, for example in the foundational paper [KKP08]. In the latter paper, the authors formulate the homological mirror symmetry conjecture in terms of noncommutative (nc for short) Hodge structures (see [KKP08, §3]). The main conjecture [KKP08, Conjecture 2.24] claims that there exists a nc-Hodge structure on the periodic cyclic homology of any smooth and proper dg-algebra over \mathbb{C} , which furthermore comes from a commutative one. The definition of a rational pure nc-Hodge structure consists of different structures attached to periodic cyclic homology: the de Rham structure corresponding to the Hodge filtration and the Betti structure corresponding to the rational structure given by rational Betti cohomology (see [KKP08, Definition 2.5]). The first motivation for pursuing topological K-theory came from finding a candidate for the rational structure for a nc-Hodge structure (see [KKP08, §2.2.6]). The several algebraic flavored invariants associated to dg-categories (algebraic K-theory, cyclic homology, Hochschild cohomology, Hall algebras, . . .) were intensively studied by Keller, Tsygan, Tabuada, Cisinski and Toën. Topological K-theory can itself be thought of as the noncommutative analog of rational Betti cohomology of algebraic varieties.

Main results

The aim of the paper is to give a construction of topological K-theory for dg-categories over \mathbb{C} which satisfies all the expected properties. These properties are summarized in the following main theorem. We denote by $\text{dgCat}_{\mathbb{C}}$ the category of \mathbb{C} -dg-categories, Sp the category of symmetric spectra, K_{top} is the usual nonconnective topological K-theory, \mathbf{K} the algebraic K-theory, HN the negative cyclic homology, HP the periodic cyclic homology, $\mathbf{BU} = K_{\text{top}}(*)$ the usual spectrum of topological K-theory, and $\mathbb{1}$ the dg-category with one object and \mathbb{C} as endomorphism ring.

THEOREM 1.1. *There exists a functor $\mathbf{K}^{\text{top}} : \text{dgCat}_{\mathbb{C}} \rightarrow \text{Sp}$, called the topological K-theory of noncommutative spaces, which satisfies the following properties.*

- (a) $\mathbf{K}^{\text{top}}(\mathbb{1}) \simeq \mathbf{BU}$ in the homotopy category of spectra.
- (b) If X is a separated \mathbb{C} -scheme of finite type, then there exists a functorial isomorphism

$$\mathbf{K}^{\text{top}}(\mathcal{L}_{\text{perf}}(X)) \simeq K_{\text{top}}(X(\mathbb{C})),$$

where $\mathcal{L}_{\text{perf}}(X)$ is the dg-category of perfect complexes of quasi-coherent \mathcal{O}_X -modules on X .

(c) \mathbf{K}^{top} commutes with filtered colimits, is Morita-invariant and sends exact sequences of dg-categories to distinguished triangles in the triangulated homotopy category of spectra.

(d) For any dg-category T , there exists a functorial commutative square in the homotopy category of spectra

$$\begin{array}{ccc} \mathbf{K}(T) & \xrightarrow{\text{Ch}} & \text{HN}(T) \\ \downarrow & & \downarrow \\ \mathbf{K}^{\text{top}}(T) & \xrightarrow{\text{Ch}^{\text{top}}} & \text{HP}(T) \end{array}$$

such that in the case of a separated scheme of finite type X , the map Ch^{top} for $T = \mathcal{L}_{\text{perf}}(X)$ is isomorphic to the usual topological Chern character.

Our definition of topological K-theory for dg-categories is really inspired by Friedlander and Walker’s definition of semi-topological K-theory for quasi-projective complex algebraic varieties (see [FW01, FW05]), itself inspired by the work of Thomason [Tho85].

The definition we give of topological K-theory starts with algebraic K-theory and proceeds in two steps. If T is a \mathbb{C} -dg-category, we denote by $\mathbf{K}(T)$ the nonconnective algebraic K-theory of T (defined by Schlichting [Sch06]). We also have a presheaf of spectra on the site of complex affine \mathbb{C} -schemes,

$$\underline{\mathbf{K}}(T) : \text{Spec}(A) \mapsto \mathbf{K}(T \otimes_{\mathbb{C}} A).$$

The first step consists in applying the topological realization of simplicial presheaves. Denote by $\text{Aff}_{\mathbb{C}}$ the category of complex affine schemes of finite type, by SSet the category of simplicial sets, and by $\text{SPr}(\text{Aff}_{\mathbb{C}})$ the category of simplicial presheaves on $\text{Aff}_{\mathbb{C}}$. Then the topological realization is a functor

$$\text{ssp} : \text{SPr}(\text{Aff}_{\mathbb{C}}) \rightarrow \text{SSet}.$$

It is the left Kan extension of the functor ‘space of complex points’ $\text{Aff}_{\mathbb{C}} \rightarrow \text{SSet}$ (where space is understood as simplicial set) along the Yoneda embedding $\text{Aff}_{\mathbb{C}} \rightarrow \text{SPr}(\text{Aff}_{\mathbb{C}})$. The topological realization extends naturally to presheaves of spectra and is a left Quillen functor for the \mathbf{A}^1 -étale local model structure on simplicial presheaves. We denote by $|-| := \mathbb{L}\text{ssp}$ the left derived functor with respect to this model structure.

DEFINITION 1.2 (see Definition 4.1). The semi-topological K-theory of a \mathbb{C} -dg-category T is the spectrum $\mathbf{K}^{\text{st}}(T) := |\underline{\mathbf{K}}(T)|$.

We can see that there exists a canonical map $\mathbf{K}(T) \rightarrow \mathbf{K}^{\text{st}}(T)$. When $T = \mathcal{L}_{\text{perf}}(X)$ is the dg-category of perfect complexes on a smooth complex algebraic variety, the map $\mathbf{K}_0(X) \rightarrow \mathbf{K}_0^{\text{st}}(X)$ is the quotient map given by the algebraic equivalence relation, i.e. two algebraic vector bundles $E \rightarrow X$ and $E' \rightarrow X$ are algebraically equivalent if there exist a connected complex algebraic

curve C and a vector bundle $E'' \rightarrow C \times X$ such that we recover E and E' by restricting E'' to some \mathbb{C} -points of C .

Using the proper topology (we could also have used the *cdh*-topology of Voevodsky), and the proper local model structure on simplicial presheaves, we show that the topological realization of a presheaf remains unchanged if we restrict this presheaf to smooth schemes. Denote by l^* the restriction to smooth schemes.

THEOREM 1.3 (see Theorem 3.18). *Let $F \in \text{SPr}(\text{Aff}_{\mathbb{C}})$. Then there exists a canonical isomorphism $|l^*F| \simeq |F|$ in the homotopy category of simplicial sets.*

To prove this, as announced in the abstract, we prove a homotopical generalization of Deligne’s cohomological proper descent which can be expressed by saying that *ssp* is a left Quillen functor with respect to the proper local model structure on simplicial presheaves.

PROPOSITION 1.4 (see Proposition 3.21). *For every proper hypercovering $Y_{\bullet} \rightarrow X$ of a scheme, the induced map, $\text{hocolim}_{\Delta^{\text{op}}} |Y_{\bullet}| \rightarrow |X|$, is an isomorphism in the homotopy category of simplicial sets.*

Theorem 1.3 allows us to prove the following result. Denote by **bu** the connective cover of **BU**.

THEOREM 1.5 (see Theorem 4.6). *There exists an isomorphism $\mathbf{K}^{\text{st}}(\mathbb{1}) \simeq \mathbf{bu}$ in the homotopy category of spectra.*

Using the Tabuada–Cisinski theorem on the corepresentability of K-theory inside the motivic category of dg-categories, we define a canonical structure of ring spectrum on $\mathbf{K}(\mathbb{1})$ and a structure of $\mathbf{K}(\mathbb{1})$ -module on $\mathbf{K}(T)$ for any dg-category T . Therefore $\mathbf{K}^{\text{st}}(T)$ is a **bu**-module.

The second step in the process of the definition of topological K-theory is the inversion of a Bott element $\beta \in \pi_2(\mathbf{bu})$.

DEFINITION 1.6 (see Definition 4.13). *The topological K-theory of a \mathbb{C} -dg-category T is the spectrum $\mathbf{K}^{\text{top}}(T) = \mathbf{K}^{\text{st}}(T)[\beta^{-1}]$.*

This definition is of course motivated by the agreement of the Bott inverted Friedlander and Walker semi-topological K-theory with the usual topological K-theory of complex points (see [FW03, Theorem 5.8]), which was also motivated by Thomason’s work on the Bott inverted algebraic K-theory with finite coefficients. Point (b) of Theorem 1.1 is proved by using Riou’s Spanier–Whitehead duality in the motivic homotopy category of smooth schemes (see Proposition 4.32). Finally, point (d) of Theorem 1.1, namely the construction of the topological Chern map, is achieved using Kassel’s Künneth formula for periodic cyclic homology and once again Theorem 1.3.

Finally, in § 4.3, we give a convenient description of the connective semi-topological K-theory of any dg-category in terms of the stack \mathcal{M}^T of perfect T^{op} -dg-modules. This stack is a stack of E_{∞} -spaces because of the sum of dg-modules. It turns out that its topological realization $|\mathcal{M}^T|$ is a group-like E_{∞} -space and one can consider it as a connective spectrum. The proof is based on the existence of an \mathbf{A}^1 -equivalence between the stack in E_{∞} -spaces \mathcal{M}^T and the Waldhausen construction of the category of perfect T^{op} -dg-modules (see Proposition 4.17 and Theorem 4.21).

Applications and related works

As was said above, the semi-topological K-theory of quasi-projective complex algebraic varieties has already been defined by Friedlander and Walker. However, let us mention that their operation

of ‘semi-topologization’ seems to differ from our topological realization. An indirect comparison is nevertheless given by point (b) of Theorem 1.1.

The construction of the topological K-theory of dg-categories over \mathbb{C} and the existence of the topological Chern map can be used in order to state the lattice conjecture claiming that this new invariant gives a rational structure on the periodic homology of a smooth proper dg-category.

CONJECTURE 1.7 (Lattice conjecture). For every smooth proper \mathbb{C} -dg-category T , the Chern map $\text{Ch}^{\text{top}} \wedge_{\mathbb{S}} \text{HC} : \mathbf{K}^{\text{top}}(T) \wedge_{\mathbb{S}} \text{HC} \rightarrow \text{HP}(T)$ is an isomorphism in the homotopy category of spectra.

By points (b) and (d) in Theorem 1.1, the lattice conjecture is true for dg-categories of the form $\mathcal{L}_{\text{perf}}(X)$ for X any separated scheme of finite type over \mathbb{C} .

As an application in §4.7 we show that the lattice conjecture is true for finite-dimensional associative \mathbb{C} -algebras, but if we replace \mathbf{K}^{top} by another invariant $\tilde{\mathbf{K}}^{\text{top}}$ which is the Bott inverted connective semi-topological K-theory. The proof consists in showing the invariance of $\tilde{\mathbf{K}}^{\text{top}}$ under infinitesimal extension. It has the following consequence about the periodic homology groups of a finite-dimensional algebra. If B is such an algebra, denote by Vect^B the stack of projective right modules of finite type over B , and by $|\text{Vect}^B|^{ST}$ the stabilization of its topological realization with respect to the B -module B .

PROPOSITION 1.8 (see Corollary 4.42). The Chern map $\tilde{\mathbf{K}}^{\text{top}}(B) \rightarrow \text{HP}(B)$ induces an isomorphism of \mathbb{C} -vector spaces for $i = 0, 1$,

$$\text{colim}_{k \geq 0} \pi_{i+2k} |\text{Vect}^B|^{ST} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \text{HP}_i(B)$$

where the colimit is induced by the action of the Bott element β on homotopy groups, $\pi_i |\text{Vect}^B|^{ST} \xrightarrow{\times \beta} \pi_{i+2} |\text{Vect}^B|^{ST}$.

In the case where B is moreover of finite homological dimension, i.e. smooth as a dg-category, we prove a similar formula but in terms of the stack Vect_B of finite-dimensional modules over B ; see Corollary 4.44. The stack Vect_B has already been thought of by mathematicians as reflecting the noncommutative geometrical property of the algebra B , or in other words as a noncommutative spectrum. Therefore our formula gives a somehow geometrical interpretation of the periodic cyclic homology groups in this case.

We believe the lattice conjecture is true for objects from very diverse origins like smooth proper DM-stacks over \mathbb{C} and categories of matrix factorizations. We notice that the latter example requires the use of 2-periodic dg-categories which have their own homotopy theory and the topological realization has also to be adapted to this 2-periodic context.

Another interesting question raised by topological K-theory is the behavior of the Chern map $\mathbf{K}^{\text{top}}(T) \rightarrow \text{HP}(T)$ with respect to the Gauss–Manin connection supported by the periodic homology of a family of dg-categories parametrized by a smooth affine variety (see [TVe09, Tsy07]). We believe the map Ch^{top} is flat with respect this connection. We can also mention [KKP08, § 2.2.5] for the relation with the associated variation of nc-Hodge structures.

Point (d) in Theorem 1.1 allows us to propose a general definition of the Deligne cohomology of smooth and proper dg-categories over \mathbb{C} . In the commutative case, Deligne cohomology of degree $2p$ of a smooth projective variety with coefficients in the p th Deligne complex $\mathbb{Z}(p)_{\mathcal{D}}$ is given by the extension of integral (p, p) -classes with its p th intermediate Griffiths’s jacobian. This suggests the following definition of Deligne cohomology for a smooth and proper \mathbb{C} -dg-category T :

$$\text{H}_{\mathcal{D}}(T) := \mathbf{K}^{\text{top}}(T) \times_{\text{HP}(T)}^h \text{HN}(T).$$

As usual, by cohomology we mean the spectrum whose stable homotopy groups give the actual cohomology groups. This new invariant comes with a map $\mathbf{K}(T) \rightarrow H_{\mathcal{D}}(T)$ which can be thought of as a mix between usual and secondary characteristic classes. Let us mention also Marcolli and Tabuada's work [MT12] on intermediate jacobians for dg-categories.

Finally, let us mention another possible relation with the work of Freed [Fre09] where for problems of dimension reduction of certain Chern–Simons theories, the author raises the question of finding a refinement of Hochschild homology defined over the integers. The topological K-theory endowed with its Chern map seems to be a good candidate for this refinement.

Notations and conventions

- We handle set theoretical issues by choosing two Grothendieck universes $\mathbb{U} \in \mathbb{V}$ in the sense of [SGA4]. We suppose that \mathbb{U} contains the set \mathbb{N} of natural numbers.
- In general, we consider objects (sets, simplicial sets, topological spaces, ...) which are \mathbb{V} -small. We denote by Set the category of \mathbb{V} -small sets, SSet the category of \mathbb{V} -small simplicial sets, Top the category of \mathbb{V} -small topological spaces, and Sp the category of \mathbb{V} -small symmetric spectra.
- If C is any category, we denote by C^{op} the opposite category of C . If D is any another category, we denote by C^D the category of functors and natural transformations $D \rightarrow C$.
- Hovey's book [Hov99] is our reference for definitions and results on model categories. If M is a model category, we denote by $\text{Ho}(M)$ the homotopy category of M . Most of the time we call weak equivalences simply *equivalences*. If M is a cofibrantly generated model category, and C any category, we will use several times the projective model structure on functors M^C . Its existence is proven in [Hir09, § 11.6].
- The category SSet is endowed with its standard model structure (see [Hov99, ch. 3]). The category Sp is endowed with its standard stable model structure (see [HSS00]), which is Quillen equivalent to the usual stable model category of spectra, but has the advantage of being model monoidal. The category Top is endowed with its standard model structure (e.g. see [Hov99, § 2.4]).
- All the spectra we consider are symmetric spectra. We will abusively forget the word 'symmetric' in front of spectra for ease of reading, but we will always mean symmetric spectra.
- When we are dealing with symmetric ring spectra, by convention we will mean strictly associative symmetric ring spectra. If A is a symmetric ring spectrum, the term A -modules means by convention right A -modules. The notation $A\text{-Mod}_{\mathfrak{s}}$ stands for the category of right A -module spectra, which we sometimes call just A -modules.
- If C is a category and $x, y \in C$ two objects of C , we denote by $\text{Hom}_C(x, y)$ the set of morphisms or maps from x to y in C . If C is enriched in a category V different than the category of sets, we denote by $\underline{\text{Hom}}_C(x, y)$ the object of maps from x to y , specifying if needed the category V . If C is moreover endowed with a model structure compatible with the enrichment, we denote by $\mathbb{R}\underline{\text{Hom}}_C$ its derived internal hom. In the particular case of $V = \text{SSet}$, we denote the internal homs by Map_C and $\mathbb{R}\text{Map}_C$.
- In a category with final object we denote 'this' final object by $*$.
- We denote by Δ the standard simplicial category of finite ordinals $[0], [1], [2], \dots$ with increasing maps as morphisms. If nothing is specified S^1 refers to the standard model $S^1 = \Delta^1 / \partial \Delta^1$ of the simplicial circle, pointed by its zero simplex.

- By convention, all schemes are *separated of finite type over the base*. If k is a commutative noetherian ring, we denote by Aff_k the category of affine k -schemes of finite type over k , and by Sch_k the category of separated k -schemes of finite type over k .
- If C is a \mathbb{V} -small category and V a locally \mathbb{V} -small category, we denote by $\text{Pr}(C, V)$ the category of presheaves on C with values in V . In the particular case of $V = \text{SSet}$, we denote $\text{Pr}(C, V) =: \text{SPr}(C)$. In the particular case of $V = \text{Sp}$, we denote $\text{Pr}(C, \text{Sp}) =: \text{Sp}(C)$.
- *In the particular cases of $V = \text{SSet}, \text{Sp}$ endowed with their standard model structure, the category $\text{Pr}(\text{Aff}_k, V)$ is by default endowed with its projective model structure, for which the equivalences and fibrations are levelwise ones (cf. [Hir09, § 11.6]). Unless otherwise stated, $\text{SPr}(\text{Aff}_k)$ and $\text{Sp}(\text{Aff}_k)$ refer to the projective model structures, also called global model structures by opposition with the local ones.*
- If M is a model category, and $F : I \rightarrow M$ a diagram in M (with I a category), we denote by $\text{hocolim}_I F$ the homotopy colimit of F , i.e. the left derived functor of the functor $\text{colim}_I : M^I \rightarrow M$ for the projective model structure. We denote by $\text{holim}_I F$ the homotopy limit of F , i.e. the right derived functor of the functor $\text{lim}_I : M^I \rightarrow M$ for the injective model structure. We denote by $A \overset{h}{\times}_C B$ the homotopy pullback and by $A \overset{h}{\amalg}_C B$ the homotopy pushout.
- With a left Quillen functor $f : M \rightarrow N$ between model categories, we will several times abusively say that $\mathbb{L}f : \text{Ho}(M) \rightarrow \text{Ho}(N)$ commutes with homotopy colimits, meaning that for all small categories I , and all I -diagrams $F : I \rightarrow M$, the map $\text{hocolim}_I \mathbb{L}f(F) \rightarrow \mathbb{L}f(\text{hocolim}_I F)$ is an isomorphism in $\text{Ho}(N)$. We will practice an analogous abuse for homotopy limits.
- In this text, unless stated otherwise, the sets of adjoint pairs of functors written horizontally are written such that every functor F is left adjoint of the functor just below F .

2. Preliminaries

In this section we set up some notation, definitions and results that will be used in the definition of the topological K-theory and its Chern character. The first part deals with particular models for homotopy coherently associative monoids (and their commutative analog) and their link with the homotopy theory of symmetric spectra. In the second part we set definitions and recall the properties of algebraic K-theory of noncommutative spaces (connective and nonconnective). Finally, in the third part we use Cisinski and Tabuada’s result on nonconnective K-theory of noncommutative spaces to redefine the ring structure on algebraic K-theory and the Chern map in a linear fashion with respect to the ring spectrum of algebraic K-theory.

2.1 Monoids up to homotopy and connective spectra

We will use particular models for homotopy associative monoids and homotopy commutative monoids known as Δ - and Γ -spaces, respectively. They are quite convenient if one wants to handle algebraic structures up to coherent homotopy while staying in the realm of model categories, without referring to Lurie’s ∞ -operads and ∞ -categories. We recall basic results about Δ -spaces, Γ -spaces, group completion, and the link between the homotopy theory of very special Γ -spaces and the homotopy theory of connective spectra. At the end we recall how to define the Waldhausen K-theory spectrum using the group completion of Γ -spaces.

Let Γ be the skeletal category of finite pointed sets with objects the sets $\underline{n} = \{0, \dots, n\}$ with 0 as basepoints for all $n \in \mathbb{N}$ and with morphisms all pointed maps of sets.

DEFINITION 2.1. Let M be a model category.

- A Δ -object (respectively, a Γ -object) in M is a functor $\Delta^{\text{op}} \rightarrow M$ sending $[0]$ to $*$ (respectively, a functor $\Gamma \rightarrow M$ sending $\underline{0}$ to $*$). Morphisms being natural transformations of functors, we denote by $\Delta - M$ (respectively, by $\Gamma - M$) the category of Δ -objects (respectively, of Γ -objects) in M . For $E \in \Delta - M$ (respectively, $F \in \Gamma - M$), we adopt the notation $E([n]) = E_n$ and $F(\underline{n}) = F_n$.
- A Δ -object E in M is called special if all the Segal maps are weak equivalences in M , i.e. if for all $[n] \in \Delta$ the map

$$p_0^* \times \cdots \times p_{n-1}^* : E_n \longrightarrow E_1^{\times n} = E_1 \overset{h}{\times} \cdots \overset{h}{\times} E_1$$

is a weak equivalence in M where $p_i : [1] \rightarrow [n]$, $p_i(0) = i$ and $p_i(1) = i + 1$. We denote by $s\Delta - M$ the full subcategory of $\Delta - M$ consisting of special Δ -objects in M .

- A Γ -object F in M is called special if for all $\underline{n} \in \Gamma$ the map

$$q_*^1 \times \cdots \times q_*^n : F_n \longrightarrow F_1^{\times n}$$

is a weak equivalence in M , where $q^i : \underline{n} \rightarrow \underline{1}$, $q^i(j) = 1$ if $j = i$ and $q^i(j) = 0$ if $j \neq i$. We denote by $s\Gamma - M$ the full subcategory of $\Gamma - M$ consisting of special Γ -objects in M .

- If $E \in s\Delta - M$, we say that E is very special if the map

$$p_0^* \times d_1^* : E_2 \longrightarrow E_1 \overset{h}{\times} E_1$$

is a weak equivalence in M , where $d_1 : [1] \rightarrow [2]$ is the face map which avoids 1 in $[2]$. We denote by $vs\Delta - M$ the full subcategory of $s\Delta - M$ consisting of very special Δ -objects.

- If $F \in s\Gamma - M$, we say that F is very special if the map

$$q_*^1 \times \mu_* : F_2 \longrightarrow F_1 \overset{h}{\times} F_1$$

is a weak equivalence in M , where $\mu : \underline{2} \rightarrow \underline{1}$ is the map defined by $\mu(1) = 1$ and $\mu(2) = 1$. We denote by $vs\Gamma - M$ the full subcategory of $s\Gamma - M$ consisting of very special Γ -objects.

Remark 2.2. – If we take $M = \text{SSet}$, the Δ -objects and Γ -objects are usually called Δ -spaces and Γ -spaces, e.g. in [BF78].

- The special Δ -objects in a model category M are particular models for associative monoids up to coherent homotopy in M (or A_∞ -monoids). If we take $M = \text{Set}$ the category of \mathbb{U} -small sets, with isomorphisms of sets as weak equivalences, then the category $s\Delta - \text{Set}$ is equivalent to the category of monoids in sets. Namely, a Δ -set E maps to the monoid E_1 with composition law given by

$$E_1 \times E_1 \xrightarrow{(d_0^* \times d_2^*)^{-1}} E_2 \xrightarrow{d_1^*} E_1.$$

The associativity and unital conditions are recovered via the simplicial identities. The special Γ -objects are particular models for commutative monoids up to coherent homotopy in M (or E_∞ -monoids). The composition law is recovered by the map μ , and the commutativity is encoded by the map $\underline{2} \rightarrow \underline{2}$ interchanging 1 and 2.

- In the special case where M is the projective model category of simplicial presheaves on a site, we can then replace homotopy products by products in Definition 2.1. In this case, a special Δ -object E is very special if and only if the monoid $\pi_0 E_1$ is a group (i.e. every element has a two-sided inverse; see [Bla13, Lemma 1.3]).

Recall that we have at least three interesting model structures on $\Delta - M$ for any left proper combinatorial model category M .

- The *projective* or *strict model structure* for which weak equivalences and fibrations are levelwise weak equivalences and levelwise fibrations, respectively. We denote this model structure by $\Delta - M$. In all that follows, by default ‘equivalence in $\Delta - M$ ’ will refer to levelwise weak equivalence.
- The *special model structure* which is the Bousfield localization of the strict one with respect to the set of maps

$$\left(\bigsqcup_{i=0}^{n-1} h_{p_i} : h_{[1]} \sqcup \cdots \sqcup h_{[1]} \longrightarrow h_{[n]} \right)_{n \geq 1} \quad \square \quad (\text{generating cofibrations of } M)$$

where \square is the box product of [Hov99, Theorem 3.3.2]. We denote it by $\Delta - M^{\text{sp}}$. The fibrant objects for this model structure are the special Δ -objects which are moreover levelwise fibrant.

- The *very special model structure* which is a Bousfield localization of the special one with respect to the map

$$(h_{p_0} \sqcup h_{d_1} : h_{[1]} \sqcup h_{[1]} \longrightarrow h_{[2]}) \quad \square \quad (\text{generating cofibrations of } M).$$

We denote it by $\Delta - M^{\text{vsp}}$. The fibrant objects for this model structure are the very special Δ -objects which are, moreover, levelwise fibrant.

We have derived identity functors

$$\text{Ho}(\Delta - M) \begin{matrix} \xrightarrow{\mathbb{L}\text{id}} \\ \xleftarrow{\mathbb{R}\text{id}} \end{matrix} \text{Ho}(\Delta - M^{\text{sp}}) \begin{matrix} \xrightarrow{\mathbb{L}\text{id}'} \\ \xleftarrow{\mathbb{R}\text{id}'} \end{matrix} \text{Ho}(\Delta - M^{\text{vsp}}).$$

We denote by $\text{mon} := \mathbb{R}\text{id}\mathbb{L}\text{id}$ the free homotopy associative monoid functor and by $(-)^+ := \mathbb{R}\text{id}'\mathbb{L}\text{id}'$ the homotopy group completion.

We have similar model structures for Γ -objects: a projective model structure $\Gamma - M$, a special model structure $\Gamma - M^{\text{sp}}$, and a very special model structure $\Gamma - M^{\text{vsp}}$ with corresponding free homotopy commutative monoid and homotopy commutative group completion functors:

$$\text{Ho}(\Gamma - M) \xrightarrow{\text{com}} \text{Ho}(\Gamma - M^{\text{sp}}) \xrightarrow{(-)^+} \text{Ho}(\Gamma - M^{\text{vsp}}).$$

(We give the group completion the same notation as for associative monoids because one can show they are actually equivalent, which can be expressed by the following.)

We have a fully faithful functor from homotopy commutative monoids to homotopy associative monoids given by composition with the functor

$$\alpha : \Delta^{\text{op}} \longrightarrow \Gamma,$$

defined on objects by $\alpha([n]) = \underline{n}$. And for any map $f : [n] \longrightarrow [m]$ in Δ we define $\alpha(f) = g : \underline{m} \longrightarrow \underline{n}$ by

$$g(i) = \begin{cases} 0 & \text{if } 0 \leq i \leq f(0), \\ j & \text{if } f(j-1) < i \leq f(j), \\ 0 & \text{if } f(n) < i. \end{cases}$$

One can verify that $\alpha(p_i) = q^{i+1}$ for $i = 0, \dots, n-1$, and $\alpha(d_1) = \mu$, so that the fully faithful functor

$$\alpha^* : \Gamma - M \longrightarrow \Delta - M$$

sends special Γ -objects to special Δ -objects and also very special objects to such. Hence we obtain a diagram

$$\begin{array}{ccccc} \mathrm{Ho}(\Gamma - M) & \xrightarrow{\mathrm{com}} & \mathrm{Ho}(\Gamma - M^{\mathrm{sp}}) & \xrightarrow{(-)^+} & \mathrm{Ho}(\Gamma - M^{\mathrm{vsp}}) \\ \alpha^* \downarrow & & \alpha^* \downarrow & & \alpha^* \downarrow \\ \mathrm{Ho}(\Delta - M) & \xrightarrow{\mathrm{mon}} & \mathrm{Ho}(\Delta - M^{\mathrm{sp}}) & \xrightarrow{(-)^+} & \mathrm{Ho}(\Delta - M^{\mathrm{vsp}}) \end{array}$$

The left square is not commutative anymore but we can actually show that the right square is commutative up to canonical isomorphism.

Remark 2.3. Working with $M = \mathbf{SSet}$ or with the global model category of simplicial presheaves on a category, we can make the following observation. By Segal’s theorem [Seg74, Proposition 1.5], the group completion functor $(-)^+$ has as model the composite functor

$$\mathrm{Ho}(\Delta - \mathbf{SSet}^{\mathrm{sp}}) \xrightarrow{|\cdot|} \mathrm{Ho}(\mathbf{SSet}_*) \xrightarrow{\mathbb{R}\Omega_\bullet} \mathrm{Ho}(\Delta - \mathbf{SSet}^{\mathrm{vsp}}),$$

where $|\cdot|$ is the realization of bisimplicial sets and for a pointed fibrant simplicial set (X, x) the simplicial set $\Omega_n X$ is the simplicial set of maps from Δ^n to X which send all vertices to x . We can indeed say more: the composite functor $(-)^+ \circ \mathrm{mon}$ has as model the functor $\mathbb{R}\Omega_\bullet \circ |\cdot|$.

Example 2.4. The following example will be important to us in this paper. If C is any Waldhausen category, we have a Δ -space

$$\mathcal{K}_\bullet(C) := \mathrm{Nw}S_\bullet C$$

where Nw is the nerve of weak equivalences and S_\bullet is Waldhausen’s S-construction. Level 1 is $\mathrm{Nw}S_1 C$ which is equivalent to $\mathrm{Nw}C$. This Δ -space is not special in general. Algebraic K-theory is indeed a way to make it very special. The algebraic K-theory space of C is defined by the pointed simplicial set

$$K(C) := \Omega|\mathrm{Nw}S_\bullet C|,$$

where Ω means Ω_1 in the notation of Remark 2.3, i.e. the simplicial set of loops. The basepoint is taken to be the zero object of C . Algebraic K-theory defines a functor

$$K : \mathbf{WCat} \longrightarrow \mathbf{SSet}_*$$

from \mathbb{V} -small Waldhausen categories to \mathbb{V} -small simplicial sets. By Segal’s theorem (see Remark 2.3), the algebraic K-theory of C is the first level of the group completion

$$(\mathrm{mon} \mathcal{K}_\bullet(C))^+ \simeq \mathbb{R}\Omega_\bullet|\mathrm{Nw}S_\bullet C|.$$

Moreover, one can verify directly using adjoint functors that

$$\pi_0(\mathrm{mon} \mathcal{K}_\bullet(C))_1^+ \simeq (\mathrm{mon} \pi_0 \mathcal{K}^{(1)}(C))^+.$$

The free monoid of $\pi_0 \mathcal{K}^{(1)}(C)$ is the monoid in which we identify a with the product of a' and a'' each time there is a cofibration sequence $a' \hookrightarrow a \twoheadrightarrow a''$. It follows that this product is commutative and coincides with the sum in C , and that $\pi_0 K(C) = K_0(C)$ is the Grothendieck group of C .

We recall the equivalence between the homotopy theory of very special Γ -spaces and the homotopy theory of connective spectra. This was first proven by Segal [Seg74] and upgraded in the language of model categories in [BF78]. [BF78, Theorem 5.8] can be directly generalized from

Γ -spaces and spectra to Γ -objects in $M = \text{SPr}(C)$ and presheaves of spectra on C . Moreover, following [Sch, Example 2.39], we can replace ordinary spectra by symmetric spectra. We denote by $\text{Sp}^{\text{con}}(C)$ the subcategory of presheaves of connective spectra. We have a pair of adjoint functors

$$\Gamma - \text{SPr}(C) \begin{matrix} \xrightarrow{\mathcal{B}} \\ \xleftarrow{\mathcal{A}} \end{matrix} \text{Sp}^{\text{con}}(C).$$

Recall from [BF78, § 5] that a Γ -space can be extended to a functor from symmetric spectra to symmetric spectra. The functor \mathcal{B} is defined on an object $E \in \Gamma - \text{SPr}(C)$ by

$$\mathcal{B}E = E(\mathbb{S}),$$

the value of E on the sphere spectrum, which is a connective spectrum for every Γ -object E . This functor is really identical to Segal’s functor from special Γ -spaces to spectra, defined using iterations of realization of simplicial spaces. The functor \mathcal{B} preserves weak equivalences between all Γ -spaces, not just cofibrants. We endow the category $\text{Sp}(C)$ of presheaves of symmetric spectra on C with the projective model structure and we denote by $\text{Ho}(\text{Sp}^{\text{con}}(C))$ the subcategory of $\text{Ho}(\text{Sp}(C))$ consisting of connective symmetric spectra.

THEOREM 2.5. *The adjoint pair $(\mathcal{B}, \mathcal{A})$ is a Quillen pair for the very special model structure on $\Gamma - \text{SPr}(C)$. Moreover, it is a Quillen equivalence, inducing an equivalence of categories,*

$$\text{Ho}(\Gamma - \text{SPr}(C)^{\text{vsp}}) \begin{matrix} \xrightarrow{\mathbb{L}\mathcal{B}} \\ \xleftarrow{\mathbb{R}\mathcal{A}} \end{matrix} \text{Ho}(\text{Sp}^{\text{con}}(C)).$$

Remark 2.6. Following [Bla13, § 1.2], we give a model for the algebraic K-theory spectrum of a Waldhausen category. This model has the advantage of being canonically delooped using the functor \mathcal{B} of Theorem 2.5, compared to the way Waldhausen has defined his K-theory spectrum. We just recall the construction; for more details and a comparison with Waldhausen’s spectrum, we refer to [Bla13, § 1.2].

Let C be any Waldhausen category. The axioms setting the structure of a Waldhausen category imply that C has finite sums. We can therefore construct a special Γ -object in the category WCat of \mathbb{V} -small Waldhausen categories denoted by $B_W C$ such that there is an equivalence of categories $(B_W C)_1 \simeq C$ and the composition law is given by the sum in C . Moreover, there is an equivalence of categories $(B_W C)_n \simeq C^n$ for any $n \geq 1$. Since the algebraic K-theory space functor $K : \text{WCat} \rightarrow \text{SSet}_*$ commutes with finite products, we obtain a special Γ -space by taking K levelwise:

$$K^\Gamma(C) := K(B_W C).$$

Because $\pi_0 K^\Gamma(C)_1 \simeq K_0(C)$ is a group, the Γ -object $K^\Gamma(C)$ is very special and thus gives a connective symmetric spectrum

$$\tilde{\mathbf{K}}(C) := \mathcal{B}(K^\Gamma(C)).$$

This defines a functor

$$\tilde{\mathbf{K}} : \text{WCat} \rightarrow \text{Sp}^{\text{con}}.$$

For any Waldhausen category C , there exists a map of special Γ -spaces,

$$NwB_W C \rightarrow K^\Gamma(C) \tag{1}$$

given by the adjoint of the map $S^1 \wedge NwB_W C \rightarrow |NwS_\bullet B_W C|$. We recall (see [Bla13, Lemma 1.10]) that in the special case where all cofibrations are split in the Waldhausen

category C , the map of very special Γ -spaces,

$$(NwB_W C)^+ \longrightarrow K^\Gamma(C),$$

is a levelwise equivalence.

2.2 Algebraic K-theory of noncommutative spaces

In all that follows, k is a commutative associative unital noetherian base ring.

Actually our definition of topological K-theory is developed only over the complex field \mathbb{C} . Thus, from § 3, the base ring will be \mathbb{C} . We fix some notation for dg-categories. We denote by $C(k)$ the category of \mathbb{U} -small unbounded cochain complexes of k -modules. We denote by dgCat_k the category of \mathbb{U} -small k -dg-categories, i.e. of \mathbb{U} -small $C(k)$ -enriched categories, and $C(k)$ -enriched functors between them. By convention, the expression dg-category always refers to a \mathbb{U} -small k -dg-category.

We endow dgCat_k with the standard model structure [Tab07, Theorem 1.8] for which equivalences are quasi-equivalences; we denote this model structure by dgCat_k . We denote by dgMor_k the Morita model structure on dgCat_k [Tab07, Theorem 2.27], for which the equivalences are Morita equivalences. If T is a dg-category, we denote by T^{op} its opposite dg-category, and by $T^{\text{op}} - \text{Mod}$ the category of T^{op} -dg-modules, endowed with its projective model structure. Its homotopy category is denoted by $D(T)$ and called the derived category of T . A map $T \longrightarrow T'$ in dgCat_k is a Morita equivalence if the induced map $D(T) \longrightarrow D(T')$ is an equivalence of categories.

We consider $\text{Perf}(T)$ the category of cofibrant and perfect objects in $T^{\text{op}} - \text{Mod}$, i.e. the subcategory of $T^{\text{op}} - \text{Mod}$ consisting of cofibrant T^{op} -dg-modules which are perfect (or compact) as objects of the derived category $D(T)$. We endow $\text{Perf}(T)$ with the structure of a Waldhausen category induced by the model structure of $T^{\text{op}} - \text{Mod}$, i.e. a map is a weak equivalence (respectively, a cofibration) in $\text{Perf}(T)$ if it is so in $T^{\text{op}} - \text{Mod}$. The axioms of a Waldhausen category structure are satisfied essentially because the homotopy pushout of two perfect dg-modules over a third perfect dg-module is again perfect. Moreover, the Waldhausen category $\text{Perf}(T)$ satisfies the saturation axiom and the extension axiom and has a cylinder functor which satisfies the cylinder axiom.

Let $f : T \longrightarrow T'$ be a map in dgCat_k . Then f induces a Quillen pair

$$T^{\text{op}} - \text{Mod} \begin{array}{c} \xrightarrow{f_!} \\ \xleftarrow{f^*} \end{array} T'^{\text{op}} - \text{Mod}$$

where f^* is defined on objects by composition with f . As a left Quillen functor, the direct image $f_!$ preserves perfect dg-modules and induces an exact functor still denoted by $f_!$:

$$f_! : \text{Perf}(T) \longrightarrow \text{Perf}(T').$$

This defines a lax functor $\text{dgCat}_k \longrightarrow \text{WCat}$, and, applying the canonical strictification procedure, we obtain a functor

$$\text{Perf} : \text{dgCat}_k \longrightarrow \text{WCat}.$$

DEFINITION 2.7.

- (a) The algebraic K-theory space of a dg-category T is the pointed simplicial set $K(T) := K(\text{Perf}(T))$. This defines a functor

$$K : \text{dgCat}_k \longrightarrow \text{SSet}_*.$$

- (b) The *connective algebraic K-theory spectrum of a dg-category* T is the spectrum $\tilde{\mathbf{K}}(T) := \tilde{\mathbf{K}}(\text{Perf}(T))$. This defines a functor

$$\tilde{\mathbf{K}} : \text{dgCat}_k \longrightarrow \text{Sp}^{\text{con}}.$$

Remark 2.8. If $T = A$ is any associative k -algebra, one can consider vector bundles on $\text{Spec}(A)$, or in other words, projective (right) A -modules of finite type. This forms a Waldhausen category $\text{Vect}(A)$ with weak equivalences being isomorphisms and cofibrations being monomorphisms. One can show (using [TT90, Theorem 1.11.7]) that there is a weak equivalence of simplicial sets $K(\text{Vect}(A)) \simeq K(\text{Perf}(A))$, and thus a weak equivalence on the associated connective K-theory spectra too.

We now recall the main properties of connective K-theory: filtered colimits, Morita invariance and additivity on split short exact sequences of dg-categories.

- A short sequence $T' \xrightarrow{i} T \xrightarrow{p} T''$ of dg-categories is called *exact* if the sequence of triangulated perfect derived categories $D_{\text{pe}}(T') \xrightarrow{i_!} D_{\text{pe}}(T) \xrightarrow{p_!} D_{\text{pe}}(T'')$ is exact, i.e. that $i_!$ is fully faithful and that $p_!$ induces an equivalence up to factors between $D_{\text{pe}}(T'')$ and the Verdier quotient $D_{\text{pe}}(T)/D_{\text{pe}}(T')$.
- A short sequence $T' \xrightarrow{i} T \xrightarrow{p} T''$ of dg-categories is called *strictly split exact* if it is exact, and if, moreover, the functor $i_!$ has a right adjoint denoted by i^* , the functor $p_!$ has a right adjoint denoted by p^* such that $i^*i_! \simeq \text{id}_{\text{Perf}(T')}$ and $p_!p^* \simeq \text{id}_{\text{Perf}(T'')}$ via the adjunction maps.
- A short sequence $T' \xrightarrow{i} T \xrightarrow{p} T''$ of dg-categories is called *split exact* if it is isomorphic in $\text{Ho}(\text{dgMor}_k)$ to a strictly split short exact sequence.

PROPOSITION 2.9.

- (a) The functor $\tilde{\mathbf{K}}$ commutes with filtered homotopy colimits in dgCat_k .
- (b) The functor $\tilde{\mathbf{K}}$ sends Morita equivalences in dgCat_k to isomorphisms in $\text{Ho}(\text{Sp})$.
- (c) Let $T' \xrightarrow{i} T \xrightarrow{p} T''$ be a split short exact sequence of dg-categories. Then the morphism

$$i_! + p^* : \tilde{\mathbf{K}}(T') \oplus \tilde{\mathbf{K}}(T'') \longrightarrow \tilde{\mathbf{K}}(T)$$

is an isomorphism in $\text{Ho}(\text{Sp})$.

Proof. These facts are well known, but for details see [Bla13, Proposition 2.8]. □

We recall how to define nonconnective algebraic K-theory of dg-categories using the construction of Tabuada and Cisinski [CT11]. This definition coincides with Schlichting’s construction thanks to [CT11, Proposition 6.6].

The main ingredient of nonconnective K-theory is the countable sum completion functor or flasque envelope:

$$\mathcal{F} : \text{dgCat}_k \longrightarrow \text{dgCat}_k,$$

(see [CT11, § 6] for this construction). It comes with a quasi-fully faithful functor $T \longrightarrow \mathcal{F}(T)$. The essential property of the dg-category $\mathcal{F}(T)$ is that it admits countable sums, and thus satisfies $\tilde{\mathbf{K}}(\mathcal{F}(T)) = 0$. One can define the suspension functor of dg-categories

$$\mathcal{S} : \text{dgCat}_k \longrightarrow \text{dgCat}_k,$$

by $\mathcal{S}(T) := \mathcal{F}(T)/T$, where the quotient is calculated in $\text{Ho}(\text{dgCat}_k)$. The sequence of spectra $(\mathbf{K}(\mathcal{S}^n(T)))_{n \geq 0}$ actually forms a spectrum in the monoidal category Sp of symmetric spectra (see [CT11, Proposition 7.2]) and we take the zeroth level of the associated Ω -spectrum to define nonconnective \mathbf{K} -theory.

DEFINITION 2.10 [CT11, Proposition 7.5]. The *nonconnective algebraic K-theory of a dg-category* T is the spectrum

$$\mathbf{K}(T) := \text{hocolim}_{n \geq 0} \tilde{\mathbf{K}}(\mathcal{S}^n(T))[-n].$$

This defines a functor

$$\mathbf{K} : \text{dgCat}_k \longrightarrow \text{Sp}.$$

By definition, for every $T \in \text{dgCat}_k$ we have a natural map $\tilde{\mathbf{K}}(T) \longrightarrow \mathbf{K}(T)$. The main properties of nonconnective \mathbf{K} -theory are stated in the following proposition.

PROPOSITION 2.11.

- (a) For every triangulated dg-category T , the natural map $\tilde{\mathbf{K}}(T) \longrightarrow \mathbf{K}(T)$ induces an isomorphism on π_i for all $i \geq 0$. Therefore $\tilde{\mathbf{K}}(T)$ is the connective covering of the spectrum $\mathbf{K}(T)$.
- (b) The functor \mathbf{K} commutes with filtered homotopy colimits in dgCat_k .
- (c) The functor \mathbf{K} sends Morita equivalences to isomorphism in $\text{Ho}(\text{Sp})$.
- (d) Let $T' \xrightarrow{i} T \xrightarrow{p} T''$ be an exact sequence of dg-categories. Then the induced sequence

$$\mathbf{K}(T') \xrightarrow{i} \mathbf{K}(T) \xrightarrow{p} \mathbf{K}(T'')$$

is a distinguished triangle in $\text{Ho}(\text{Sp})$.

In fact, it can be shown that there is essentially a unique way to extend $\tilde{\mathbf{K}}$ to a nonconnective invariant satisfying all these properties (see [Rob13, Theorem 1.9]).

Proof. These facts are well known and result from Schlichting’s theory. For details and references, see [Bla13, Proposition 1.15]. □

Notation 2.12. If X is a scheme of finite type over k , we denote by $\mathcal{L}_{\text{perf}}(X)$ the dg-category of perfect complexes of quasi-coherent \mathcal{O}_X -modules. Following [Toë07, 8.3], this dg-category can be defined starting with the injective $C(k)$ -model category $C(\text{QCoh}(X))$ of complexes of quasi-coherent sheaves of \mathcal{O}_X -modules on X (in the sense of [Hov01]) and then taking the associated k -dg-category of cofibrant-fibrant object $\mathcal{L}_{\text{qcoh}}(X) := \text{Int}(C(\text{QCoh}(X)))$. This latter dg-category is only \mathbb{V} -small, but its full sub-dg-category consisting of perfect complexes which is defined to be $\mathcal{L}_{\text{perf}}(X)$ is equivalent to a \mathbb{U} -small dg-category. We recall that a complex of quasi-coherent \mathcal{O}_X -modules is said to be perfect if it is locally quasi-isomorphic to a bounded complex of locally free \mathcal{O}_X -modules. These objects are the compact objects of the derived category $D_{\text{qcoh}}(X)$ of quasi-coherent sheaves on X .

The assignment $X \mapsto \mathcal{L}_{\text{qcoh}}(X)$ can be arranged into a functor from \mathbb{U} -small k -schemes into \mathbb{V} -small dg-categories

$$\mathcal{L}_{\text{qcoh}} : \text{Sch}_k^{\text{op}} \longrightarrow \text{dgCat}_k^{\mathbb{V}}.$$

In order to obtain such a functor, we adopt the definition of quasi-coherent modules given in [TVe08, § 1.3.7]. The definition of perfect complexes is easily formulated in this setting: these are collections of quasi-coherent complexes which are locally quasi-isomorphic to a bounded

complex of (locally) free modules. In this way, for any morphism $f : X \rightarrow Y$ in Sch_k the pullback functor $f^* : \mathcal{L}_{\text{qcoh}}(Y) \rightarrow \mathcal{L}_{\text{qcoh}}(X)$ preserves perfect complexes and we obtain a map $f^* : \mathcal{L}_{\text{perf}}(Y) \rightarrow \mathcal{L}_{\text{perf}}(X)$. This defines a functor

$$\mathcal{L}_{\text{perf}} : \text{Sch}_k^{\text{op}} \rightarrow \text{dgCat}_k.$$

Following Schlichting [Sch06, §6.5] and the work of Thomason and Trobaugh, we define the algebraic K-theory of a scheme $X \in \text{Sch}_k$ to be the algebraic K-theory of its dg-category of perfect complexes $\mathbf{K}(X) := \mathbf{K}(\mathcal{L}_{\text{perf}}(X))$. This defines a functor

$$\mathbf{K} : \text{Sch}_k^{\text{op}} \rightarrow \text{Sp}.$$

Schlichting proved [Sch06, Theorem 5] that for any quasi-compact quasi-separated scheme X over k , the algebraic K-theory $\mathbf{K}(X)$ coincides with Thomason–Trobaugh nonconnective K-theory of X [TT90, Definition 6.4, p. 360].

2.3 Algebraic Chern character

We set notation for cyclic homology of noncommutative spaces. All the different versions of cyclic homology are defined from a single object called the mixed complex associated to a dg-category which was defined by Keller [Kel99]. The mixed complex is a functor

$$\text{Mix} = \text{Mix}(- | k) : \text{dgCat}_k \rightarrow \Lambda - \text{Mod},$$

with values in the category of Λ -dg-modules, where Λ is the k -dg-algebra generated by a single element B in degree -1 satisfying the relation $B^2 = 0$ and $d(B) = 0$. A Λ -dg-module is commonly called a mixed complex. If T is a locally cofibrant k -dg-category, we define a precyclic complex of k -modules in the sense of [Kel98a, §2.1], denoted by $C(T)$, by

$$C(T)_n = \bigoplus_{x_0, x_1, \dots, x_n \in T} T(x_n, x_0) \otimes_k T(x_{n-1}, x_n) \otimes_k T(x_{n-2}, x_{n-1}) \otimes_k \cdots \otimes_k T(x_0, x_1)$$

where the sum runs over all sequences of cardinal $n + 1$ of objects in T . The differential $d : C(T)_n \rightarrow C(T)_{n-1}$ is defined by

$$d(f_n \otimes f_{n-1} \otimes \cdots \otimes f_0) = f_{n-1} \otimes \cdots \otimes f_1 \otimes f_0 f_n + \sum_{i=1}^n (-1)^i f_n \otimes \cdots \otimes f_{i-1} f_i \otimes \cdots \otimes f_0,$$

where the f_i are homogeneous elements. The cyclic operator $t : C(T)_n \rightarrow C(T)_n$ is defined by

$$t(f_n \otimes \cdots \otimes f_0) = (-1)^{n+1} f_0 \otimes f_n \otimes \cdots \otimes f_1.$$

The mixed complex associated to the precyclic complex $C(T)$ (in the sense of [Kel98a, §2.1]) is by definition the mixed complex of T , denoted by $\text{Mix}(T)$. The object $\text{Mix}(T)$ is a relative invariant in the sense that it depends crucially on the ground ring. In an abuse of notation we omit to mention the ground ring in the notation $\text{Mix}(T)$ and, by default, it refers to mixed complex and cyclic homology relative to the ground field k over which our dg-categories are defined. Keller has proved that Mix is a localizing invariant in the sense of Tabuada [Tab07].

THEOREM 2.13 [Kel99]. *The functor Mix commutes with filtered homotopy colimits in $\text{Ho}(\text{dgCat}_k)$, sends Morita equivalences to equivalences and sends exact sequences of dg-categories to distinguished triangles in the derived category of mixed complexes.*

We denote by

$$H : C(\mathbb{Z}) \longrightarrow \mathrm{Sp}$$

the standard functor from (unbounded, cohomological) complexes of \mathbb{Z} -modules to symmetric spectra. We refer to [SS03, after Corollary B.1.8] or to [Bla13, § 1.4.1] for a construction of H . We consider $C(\mathbb{Z})$ and Sp as monoidal categories with their usual monoidal structure given by the dg-tensor product and the smash product, respectively. The functor H is then a lax monoidal functor and induces a functor on monoids and modules over monoids. Therefore we have a ring spectrum Hk , an Hk -algebra spectrum $H\Lambda$, and an induced functor

$$H : \Lambda - \mathrm{Mod} \longrightarrow H\Lambda - \mathrm{Mod}_{\mathbb{S}}$$

with values in the category of $H\Lambda$ -modules spectra. We endow $\Lambda - \mathrm{Mod}$ and $H\Lambda - \mathrm{Mod}_{\mathbb{S}}$ with the model structure given by [SS00, § 4] where a map is an equivalence if it is an equivalence of the underlying modules and spectra, respectively. By definition H preserves equivalences.

We denote by $\mathrm{HH} := H \circ \mathrm{Mix}$ the composite functor

$$\begin{array}{ccc} & \mathrm{HH} & \\ & \curvearrowright & \\ \mathrm{dgCat}_k & \xrightarrow{\mathrm{Mix}} \Lambda - \mathrm{Mod} & \xrightarrow{H} H\Lambda - \mathrm{Mod}_{\mathbb{S}} \end{array}$$

and we call it Hochschild homology over k . It is still a localizing invariant (i.e. it satisfies the conditions of Theorem 2.13). It is also a lax monoidal functor because H and Mix are. In the following definition we consider Hk as an $H\Lambda$ -module via the natural augmentation $H\Lambda \longrightarrow Hk$.

DEFINITION 2.14. Let $T \in \mathrm{dgCat}_k$.

- The *negative cyclic homology* of T is the symmetric spectrum

$$\mathrm{HN}(T) := \mathbb{R}\underline{\mathrm{Hom}}_{H\Lambda}(Hk, \mathrm{HH}(T)).$$

- The Hk -module $\mathrm{HN}(T)$ is a module over the Hk -algebra $\mathbb{R}\underline{\mathrm{Hom}}_{H\Lambda}(Hk, Hk) \simeq Hk[u]$ with u a generator of degree -2 . We define the *periodic cyclic homology* of T or *periodic homology* of T as the symmetric spectrum

$$\mathrm{HP}(T) := \mathrm{HN}(T) \overset{h}{\wedge}_{Hk[u]} Hk[u, u^{-1}].$$

We obtain functors

$$\begin{aligned} \mathrm{HN} &: \mathrm{dgCat}_k \longrightarrow Hk[u] - \mathrm{Mod}_{\mathbb{S}}, \\ \mathrm{HP} &: \mathrm{dgCat}_k \longrightarrow Hk[u, u^{-1}] - \mathrm{Mod}_{\mathbb{S}}. \end{aligned}$$

We have by definition a map of $Hk[u]$ -modules $\mathrm{HN}(T) \longrightarrow \mathrm{HP}(T)$.

Remark 2.15. This definition of periodic homology is not the standard definition given in [Kas87, p. 210], but by [Ros94, Theorem 6.1.24] we know that for an associative unital algebra, the two definitions coincide.

Remark 2.16. The cyclic homology of schemes was defined and studied by Weibel [Wei96] using a sheafification of the Hochschild complex of an algebra. If X is a scheme we denote by $\mathrm{HH}(X)$ Weibel’s Hochschild homology of X (the spectral version). Following Keller [Kel99, § 1.10, p. 10] we define the Hochschild homology of a scheme X to be the Hochschild homology spectrum

$\mathrm{HH}(\mathcal{L}_{\mathrm{perf}}(X))$ of its dg-category of perfect complexes (see Remark 2.12). By the comparison result [Kel98b, Theorem 5.2], we know that if X is quasi-compact separated over a field k , then there exists a canonical isomorphism

$$\mathrm{HH}(\mathcal{L}_{\mathrm{perf}}(X)) \simeq \mathrm{HH}(X)$$

in the derived category of Λ -module spectra. Therefore there are no ambiguities in the meaning of the Hochschild homology of a separated scheme X of finite type over a field k .

Notation 2.17. Let $\mathbf{E} : \mathrm{dgCat}_k \rightarrow V$ be a functor, and we define a new functor

$$\underline{\mathbf{E}} : \mathrm{dgCat}_k \rightarrow \mathrm{Pr}(\mathrm{Aff}_k, V)$$

by $\underline{\mathbf{E}}(T)(\mathrm{Spec}(A)) = \mathbf{E}(T, A) := E(T \otimes_k^{\mathbb{L}} A) := E(QT \otimes_k A)$ where $Q : \mathrm{dgCat}_k \rightarrow \mathrm{dgCat}_k$ is a functor which satisfies the following properties.

- There exists a natural transformation in $T, Q(T) \rightarrow T$ which is an equivalence (for the standard model model structure).
- For all $T \in \mathrm{dgCat}_k$, the dg-category $Q(T)$ is flat¹ over k .

Such a functor is, for example, given by a cofibrant replacement functor. In this paper, we are primarily interested in the case where k is equal to the complex numbers \mathbb{C} , so we could just take Q to be the identity functor. We will use this notation for all classical invariants of dg-categories $\mathbf{E} = \mathbf{K}, \underline{\mathbf{K}}, \mathrm{HH}, \mathrm{HC}, \mathrm{HN}, \mathrm{HP}$.

We now use the motivic category of Tabuada (see [Tab08, CT11]) and both the Tabuada and Cisinski–Tabuada theorems of corepresentability of K-theory inside the motivic category. Following Cisinski and Tabuada, these results enable us to define the algebraic Chern character in a \mathbf{k} -linear fashion, where \mathbf{k} is the ring in presheaves of spectra given by $\mathbf{k}(\mathrm{Spec}(A)) = \mathbf{K}(A)$, the algebraic K-theory of the commutative algebra A . The corepresentability theorem gives us a natural ring structure on \mathbf{k} and of \mathbf{k} -modules on $\underline{\mathbf{K}}(T)$ for all dg-category T .

The universal localizing motivator $\mathcal{M}_{\mathrm{loc}}(k)$ of Tabuada [Tab08, Definition 11.2] can be described (using [Tab08, Proposition 12.4]) as the derivator associated to the projective model category of presheaves of spectra on finite type dg-categories, localized with respect to Morita equivalences and short exact sequences. If $\mathrm{Hodgcat}$ stands for the derivator associated to dgMor_k , there exists a morphism of derivators

$$\mathcal{U}_l : \mathrm{Hodgcat} \rightarrow \mathcal{M}_{\mathrm{loc}}(k)$$

called the universal localizing invariant, which is universal among functors which commute with filtered homotopy colimits, send Morita equivalences to equivalences and short exact sequences to triangles (see [Tab08, Theorem 11.5]). In [CT12], the authors build a symmetric monoidal structure on $\mathcal{M}_{\mathrm{loc}}(k)$ out of the Day convolution product and the derived tensor product of dg-categories, using the fact that a special class of finite type dg-categories are invariant under the latter product. Moreover, this class must satisfy some flatness properties (see [CT12, § 7.1, properties (a)–(f)]). In this subsection, we just deal with the homotopy category $\mathcal{M}_{\mathrm{loc}}(k)(*) = \mathrm{Ho}(\mathbb{M}_{\mathrm{loc}}(k))$. We state a purely categorical version of Tabuada’s universal property, which is sufficient for our purpose. We also replace the category of symmetric spectra Sp by the category $\mathrm{Sp}(\mathrm{Aff}_k)$ of presheaves of symmetric spectra. We will thus consider presheaves over dg-categories of finite type with values in the category $\mathrm{Sp}(\mathrm{Aff}_k)$.

¹ A dg-category T is flat over k if for all $x, y \in T$, the k -modules $T(x, y)^n$ are flat k -modules for all n .

We denote by $\text{dgCat}_k^{\text{tf}} \subseteq \text{dgCat}_k$ a subcategory of \mathbb{U} -small dg-categories which satisfy properties (a)–(f) in [CT12, §7.1]. Roughly speaking, this means that any dg-category in $\text{dgCat}_k^{\text{tf}}$ is of finite type in the sense of [TVa07, Definition 2.4], locally flat over k , and that the subcategory $\text{dgCat}_k^{\text{tf}}$ is stable by tensor product and by the endofunctor Q we fixed above in Notation 2.17. We consider the category $\text{Pr}(\text{dgCat}_k^{\text{tf}}, \text{Sp}(\text{Aff}_k))$ of presheaves over $\text{dgCat}_k^{\text{tf}}$ with values in $\text{Sp}(\text{Aff}_k)$. We endow the category $\text{Sp}(\text{Aff}_k)$ with the injective model structure for which we refer to the general existence theorem [Lur09, Proposition A.2.8.2]. We endow the category $\text{Pr}(\text{dgCat}_k^{\text{tf}}, \text{Sp}(\text{Aff}_k))$ with the projective model structure. We denote by

$$\underline{h} : \text{dgCat}_k^{\text{tf}} \longrightarrow \text{Pr}(\text{dgCat}_k^{\text{tf}}, \text{Sp}(\text{Aff}_k))$$

the Yoneda embedding given by $\underline{h}_T(T')(\text{Spec}(A)) = \text{Hom}_{\text{dgCat}_k}(T', T \otimes_k A)$. Let \mathcal{R} be the set of maps in $\text{Pr}(\text{dgCat}_k^{\text{tf}}, \text{Sp}(\text{Aff}_k))$ which are of the form:

- (i) $\underline{h}_T \longrightarrow \underline{h}_{T'}$, with $T \longrightarrow T'$ being a Morita equivalence;
- (ii) $\text{Cone}(\underline{h}_i) \longrightarrow \underline{h}_{T''}$, for $T' \xrightarrow{i} T \xrightarrow{p} T''$ an exact sequence of dg-categories.

We define $\mathbb{M}_{\text{loc}}(k) := L_{\mathcal{R}} \text{Pr}(\text{dgCat}_k^{\text{tf}}, \text{Sp}(\text{Aff}_k))$ as the $\text{Sp}(\text{Aff}_k)$ -enriched left Bousfield localization in the sense of [Bar07]. Therefore $\mathbb{M}_{\text{loc}}(k)$ is a $\text{Sp}(\text{Aff}_k)$ -model category in the sense of [Hov99, Definition 4.2.18], and we denote by $\mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}$ its derived Hom enriched in $\text{Sp}(\text{Aff}_k)$. By [CT12, §7.5], it is also a monoidal model category in the sense of [Hov99, Definition 4.2.6], and we denote by \wedge its monoidal product. The unit is given by the object \underline{k} which is defined by $\underline{k}(T) = \text{Hom}_{\text{dgCat}_k}(T, k)$ and constant as a presheaf on Aff_k . The following proposition follows from general properties of left Bousfield localization (see [Hir09, Propostion 3.3.18]).

PROPOSITION 2.18. *The Yoneda embedding $\underline{h} : \text{dgCat}_k^{\text{tf}} \longrightarrow \mathbb{M}_{\text{loc}}(k)$ has the following properties.*

- (i) *For all $T \in \text{dgCat}_k$, \underline{h}_T is cofibrant in $\mathbb{M}_{\text{loc}}(k)$.*
- (ii) *\underline{h} sends Morita equivalences to equivalences in $\mathbb{M}_{\text{loc}}(k)$.*
- (iii) *\underline{h} sends exact sequences to fibration sequences in $\mathbb{M}_{\text{loc}}(k)$.*

Moreover, \underline{h} is universal with respect to these three properties. More precisely, for all $\text{Sp}(\text{Aff}_k)$ -model categories V , the functor

$$\underline{h}^* : \text{Hom}_!(\mathbb{M}_{\text{loc}}(k), V) \longrightarrow \text{Hom}_*(\text{dgCat}_k^{\text{tf}}, V)$$

is an equivalence of categories, where $\text{Hom}_!$ is the category of $\text{Sp}(\text{Aff}_k)$ -enriched left Quillen functors, and Hom_* is the category of functors which satisfy properties (i)–(iii) above.

Remark 2.19. Let V be a $\text{Sp}(\text{Aff}_k)$ -model category and $E : \text{dgCat}_k \longrightarrow V$ a functor which sends Morita equivalences to equivalences and commutes with filtered colimits. Then E restricts to a functor $E| : \text{dgCat}_k^{\text{tf}} \longrightarrow V$. The extension of $E|$ to $\mathbb{M}_{\text{loc}}(k)$ is denoted by

$$E_{|, !} : \mathbb{M}_{\text{loc}}(k) \longrightarrow \text{Sp}(\text{Aff}_k).$$

Every dg-category T can be written as a filtrant colimit of dg-categories in $\text{dgCat}_k^{\text{tf}}$. Therefore there exist a filtrant diagram $\{T_\alpha\}$ of dg-categories in $\text{dgCat}_k^{\text{tf}}$ and an equivalence of dg-categories $T \simeq \text{colim}_\alpha T_\alpha$ in dgCat_k . On the other hand, the object $\underline{h}_T \in \mathbb{M}_{\text{loc}}(k)$ is defined by

$$\underline{h}_T(T')(A) = \text{Hom}_{\text{dgCat}_k}(T', T \otimes_k^{\mathbb{L}} A)$$

and represents the dg-category T in $\mathbb{M}_{\text{loc}}(k)$. Since the standard Yoneda functor is equivalent to the homotopical Yoneda embedding (see [TVe05, Lemma 4.2.2]), we have an equivalence $\underline{h}_T \simeq \text{colim}_\alpha \underline{h}_{T_\alpha}$ in $\mathbb{M}_{\text{loc}}(k)$. We therefore have equivalences

$$E_{|\cdot|}(\underline{h}_T) \simeq \text{colim}_\alpha E_{|\cdot|}(\underline{h}_{T_\alpha}) \simeq \text{colim}_\alpha E_{|\cdot|}(T_\alpha) \simeq E(T).$$

We conclude that the extension $E_{|\cdot|}$ coincides up to equivalence with E on dgCat_k . This can be rephrased in the following way. We denote by $\text{Hom}_{\text{Mor}}(\text{dgCat}_k^{\text{tf}}, V)$ the projective model category of functors which send Morita equivalences to equivalences, and by $\text{Hom}_!(\text{dgCat}_k, \text{Sp})$ the projective model category of functors which commute with filtered colimits and send Morita equivalences to equivalences. Then the inclusion functor $\text{dgCat}_k^{\text{tf}} \rightarrow \text{dgCat}_k$ induces an equivalence of categories

$$\text{Ho}(\text{Hom}_!(\text{dgCat}_k, \text{Sp})) \xrightarrow{\sim} \text{Ho}(\text{Hom}_{\text{Mor}}(\text{dgCat}_k^{\text{tf}}, \text{Sp})). \tag{2}$$

Every functor $E \in \text{Hom}_{\text{Mor}}(\text{dgCat}_k^{\text{tf}}, \text{Sp})$ can be uniquely extended up to isomorphism to an object of $\text{Ho}(\text{Hom}_!(\text{dgCat}_k, \text{Sp}))$. By an abuse of notation we denote by $\underline{\text{HH}}$ the functor $\underline{\text{HH}}_!$.

THEOREM 2.20 [CT11, Theorem 7.16]. *For all $T \in \text{dgCat}_k^{\text{tf}}$, there exists a functorial isomorphism in T ,*

$$\mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}(k, \underline{h}_T) \simeq \underline{\mathbf{K}}(T).$$

Because of formula (2), the isomorphism of Theorem 2.20 lifts to a unique isomorphism in $\text{Ho}(\text{Hom}_!(\text{dgCat}_k, \text{Sp}))$.

We consider the object \underline{h}_k in $\mathbb{M}_{\text{loc}}(k)$. It is given by $\underline{h}_k(T)(\text{Spec}(A)) = \text{Hom}_{\text{dgCat}_k}(T, A)$ for all $T \in \text{dgCat}_k$ and all $\text{Spec}(A) \in \text{Aff}_k$. The object \underline{h}_k admits a monoidal structure in $\mathbb{M}_{\text{loc}}(k)$ because by definition of the monoidal structure in $\mathbb{M}_{\text{loc}}(k)$ we have an isomorphism $\underline{h}_k \wedge \underline{h}_k \simeq \underline{h}_{k \otimes^{\mathbb{L}} k}$ in $\mathbb{M}_{\text{loc}}(k)$. For all $T \in \text{dgCat}_k^{\text{tf}}$, we have an \underline{h}_k -module \underline{h}_T with \underline{h}_k -action given by the natural isomorphism $\underline{h}_T \wedge \underline{h}_k \simeq \underline{h}_{T \otimes_k k} \simeq \underline{h}_T$. We then have the following objects.

- A monoidal model category $\mathcal{V} = \text{Sp}(\text{Aff}_k)$.
- Two $\text{Sp}(\text{Aff}_k)$ -enriched monoidal model categories

$$\mathcal{M} = \mathbb{M}_{\text{loc}}(k) \quad \text{and} \quad \mathcal{N} = \text{Pr}(\text{Aff}_k, H\Lambda - \text{Mod}_{\mathbb{S}}).$$

Units are denoted by $\mathbb{1}_{\mathcal{M}}$ and $\mathbb{1}_{\mathcal{N}}$. They are cofibrant objects in \mathcal{M} and \mathcal{N} , respectively. Thus we have two monoidal functors $\mathcal{V} \rightarrow \mathcal{M}$ and $\mathcal{V} \rightarrow \mathcal{N}$ given by the product with units. Their right adjoints $\underline{\text{Hom}}(\mathbb{1}, -)$ (\mathcal{V} -enriched Hom) are lax monoidal functors and therefore send monoids to monoids.

- A \mathcal{V} -enriched left Quillen lax monoidal functor $F : \mathcal{M} \rightarrow \mathcal{N}$ given by $F = \underline{\text{HH}}$. (In this case $F(\mathbb{1}_{\mathcal{M}}) \simeq \mathbb{1}_{\mathcal{N}}$.)
- A cofibrant monoid $a = \underline{h}_k$ in \mathcal{M} and a cofibrant a -module m given by $m = \underline{h}_T$.
- Two morphisms in \mathcal{V} given by the functoriality of $\mathbb{L}F$:

$$\begin{aligned} t & : \mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a) \rightarrow \mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a)), \\ u & : \mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m) \rightarrow \mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(m)), \end{aligned}$$

where $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}$ and $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}$ are the $\text{Ho}(\mathcal{V})$ -enriched Homs.

The general results of [SS00] on the existence of model structures on categories of monoids and modules must satisfy a special axiom called the monoid axiom, which is not satisfied by the

model category $\mathbb{M}_{\text{loc}}(k)$. However, by Hovey’s results [Hov98, Theorems 3.3, 2.1], monoids and module objects in $\mathbb{M}_{\text{loc}}(k)$ can be endowed with nice homotopical properties, which are sufficient for our purpose. Roughly speaking, monoids do not form a model category in the usual sense, but a so-called semi-model category, and we can therefore consider the homotopy category of monoids. In this situation, if M is a monoid, there exists a functorial cofibrant replacement $QM \xrightarrow{\sim} M$ and if M is cofibrant, there exists a functorial fibrant replacement $M \xrightarrow{\sim} RM$. The homotopy category of monoids is then the quotient of cofibrant-fibrant monoids by the usual homotopy relation. We can therefore make the following statements.

- The objects $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ and $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a))$ are monoids.
- The object $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, m)$ is a $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -module.
- The object $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(m))$ is a $\mathbb{R}\underline{\text{Hom}}_{\mathcal{N}}(F(\mathbb{1}_{\mathcal{M}}), F(a))$ -module and therefore a $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -module in \mathcal{V} via t .
- The map u is a map of $\mathbb{R}\underline{\text{Hom}}_{\mathcal{M}}(\mathbb{1}_{\mathcal{M}}, a)$ -modules.

Applying this in our context, we obtain the following objects for any dg-category $T \in \text{dgCat}_k^{\text{tf}}$.

- A presheaf of ring spectra $\mathbf{k} := \underline{\mathbf{K}}(*) \simeq \mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}(\underline{k}, \underline{h}_k)$.
- A presheaf of \mathbf{k} -modules $\underline{\mathbf{K}}(T) \simeq \mathbb{R}\underline{\text{Hom}}_{\mathbb{M}_{\text{loc}}(k)}(\underline{k}, \underline{h}_T)$.
- A presheaf of \mathbf{k} -modules $\underline{\text{HN}}(T) = \mathbb{R}\underline{\text{Hom}}(\underline{k}, \underline{\text{HH}}(T)) \simeq \mathbb{R}\underline{\text{Hom}}(\underline{\text{HH}}(\underline{k}), \underline{\text{HH}}(\underline{h}_T))$, where $\mathbb{R}\underline{\text{Hom}}$ is the derived Hom of $\text{Pr}(\text{Aff}_k, H\Lambda - \text{Mod}_{\mathbb{S}})$.
- A map of \mathbf{k} -modules $\underline{\mathbf{K}}(T) \rightarrow \underline{\text{HN}}(T)$.

We denote by $\mathbf{k} - \text{Mod}_{\mathbb{S}}$ the category of \mathbf{k} -modules objects in $\text{Sp}(\text{Aff}_k)$.

DEFINITION 2.21. Let $T \in \text{dgCat}_k^{\text{tf}}$. The algebraic Chern character associated to T is the map

$$\text{Ch}_T : \underline{\mathbf{K}}(T) \rightarrow \underline{\text{HN}}(T)$$

in $\text{Ho}(\mathbf{k} - \text{Mod}_{\mathbb{S}})$, which is defined above. Taking appropriate fibrant replacements, it defines a map in $\text{Ho}(\mathbf{k} - \text{Mod}_{\mathbb{S}}^{\text{dgCat}_k^{\text{tf}}})$,

$$\text{Ch} : \underline{\mathbf{K}} \rightarrow \underline{\text{HN}}.$$

By formula (2), the map Ch lifts to a unique map in $\text{Ho}(\mathbf{k} - \text{Mod}_{\mathbb{S}}^{\text{dgCat}_k})$ (up to isomorphism).

Remark 2.22. There exists an additive version $\mathbb{M}_a(k)$ of the model category $\mathbb{M}_{\text{loc}}(k)$ where we replace exact sequences of dg-categories by *split* exact sequences. Connective K-theory then becomes corepresentable as a functor $\mathbb{M}_a(k) \rightarrow \text{Sp}(\text{Aff}_k)$, and the same construction as above gives a connective Chern map,

$$\text{Ch}_c : \tilde{\underline{\mathbf{K}}} \rightarrow \underline{\text{HN}}$$

in $\text{Ho}(\tilde{\mathbf{k}} - \text{Mod}_{\mathbb{S}}^{\text{dgCat}_k})$, where $\tilde{\mathbf{k}}$ is the presheaf of connective K-theory. By construction, the map

$$\tilde{\underline{\mathbf{K}}} \rightarrow \underline{\mathbf{K}} \xrightarrow{\text{Ch}} \underline{\text{HN}}$$

is isomorphic to Ch_c in $\text{Ho}(\text{Sp}^{\text{dgCat}_k})$.

Remark 2.23. By definition, the Chern map of Definition 2.21 coincides up to isomorphism in $\text{Ho}(\text{Sp}(\text{Aff}_k)^{\text{dgCat}_k})$ with the Tabuada Chern map as defined in [Tab13, p. 4], and we therefore know that it coincides with the classic version of the Chern map by [Tab13, Theorem 2.8].

3. Topological realization over complex numbers

The topological realization of a simplicial presheaf defined over \mathbb{C} -schemes is a topological space functorially associated to this presheaf. In some sense the topological realization of a simplicial presheaf is the analog of the ‘espace étalé’ associated to a sheaf over a topological space. We begin by considering the functor which associates to any \mathbb{C} -scheme its space of complex points. This functor extends in a canonical way to all simplicial presheaves. The resulting functor is the topological realization, and could also be called the Betti realization. Topological realization has already been studied by Simpson [Sim96], by Morel and Voevodsky [MV99] at the level of the motivic homotopy category, and by Dugger and Isaksen [DI01] where the authors proved its compatibility with the homotopy theory of stacks (in the étale or Nisnevich topology). In this section we set up definitions, properties and results about the topological realization. The most original part is the last one in which it is proven that the topological realization of a simplicial presheaf is unchanged if we first restrict this presheaf to smooth schemes. This uses a homotopical generalization of Deligne’s cohomological proper descent.

3.1 Generalities about the topological realization

We recall that $\text{Aff}_{\mathbb{C}}$ stands for the category of affine \mathbb{C} -schemes of finite type. We denote by

$$\text{sp} : \text{Aff}_{\mathbb{C}} \longrightarrow \text{Top}$$

the functor which associates to an affine \mathbb{C} -scheme X its topological space of complex points. Let $X = \text{Spec}(A)$ with presentation $A = \mathbb{C}[X_1, \dots, X_n]/\mathfrak{a}$, with \mathfrak{a} an ideal of the polynomial algebra $\mathbb{C}[X_1, \dots, X_n]$. Then this presentation defines a closed immersion $i : X \hookrightarrow \mathbf{A}^n$ and $\text{sp}(X) = i(X)(\mathbb{C})$ endowed with the induced transcendental topology of the space $\mathbf{A}^n(\mathbb{C})$. Composing with the singular complex functor $S : \text{Top} \longrightarrow \text{SSet}$, we obtain a simplicial set-valued realization denoted by ssp . We have the commutative triangle

$$\begin{array}{ccc} \text{Aff}_{\mathbb{C}} & \xrightarrow{\text{sp}} & \text{Top} \\ & \searrow \text{ssp} & \downarrow S \\ & & \text{SSet} \end{array}$$

DEFINITION 3.1. We still denote by ssp (respectively, sp) and call *topological realization* the SSet -enriched left Kan extension² of the functor $\text{ssp} : \text{Aff}_{\mathbb{C}} \longrightarrow \text{SSet}$ (respectively, of the functor sp)

² We briefly recall the notion of enriched left Kan extension. Let V be a monoidal category, $F : C \longrightarrow D$ a functor with C any category and D a V -enriched category, which is moreover cocomplete in the enriched sense, $G : C \longrightarrow C'$ a functor with C' another cocomplete V -enriched category. The V -enriched left Kan extension of F along G is the unique V -enriched functor $\tilde{F} : C' \longrightarrow D$ (up to isomorphism) which commutes to V -enriched colimits and such that the triangle

$$\begin{array}{ccc} C & \xrightarrow{G} & C' \\ & \searrow F & \downarrow \tilde{F} \\ & & D \end{array}$$

commutes up to isomorphism. In our case, we do not need to consider general enriched colimits, but only colimits of the form (5) later in this subsection.

along the Yoneda embedding $h : \text{Aff}_{\mathbb{C}} \hookrightarrow \text{SPr}(\text{Aff}_{\mathbb{C}})$. We then obtain functors

$$\begin{array}{ccc} \text{Aff}_{\mathbb{C}} & \xrightarrow{h} & \text{SPr}(\text{Aff}_{\mathbb{C}}) & \xrightarrow{\text{sp}} & \text{Top} \\ & & \searrow \text{ssp} & & \downarrow s \\ & & & & \text{SSet} \end{array}$$

General properties of the functor ssp are summarized in the following proposition. Analogous properties hold for sp .

PROPOSITION 3.2. (i) *The functor $\text{ssp} : \text{SPr}(\text{Aff}_{\mathbb{C}}) \rightarrow \text{SSet}$ commutes with colimits and its right adjoint is the functor R such that for any simplicial set K , $R(K) : X \mapsto \text{Map}(\text{ssp}(X), K)$. The adjoint pair (ssp, R) is a Quillen pair for the global model structure on $\text{SPr}(\text{Aff}_{\mathbb{C}})$.*

(ii) *The functor ssp commutes with finite homotopy products. In particular, ssp is a symmetric monoidal functor for the cartesian model structure on $\text{SPr}(\text{Aff}_{\mathbb{C}})$ and SSet .*

(iii) *For all $K \in \text{SSet}$, and all $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})$, there exists a canonical isomorphism $\text{ssp}(K \times E) \simeq K \times \text{ssp}(E)$. For all $K \in \text{SSet}_*$ and all $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})_*$ there exists a canonical isomorphism $\text{ssp}(K \wedge E) \simeq K \wedge \text{ssp}(E)$.*

(iv) *Let $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})$. We denote by $\pi_0^{\text{pr}} E$ the presheaf obtained by applying π_0 levelwise. Then there exists a canonical isomorphism of sets $\pi_0 \text{ssp}(E) \simeq \pi_0 \text{ssp}(\pi_0^{\text{pr}} E)$.*

Proof. (i) R is right adjoint to ssp by definition. By its definition, R sends (trivial) fibrations to (trivial) fibrations. The pair (ssp, R) is therefore a Quillen pair.

(ii) This follows from formula (5) of Remark 3.3 below, and the fact that in a topos (here the topos of simplicial presheaves on $\text{Aff}_{\mathbb{C}}$) products commute with colimits.

(iii) The proof of the pointed case is similar to that of the unpointed case. It comes from the fact that $K \times E = \coprod_K E$ and that the functor ssp is a left adjoint and therefore commutes with colimits.

(iv) Let $A \in \text{Set}$. We have canonical isomorphisms

$$\begin{aligned} \text{Hom}_{\text{Set}}(\pi_0 \text{ssp}(\pi_0^{\text{pr}} E), A) &\simeq \text{Hom}_{\text{SSet}}(\text{ssp}(\pi_0^{\text{pr}} E), A) \\ &\simeq \text{Hom}_{\text{SPr}(\text{Aff}_{\mathbb{C}})}(\pi_0^{\text{pr}} E, R(A)) \\ &\simeq \text{Hom}_{\text{SPr}(\text{Aff}_{\mathbb{C}})}(E, R(A)) \\ &\simeq \text{Hom}_{\text{SSet}}(\text{ssp}(E), A) \\ &\simeq \text{Hom}_{\text{Set}}(\pi_0 \text{ssp}(E), A). \end{aligned}$$

We conclude by the Yoneda lemma. □

Remark 3.3. Every presheaf of sets $F \in \text{Pr}(\text{Aff}_{\mathbb{C}})$ can be written as a colimit of representable presheaves. More precisely, the map

$$\text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} h_X \xrightarrow{\sim} F \tag{3}$$

is an isomorphism of presheaves. Therefore, since ssp commutes with colimits, we have the formula

$$\text{ssp}(F) \simeq \text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} \text{ssp}(X). \tag{4}$$

In § 3.3 we will use a homotopical version of the previous fact. Now take $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})$ a simplicial presheaf. Formula (3) is not yet available. This comes from the very homotopical

flavor of the object E and $\text{Aff}_{\mathbb{C}}/E$ should really be considered as an ∞ -category in order to obtain a similar formula, with a colimit in a higher sense. An alternative way is to write E as a more complicated colimit (see [Dug01, p. 8]). Indeed, E can be written as the standard coequalizer

$$E \simeq \text{colim} \left(\coprod_{X \in \text{Aff}_{\mathbb{C}}} E(X) \times h_X \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \coprod_{Y \rightarrow Z} E(Z) \times h_Y \right). \tag{5}$$

And thus we have an equivalence

$$\text{sph}(E) \simeq \text{colim} \left(\coprod_{X \in \text{Aff}_{\mathbb{C}}} E(X) \times \text{sph}(X) \begin{array}{c} \longleftarrow \\ \longleftarrow \\ \longleftarrow \end{array} \coprod_{Y \rightarrow Z} E(Z) \times \text{sph}(Y) \right).$$

3.2 \mathbf{A}^1 -étale model structure

Let C be a Grothendieck site with topology denoted by τ . Jardine (see [Jar87], and also [DHI04]) has proven the existence of a τ -local model structure on the category $\text{SPr}(C)$ of simplicial presheaves on C such that the equivalences are precisely local equivalences. Thanks to [DHI04], this model structure can be defined as the left Bousfield localization of the global model structure $\text{SPr}(C)$ with respect to the set of maps of the form

$$\text{hocolim}_{\Delta^{\text{op}}} h_{U_{\bullet}} \longrightarrow h_X$$

in $\text{SPr}(C)$, for all $X \in C$ and all τ -hypercovering $U_{\bullet} \longrightarrow X$ in C . Morally, this construction forces simplicial presheaves to satisfy τ -homotopical descent. We denote by $\text{SPr}(C)^{\tau}$ the τ -local model structure. We will use it in the particular case of $C = \text{Aff}_{\mathbb{C}}$ with the étale topology, and call this model structure the étale local model structure. The homotopy category $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}})$ is a model for the homotopy theory of stacks.

We endow $\text{Aff}_{\mathbb{C}}$ with the étale topology. Everything below is also valid for the Nisnevich topology. Following [MV99], we can build the étale \mathbf{A}^1 -homotopy theory of schemes out of the site $\text{Aff}_{\mathbb{C}}$. The étale \mathbf{A}^1 -homotopy category of schemes is by definition the left Bousfield localization of the global model category $\text{SPr}(\text{Aff}_{\mathbb{C}})$ by the set of maps of the form:

- (i) $\text{hocolim}_{\Delta^{\text{op}}} h_{U_{\bullet}} \longrightarrow h_X$, for $U_{\bullet} \longrightarrow X$ an étale hypercovering of a scheme $X \in \text{Aff}_{\mathbb{C}}$;
- (ii) a projection $h_X \times h_{\mathbf{A}^1} \longrightarrow h_X$, for $X \in \text{Aff}_{\mathbb{C}}$.

We denote this model structure by $\text{SPr}^{\text{ét}, \mathbf{A}^1}$ and call it the \mathbf{A}^1 -étale structure on simplicial presheaves. Equivalences in $\text{SPr}^{\text{ét}, \mathbf{A}^1}$ are called \mathbf{A}^1 -equivalences. We denote by

$$\text{sph} : \text{SPr}^{\text{ét}, \mathbf{A}^1} \longrightarrow \text{Top}$$

the corresponding topological realization, and by ssph its simplicial analog. We denote by Rh the right adjoint to sph . An important property of the topological realization is stated in the following theorem.

THEOREM 3.4 [DI01, Theorem 5.2]. *The adjoint pair (sph, Rh) is a Quillen pair*

$$\text{SPr}^{\text{ét}, \mathbf{A}^1} \begin{array}{c} \xrightarrow{\text{sph}} \\ \xleftarrow{Rh} \end{array} \text{Top},$$

for the \mathbf{A}^1 -étale model structure. More precisely the functor sp sends relations (i) and (ii) above (which define the \mathbf{A}^1 -étale model structure) to equivalences of spaces.

Remark 3.5.

- It follows from the above that the simplicial version $\text{ssph} : \text{SPr}^{\text{ét}, \mathbf{A}^1} \longrightarrow \text{SSet}$ is a left Quillen functor and preserves relations of type (i) and (ii) above which define $\text{SPr}^{\text{ét}, \mathbf{A}^1}$.
- Another consequence of Theorem 3.4 is that the derived functors

$$\begin{aligned} \mathbb{L} \text{ssp} &: \text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})) \longrightarrow \text{Ho}(\text{SSet}), \\ \mathbb{L} \text{ssph} &: \text{Ho}(\text{SPr}^{\text{ét}, \mathbf{A}^1}) \longrightarrow \text{Ho}(\text{SSet}) \end{aligned}$$

satisfy $\mathbb{L} \text{ssp}(E) \simeq \mathbb{L} \text{ssph}(E)$ for all simplicial presheaves E . In other words, if Q is a cofibrant replacement functor for the global model structure $\text{SPr}(\text{Aff}_{\mathbb{C}})$, then $\mathbb{L} \text{ssph}(E) = \text{ssp}(QE)$.

DEFINITION 3.6. The *derived topological realization* is the functor $\mathbb{L} \text{ssp}$ (or equivalently $\mathbb{L} \text{ssph}$) and is denoted by $|-|$. We then have a commutative triangle

$$\begin{array}{ccc} \text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})) & \xrightarrow{\mathbb{R} \text{id}} & \text{Ho}(\text{SPr}^{\text{ét}, \mathbf{A}^1}) \\ & \searrow \text{|-|} & \swarrow \text{|-|} \\ & \text{Ho}(\text{SSet}) & \end{array}$$

We denote by $|-|^{\text{top}}$ its topological analog with values in $\text{Ho}(\text{Top})$.

PROPOSITION 3.7. (i) *The derived topological realization $|-|$ commutes with homotopy colimits. Its right adjoint is denoted by $(-)_B$. For all $K \in \text{SSet}$ and all $X \in \text{Aff}_{\mathbb{C}}$, we have $K_B(X) = \mathbb{R} \text{Map}(\text{ssp}(X), K)$.*

(ii) *The functor $|-|$ commutes with homotopy finite products.*

(iii) *For all $K \in \text{SSet}$, and all $E \in \text{SPr}^{\text{ét}, \mathbf{A}^1}$, we have a canonical isomorphism $K \times |E| \simeq |K \times E|$ in $\text{Ho}(\text{SSet})$. For all $K \in \text{SSet}_*$ and all $E \in \text{SPr}_*^{\text{ét}, \mathbf{A}^1}$, we have a canonical isomorphism $K \wedge |E| \simeq |K \wedge E|$ in $\text{Ho}(\text{SSet}_*)$.*

Proof. Proofs are essentially the same as for the nonderived case, and we omit them. □

Notation 3.8. By an abuse of notation, we will omit the word ‘derived’ in front of topological realization, although we always mean the derived topological realization throughout.

3.3 π_0 of the topological realization

We give an explicit description of the set $\pi_0|E|$ for any simplicial presheaf $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})$, which will be used below. This description is based on the formula (3) given by Yoneda.

LEMMA 3.9. *Let $F \in \text{Pr}(\text{Aff}_{\mathbb{C}})$ be a presheaf of sets. Then there exists a canonical isomorphism of sets,*

$$\pi_0 \text{ssp}(F) \simeq F(\mathbb{C}) / \sim,$$

where \sim is the equivalence relation defined by $[x] \sim [y]$ if there exist a connected algebraic curve C , a map $f : h_C \longrightarrow F$ in $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}}))$, and two complex points $x', y' \in C(\mathbb{C})$ such that $f(x') = x$ and $f(y') = y$.

Proof. As there exists an isomorphism $\pi_0 \text{ssp}(F) \simeq \pi_0 \text{sp}(F)$, where sp is the realization with values in Top , it suffices to prove the statement for $\pi_0 \text{sp}(F)$. From formula (3) in Remark 3.3,

there exists a natural map $F(\mathbb{C}) \rightarrow \text{sp}(F)$ in Top and therefore a map $F(\mathbb{C}) \rightarrow \pi_0 \text{sp}(F)$ by taking π_0 . We then have a commutative square

$$\begin{CD} F(\mathbb{C}) @>>> \pi_0 \text{sp}(F) \\ @V \wr VV @VV \wr V \\ \text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} X(\mathbb{C}) @>>> \text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} \pi_0 \text{sp}(X) \end{CD}$$

where the left vertical arrow is obtained by taking complex points in (3), and the right vertical arrow is obtained by taking $\pi_0 \text{sp}$ in (3). These two arrows are therefore isomorphisms. For all $X \in \text{Aff}_{\mathbb{C}}$, the map $X(\mathbb{C}) \rightarrow \pi_0 \text{sp}(X)$ is surjective. It follows that the top map in the square is surjective. Now let $a, b \in F(\mathbb{C})$ be two complex points having the same image under $F(\mathbb{C}) \rightarrow \pi_0 \text{sp}(F)$. In $\text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} X(\mathbb{C})$, a and b correspond to two affine schemes (X, x) and (Y, y) above F , each provided with a complex point. By hypothesis, these two elements have the same image in $\text{colim}_{X \in \text{Aff}_{\mathbb{C}}/F} \pi_0 \text{sp}(X)$. Therefore there exist an affine scheme $Z \rightarrow F$ above F , and two maps $p : X \rightarrow Z$ and $q : Y \rightarrow Z$, and a continuous path between $p(x)$ and $q(y)$ in the space $\text{sp}(Z)$. It is a classical fact in algebraic geometry that, using Bertini's theorem, we can deduce the existence of a connected algebraic curve $g : C \rightarrow Z$ above Z , and two complex points $x', y' \in C(\mathbb{C})$ such that $g(x') = p(x)$ and $g(y') = q(y)$. By composition we obtain a map $f : h_C \rightarrow F$ such that $f(x') = a$ and $f(y') = b$. The proof is then complete. \square

PROPOSITION 3.10. *Let $E \in \text{SPr}(\text{Aff}_{\mathbb{C}})$ be a simplicial presheaf. Then there exists a canonical isomorphism of sets,*

$$\pi_0|E| \simeq \pi_0(E(\mathbb{C}))/\sim,$$

where \sim is the equivalence relation defined by $[x] \sim [y]$ if there exist a connected algebraic curve C , a map $f : h_C \rightarrow F$ in $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}}))$, and two complex points $x', y' \in C(\mathbb{C})$ such that $f(x') = x$ and $f(y') = y$.

Proof. We denote by $\pi_0^{\text{pr}} E$ the presheaf obtained by applying π_0 levelwise to the presheaf E . We apply Lemma 3.9 to the presheaf of sets $F = \pi_0^{\text{pr}} E$. We obtain an isomorphism $\pi_0 \text{ssp}(\pi_0^{\text{pr}} E) \simeq \pi_0^{\text{pr}} E(\mathbb{C})/\sim \simeq (\pi_0 E(\mathbb{C}))/\sim$. By Proposition 3.2, there is an isomorphism $\pi_0 \text{ssp}(\pi_0^{\text{pr}} E) \simeq \pi_0 \text{ssp}(E)$. However, by Remark 3.5, there is an isomorphism $\pi_0(\text{ssp}(E)) \simeq \pi_0|E|$, and the proof is complete. \square

3.4 Topological realization of structured presheaves

Strict groups. Let $\text{SGp}_{\mathbb{C}}$ (respectively, SGp) be the category of strict group objects in the cartesian monoidal category $\text{SPr}(\text{Aff}_{\mathbb{C}})$ (respectively, in the cartesian monoidal category SSet). We denote by

$$B : \text{SGp} \rightarrow \text{SSet}$$

the classifying space functor. For all $G \in \text{SGp}$, the space BG is defined as the homotopy colimit

$$BG := \text{hocolim } G^\bullet$$

where G^\bullet is the bisimplicial set defined by $[n] \mapsto G^n$, with faces and degeneracies induced by products and neutral elements in G . There exists a functorial cofibrant replacement of the object G^\bullet in the projective model category $\text{SSet}^{\Delta^{\text{op}}}$, which justifies our considering B as a functor between nonlocalized categories. For all $G \in \text{SGp}_{\mathbb{C}}$, we set $BG(\text{Spec}(A)) = B(G(\text{Spec}(A)))$.

Because $|-|$ commutes with products, the topological realization of a presheaf of simplicial groups is a simplicial group. We consider the diagram

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{SGp}_{\mathbb{C}}) & \xrightarrow{\mathbb{B}} & \mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})) \\ \downarrow |-| & & \downarrow |-| \\ \mathrm{Ho}(\mathrm{SGp}) & \xrightarrow{\mathbb{B}} & \mathrm{Ho}(\mathrm{SSet}) \end{array}$$

PROPOSITION 3.11. *For all $G \in \mathrm{SGp}_{\mathbb{C}}$, there exists a canonical isomorphism $\mathbb{B}|G| \simeq |\mathbb{B}G|$ in $\mathrm{Ho}(\mathrm{SSet})$.*

Proof. This follows directly from the fact that $|-|$ commutes with finite homotopy products and homotopy colimits. \square

Symmetric spectra. We extend the topological realization functor $\mathrm{ssp} : \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}}) \rightarrow \mathrm{SSet}$ to presheaves of symmetric spectra by taking the Sp -enriched left Kan extension of the composite functor

$$\mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}}) \xrightarrow{\mathrm{ssp}} \mathrm{SSet} \xrightarrow{\Sigma^{\infty}(-)_{+}} \mathrm{Sp}$$

along the infinite suspension functor $\Sigma^{\infty}(-)_{+} : \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}}) \rightarrow \mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})$. We denote this extension by $\mathrm{ssp}_{\mathbb{S}}$. We obtain a commutative diagram

$$\begin{array}{ccc} \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}}) & \xrightarrow{\mathrm{ssp}} & \mathrm{SSet} \xrightarrow{\Sigma^{\infty}(-)_{+}} \mathrm{Sp} \\ \downarrow \Sigma^{\infty}(-)_{+} & & \nearrow \mathrm{ssp}_{\mathbb{S}} \\ \mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}}) & & \end{array}$$

The right adjoint to $\mathrm{ssp}_{\mathbb{S}}$ sends a symmetric spectrum E to the presheaf $X \mapsto \underline{\mathrm{Hom}}_{\mathrm{Sp}}(\Sigma^{\infty} X_{+}, E)$. We have the same properties as with the simplicial realization.

PROPOSITION 3.12.

- For all $F \in \mathrm{Sp}$ and all $E \in \mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})$ we have a canonical isomorphism of spectra $F \wedge \mathrm{ssp}_{\mathbb{S}}(E) \simeq \mathrm{ssp}_{\mathbb{S}}(F \wedge E)$.
- For all $E, E' \in \mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})$, we have a canonical isomorphism of spectra $\mathrm{ssp}_{\mathbb{S}}(E \wedge E') \simeq \mathrm{ssp}_{\mathbb{S}}(E) \wedge \mathrm{ssp}_{\mathbb{S}}(E')$, i.e. the functor $\mathrm{ssp}_{\mathbb{S}}$ is monoidal.

We denote by $\mathrm{Sp}^{\acute{e}t, \mathbf{A}^1}$ the \mathbf{A}^1 -étale model structure on the category $\mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})$. Proposition 3.4 implies that the functor

$$\mathrm{ssph}_{\mathbb{S}} : \mathrm{Sp}^{\acute{e}t, \mathbf{A}^1} \rightarrow \mathrm{Sp}$$

is left Quillen for the \mathbf{A}^1 -étale model structure. Remark 3.5 applies also for presheaves of spectra, i.e. for all $E \in \mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})$ we have a canonical isomorphism $\mathbb{L} \mathrm{ssp}_{\mathbb{S}}(E) \simeq \mathbb{L} \mathrm{ssph}_{\mathbb{S}}(E)$.

DEFINITION 3.13. We call *spectral topological realization* and we denote by $|-|_{\mathbb{S}}$ the two derived functors $\mathbb{L} \mathrm{ssp}_{\mathbb{S}}$ and $\mathbb{L} \mathrm{ssph}_{\mathbb{S}}$. We have a commutative triangle

$$\begin{array}{ccc} \mathrm{Ho}(\mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}})) & \xrightarrow{\mathbb{R} \mathrm{id}} & \mathrm{Ho}(\mathrm{Sp}^{\acute{e}t, \mathbf{A}^1}) \\ & \searrow |-|_{\mathbb{S}} & \swarrow |-|_{\mathbb{S}} \\ & \mathrm{Ho}(\mathrm{Sp}) & \end{array}$$

We denote by $H_{\mathbb{S},B}$ the right adjoint to $|-|_{\mathbb{S}}$. For all $E \in \text{Ho}(\text{Sp})$, $H_{\mathbb{S},B}(E)$ is the presheaf

$$H_{\mathbb{S},B}(E) : X \longmapsto \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(|X|_{\mathbb{S}}, E).$$

The notation $H_{\mathbb{S},B}$ stands for the Betti cohomology. Indeed, if $E = H\mathbb{C}$ is the Eilenberg–Mac Lane spectrum of \mathbb{C} , then the presheaf $H_{\mathbb{S},B}(H\mathbb{C})$ is nothing more than the Betti cohomology. Since $\text{ssp}_{\mathbb{S}}$ is a monoidal functor, the spectral topological realization is also a monoidal functor.

Δ - and Γ -spaces. Finally, we extend the topological realization to Δ -objects in $\text{SPr}(\text{Aff}_{\mathbb{C}})$. Of course, the same can be done for Γ -objects. We use the notation of § 2.1 for model structures on Δ -objects. Let $E \in \Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})$ be a Δ -presheaf. The topological realization of E , denoted by $\text{ssp}_{\Delta}(E)$, is the Δ -space defined by the formula $\text{ssp}_{\Delta}(E)_n = \text{ssp}(E_n)$. This defines a functor

$$\text{ssp}_{\Delta} : \Delta - \text{SPr}(\text{Aff}_{\mathbb{C}}) \longrightarrow \Delta - \text{SSet}.$$

This functor admits as right adjoint the functor which associates to a Δ -space F the Δ -presheaf $R(F)_n = R(F_n)$ (where R is the right adjoint to ssp). The associated functor

$$\text{ssph}_{\Delta} : \Delta - \text{SPr}^{\text{ét}, \mathbf{A}^1} \longrightarrow \Delta - \text{SSet}$$

is left Quillen for the \mathbf{A}^1 -étale model structure.

DEFINITION 3.14. We call the derived functors $\mathbb{L}\text{ssp}_{\Delta}$ and $\mathbb{L}\text{ssph}_{\Delta}$ a realization of Δ -presheaves and denote it by $|-|_{\Delta}$. We have a commutative triangle

$$\begin{array}{ccc} \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})) & \xrightarrow{\mathbb{R}\text{id}} & \text{Ho}(\Delta - \text{SPr}^{\text{ét}, \mathbf{A}^1}) \\ & \searrow |-\!|_{\Delta} & \swarrow |-\!|_{\Delta} \\ & \text{Ho}(\Delta - \text{SSet}) & \end{array}$$

We denote its Γ -version by $|-|_{\Gamma}$.

The realization $|-|_{\Delta}$ commutes with finite homotopy products, and therefore the realization of a special (respectively, very special) Δ -object, is a special (respectively, very special) Δ -object. We have the diagram

$$\begin{array}{ccc} \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})) & \xrightarrow{|-\!|_{\Delta}} & \text{Ho}(\Delta - \text{SSet}) \\ \downarrow \text{mon} & & \downarrow \text{mon} \\ \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{sp}}) & \xrightarrow{|-\!|_{\Delta}} & \text{Ho}(\Delta - \text{SSet}^{\text{sp}}) \\ \downarrow (-)^+ & & \downarrow (-)^+ \\ \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{tsp}}) & \xrightarrow{|-\!|_{\Delta}} & \text{Ho}(\Delta - \text{SSet}^{\text{tsp}}) \end{array}$$

PROPOSITION 3.15. For all $E \in \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}}))$ we have a canonical isomorphism in $\text{Ho}(\Delta - \text{SSet})$,

$$|\text{mon}(E)|_{\Delta} \simeq \text{mon}|E|_{\Delta}.$$

Proof. It suffices to show that their right adjoints commute. But this follows from the fact that if $F \in \Delta - \text{SSet}$ is special, then the Δ -object $\mathbb{R}R(F) = \mathbb{R}\text{Map}(|-\!|, F)$ is special. \square

PROPOSITION 3.16. For all special Δ -presheaf $E \in \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}}))$ we have a canonical isomorphism in $\text{Ho}(\Delta - \text{SSet})$,

$$|E^+|_{\Delta} \simeq |E|_{\Delta}^+.$$

Proof. It suffices to show that their right adjoints commute. But it follows from the fact that if $F \in \Delta - \text{SSet}$ is very special, then the Δ -object $\mathbb{R}R(F) = \mathbb{R}\text{Map}(|-|, F)$ is very special. \square

Now we compare the topological realization of Γ -presheaves and of presheaves of connective spectra. We have the diagram

$$\begin{array}{ccc} \text{Ho}(\Gamma - \text{SPr}(\text{Aff}_{\mathbb{C}})) & \xrightarrow{|-|_{\Gamma}} & \text{Ho}(\Gamma - \text{SSet}) \\ \downarrow \mathcal{B} & & \downarrow \mathcal{B} \\ \text{Ho}(\text{Sp}(\text{Aff}_{\mathbb{C}})) & \xrightarrow{|-|_{\mathbb{S}}} & \text{Ho}(\text{Sp}) \end{array}$$

PROPOSITION 3.17. For all $E \in \text{Ho}(\Gamma - \text{SPr}(\text{Aff}_{\mathbb{C}}))$, we have a canonical isomorphism in $\text{Ho}(\text{Sp})$,

$$\mathcal{B}|E|_{\Gamma} \simeq |\mathcal{B}E|_{\mathbb{S}}.$$

Proof. This follows from the fact that \mathcal{B} is a homotopy colimit, and because $|-|_{\mathbb{S}}$ commutes with homotopy colimits. \square

We recall from §2.1 that there exists a functor

$$\alpha^* : \text{Ho}(\Gamma - \text{SPr}(\text{Aff}_{\mathbb{C}})) \longrightarrow \text{Ho}(\Delta - \text{SPr}(\text{Aff}_{\mathbb{C}})).$$

We notice that by definition of α^* , for all $E \in \Gamma - \text{SPr}(\text{Aff}_{\mathbb{C}})$, we have a canonical isomorphism

$$|\alpha^*E|_{\Delta} \simeq \alpha^*|E|_{\Gamma}.$$

3.5 Restriction to smooth schemes

We show here that the topological realization of a simplicial presheaf is unchanged if we restrict this presheaf to smooth schemes. This is a consequence of a result by Suslin and Voevodsky concerning the *cdh*-topology, and of one of our main results: the homotopical version of Deligne’s cohomological proper descent. This latter result is the analog for the proper topology of the Dugger–Isaksen theorem (see Theorem 3.4). To prove this descent result, we mimic Dugger and Isaksen’s proof and use Lurie’s proper descent theorem.

The property of ‘restriction to smooth schemes’ will be used below to show that the topological K-theory of a smooth complex algebraic variety coincides with the topological K-theory (in the dg-sense) of its dg-category of perfect complexes. It will also be used to show that the negative semi-topological K-theory of a smooth commutative algebra is zero.

We denote by $\text{Aff}_{\mathbb{C}}^{\text{liiss}}$ the category of smooth affine \mathbb{C} -schemes of finite type. We denote the inclusion by $l : \text{Aff}_{\mathbb{C}}^{\text{liiss}} \hookrightarrow \text{Aff}_{\mathbb{C}}$. The restriction on simplicial presheaves is denoted by $l^* : \text{SPr}(\text{Aff}_{\mathbb{C}}) \longrightarrow \text{SPr}(\text{Aff}_{\mathbb{C}}^{\text{liiss}})$.

THEOREM 3.18. Let $F \in \text{SPr}(\text{Aff}_{\mathbb{C}})$. Then there exists a canonical isomorphism $|l^*F| \simeq |F|$ in $\text{Ho}(\text{SSet})$.

The proof occupies the next few pages. We will use the proper topology on the category $\text{Sch}_{\mathbb{C}}$ of separated \mathbb{C} -schemes of finite type. The idea consists in using a Quillen equivalence between the proper local model category $\text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}}$ of simplicial presheaves defined over schemes,

and the proper local model category $\mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}})^{\mathrm{pro}}$ of simplicial presheaves defined over smooth schemes. Having proved this step, it remains to show that the topological realization behaves well with respect to the proper topology, which is our descent result or generalization of Deligne’s proper descent.

We denote by $\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}}$ the category of separated smooth \mathbb{C} -schemes of finite type and by

$$\begin{array}{ccc} \mathrm{Aff}_{\mathbb{C}}^{\mathrm{liss}} & \xhookrightarrow{l} & \mathrm{Aff}_{\mathbb{C}} \\ \downarrow k & & \downarrow j \\ \mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}} & \xhookrightarrow{i} & \mathrm{Sch}_{\mathbb{C}} \end{array}$$

the inclusions. We have restriction/extension functors between the étale local model categories:

$$\begin{array}{ccc} \mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{ét}} & \xrightleftharpoons[i^*]{i_!} & \mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}})^{\mathrm{ét}} \\ \uparrow j_! \downarrow j^* & & \uparrow k_! \downarrow k^* \\ \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})^{\mathrm{ét}} & \xrightleftharpoons[l^*]{l_!} & \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}}^{\mathrm{liss}})^{\mathrm{ét}} \end{array}$$

All these adjoint pairs are Quillen pairs by the general results on change of sites [DHI04, Proposition 7.2]. The restriction functor j^* preserves étale local equivalences and it is a Quillen equivalence. Indeed, this can be deduced from the analog statement for sheaves using the projective local model structure on simplicial sheaves of [Bla01, Theorem 2.1]. Theorem 3.18 can be equivalently stated in the category $\mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})$. Indeed, for all $F \in \mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})$, we have an equivalence $|\mathbb{L}j_!F| \simeq |F|$ by definition of the realization and because $\mathbb{L}j_!$ is a left adjoint. However, for all $E \in \mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})$, the counit map $\mathbb{L}j_!j^*E \rightarrow E$ is an étale local equivalence. The induced map $|\mathbb{L}j_!j^*E| \rightarrow |E|$ is therefore an equivalence by Theorem 3.4. We also have by definition of the realization an equivalence $|\mathbb{L}j_!j^*E| \simeq |j^*E|$, and thus an equivalence $|j^*E| \simeq |E|$. Consequently, we will work in the category $\mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})$.

We endow $\mathrm{Sch}_{\mathbb{C}}$ with the proper topology, i.e. a family of maps is declared to be a covering if each of the maps is proper and if this family is cojointly surjective. We define the proper topology on the category $\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}}$ by saying that a sieve $S \subseteq h_X$ of an object of $\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}}$ is a covering sieve if it contains a sieve generated by a proper covering of X in $\mathrm{Sch}_{\mathbb{C}}$. We recall that the category $\mathrm{Sch}_{\mathbb{C}}^{\mathrm{liss}}$ does not have all pullbacks in general.

Remark 3.19. By Hironaka’s theorem on resolution of singularities in characteristic zero [Hir64], every proper covering sieve S of a scheme X admits a refinement by a proper covering sieve $T \subseteq S$ generated by maps with source being smooth. Indeed, for all maps $Z \rightarrow Y$ in S , by Hironaka’s theorem, there exist a smooth scheme Z' and a proper surjective map $Z' \rightarrow Z$.

Notation 3.20. We denote by $\mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}}$ the proper local model category, i.e. the left Bousfield localization of the projective model category $\mathrm{SPr}(\mathrm{Sch}_{\mathbb{C}})$ with respect to maps

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} h_{Y_{\bullet}} \rightarrow h_X$$

for all proper hypercoverings $Y_{\bullet} \rightarrow X$ in $\mathrm{Sch}_{\mathbb{C}}$. The following is our proper descent result.

PROPOSITION 3.21. *For every proper hypercovering $Y_{\bullet} \rightarrow X$ of a scheme $X \in \mathrm{Sch}_{\mathbb{C}}$, the induced map*

$$\mathrm{hocolim}_{\Delta^{\mathrm{op}}} |Y_{\bullet}| \rightarrow |X|$$

is an isomorphism in $\text{Ho}(\text{SSet})$. We deduce that the functor

$$\text{ssp} : \text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}} \longrightarrow \text{SSet}$$

is left Quillen.

The proof of Proposition 3.21 is relatively nontrivial and is based on the purely topological fact that a proper hypercovering induces an equivalence if we take its colimit. It will be given below. We use Proposition 3.21 to prove Theorem 3.18.

The functor $i : \text{Sch}_{\mathbb{C}}^{\text{liss}} \hookrightarrow \text{Sch}_{\mathbb{C}}$ gives rise to a continuous functor on the level of sites (with the proper topology) and to a set of adjunctions between sheaves of sets in the proper topology,

$$\text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}) \begin{array}{c} \xleftarrow{i_!} \\ \xrightarrow{i^*} \\ \xleftarrow{e} \end{array} \text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}^{\text{liss}}).$$

The functor i^* is the restriction to smooth schemes. For every sheaf $F \in \text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}^{\text{liss}})$, the sheaf eF is *a priori* the sheaf associated to the presheaf $X \mapsto \text{Hom}(i_*h_X, F)$. By [DHI04, Proposition 7.2], the adjoint pair

$$\text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}} \begin{array}{c} \xleftarrow{i_!} \\ \xrightarrow{i^*} \end{array} \text{SPr}(\text{Sch}_{\mathbb{C}}^{\text{liss}})^{\text{pro}}$$

is a Quillen pair. The following proposition is a variant of [SV96, Theorem 6.2].

PROPOSITION 3.22. *The Quillen pair $(i_!, i^*)$ is a Quillen equivalence and induces an equivalence of categories,*

$$\text{Ho}(\text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}}) \begin{array}{c} \xleftarrow{\mathbb{L}i_!} \\ \xrightarrow{i^*} \end{array} \text{Ho}(\text{SPr}(\text{Sch}_{\mathbb{C}}^{\text{liss}})^{\text{pro}}).$$

Proof. It suffices to prove that the categories of sheaves of sets are actually equivalent. Indeed, by [Bla01, Theorem 2.1] there exists a projective local model structure on the category of simplicial sheaves $\text{SSh}(\text{Sch}_{\mathbb{C}})^{\text{pro}}$ and by [Bla01, Theorem 2.2] the Quillen pair formed by the inclusion of simplicial sheaves into simplicial presheaves and by the sheafification is a Quillen equivalence,

$$\text{SSh}(\text{Sch}_{\mathbb{C}})^{\text{pro}} \begin{array}{c} \xleftarrow{a} \\ \xrightarrow{\quad} \end{array} \text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}}.$$

Moreover, the category $\text{SSh}(\text{Sch}_{\mathbb{C}})^{\text{pro}}$ is the category of simplicial objects in the topos $\text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}})$, and its model structure depends only on the underlying topos $\text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}})$. The same fact holds for the model category $\text{SSh}(\text{Sch}_{\mathbb{C}}^{\text{liss}})^{\text{pro}}$.

Hence it suffices to prove that the adjoint pair

$$\text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}) \begin{array}{c} \xrightarrow{i^*} \\ \xleftarrow{e} \end{array} \text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}^{\text{liss}})$$

is an equivalence. But this is a direct consequence of [SGA4, Exposé III, Theorem 4.1]. □

LEMMA 3.23. *The restriction functor $i^* : \text{SPr}(\text{Sch}_{\mathbb{C}})^{\text{pro}} \longrightarrow \text{SPr}(\text{Sch}_{\mathbb{C}}^{\text{liss}})^{\text{pro}}$ preserves all local equivalences.*

Proof. Because of Blander’s result [Bla01, Theorem 2.2], it suffices to prove this for simplicial sheaves. Then it is a direct consequence of the fact proven above that i^* induces an equivalence of categories $i^* : \text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}) \xrightarrow{\sim} \text{Sh}_{\text{pro}}(\text{Sch}_{\mathbb{C}}^{\text{liss}})$. Indeed, the local equivalences are exactly the maps which induce isomorphisms on all homotopy sheaves for all basepoints. Moreover, the construction of homotopy sheaves in a category of simplicial objects in a topos is made in purely categorical terms, and thus they only depend on the underlying topos. \square

Proof of 3.18. Since the triangle

$$\begin{array}{ccc} \text{Sch}_{\mathbb{C}}^{\text{liss}} & \xrightarrow{i} & \text{Sch}_{\mathbb{C}} \\ & \searrow \text{ssp} & \swarrow \text{ssp} \\ & \text{SSet} & \end{array}$$

is commutative, the triangle of left Kan extension

$$\begin{array}{ccc} \text{Ho}(\text{SPr}(\text{Sch}_{\mathbb{C}}^{\text{liss}})) & \xrightarrow{\mathbb{L}i_!} & \text{Ho}(\text{SPr}(\text{Sch}_{\mathbb{C}})) \\ & \searrow |-\!| & \swarrow |-\!| \\ & \text{Ho}(\text{SSet}) & \end{array}$$

is commutative. This means that for all $G \in \text{SPr}(\text{Sch}_{\mathbb{C}}^{\text{liss}})$, there exists an isomorphism $|\mathbb{L}i_!G| \simeq |G|$ in $\text{Ho}(\text{SSet})$. Therefore if $G = \mathbb{R}i^*F$ for $F \in \text{SPr}(\text{Sch}_{\mathbb{C}})$, we have canonical isomorphisms

$$\begin{aligned} |i^*F| &\simeq |\mathbb{R}i^*F| \quad (\text{by Lemma 3.23}) \\ &\simeq |\mathbb{L}i_!\mathbb{R}i^*F| \quad (\text{from the above}) \\ &\simeq |F| \quad (\text{by Propositions 3.22 and 3.21}). \end{aligned} \quad \square$$

Proof of Proposition 3.21. The first assertion implies that ssp is left Quillen for the proper local model structure because of general facts about left Bousfield localizations.

Let $A = |X|^{\text{top}}$ and $B_{\bullet} = |Y_{\bullet}|^{\text{top}}$. The map of simplicial spaces $B_{\bullet} \rightarrow A$ is a proper hypercovering of topological spaces, i.e. a hypercovering relative to the proper topology on the category Top , for which covering families are cojointly surjective families of proper continuous maps. In order to prove Proposition 3.21 it suffices to prove that for every proper hypercovering $B_{\bullet} \rightarrow A$ between sufficiently nice topological spaces, the induced map $\text{hocolim}_{\Delta^{\text{op}}} B_{\bullet} \rightarrow A$ is a weak equivalence of spaces. By ‘sufficiently nice’ we mean locally compact Hausdorff spaces with the homotopy type of CW-complexes and such that all the components of the coskeletons $\text{cosk}_k^A B_{\bullet}$ have the homotopy type of CW complexes. These assumptions are satisfied by topological spaces which are complex points of separated schemes of finite type (see [Hir75]). The proof is then completed by Proposition 3.24. \square

PROPOSITION 3.24. *Let $B_{\bullet} \rightarrow A$ be a proper hypercovering of topological spaces such that the following properties hold.*

- The spaces A and $(B_n)_{n \geq 0}$ are Hausdorff, locally compact, and have the homotopy type of CW-complexes.
- All the components of the relative coskeletons $\text{cosk}_k^A B_{\bullet}$ have the homotopy type of CW-complexes for all $k \geq 0$.

Then the map $\text{hocolim}_{\Delta^{\text{op}}} B_{\bullet} \rightarrow A$ is an equivalence in Top .

Proof. The proof will consist of several steps which mimic the proof by Dugger and Isaksen of the open version of our statement in [DI01, Theorem 4.3] (i.e. if we replace the proper topology by the usual open covering topology). First, we reduce the statement to the case of bounded hypercoverings (in the sense of [DHI04, Definition 4.10]) using the same argument as in the proof of [DI01, Theorem 4.3]. Second, we reduce to the case of a simpler class of bounded proper hypercoverings which are nerves of proper surjective morphisms using the same argument as in the proof of [DI01, Lemma 4.2]. Third, we give a proof for this class of hypercoverings using Lurie’s proper base change theorem [Lur09, Corollary 7.3.1.18] to reduce it to the point. Then a lemma from simplicial homotopy theory provides the result for the point. The assumptions made on topological spaces are used in Lurie’s theorem and also in Toën’s theorem in order to calculate the derived global section of a constant simplicial presheaf.

We introduce some terminology. Let \mathcal{C} be any complete and cocomplete category. For any $[n] \in \Delta$ and any simplicial object C_\bullet in \mathcal{C} we denote by $sk_n C$ its n -skeleton and $\text{cosk}_n C_\bullet$ its n -coskeleton. If $\mathcal{C} = \text{Top}$, one has $(\text{cosk}_n C_\bullet)_i \simeq \text{Map}(sk_n \Delta^i, C_\bullet)$, where Map is a mapping space for $\text{Top}^{\Delta^{\text{op}}}$. There is an augmented version of these. If $C_\bullet \rightarrow D$ is an augmented simplicial object to a constant simplicial object D , then we denote by $sk_n^D C_\bullet$ and $\text{cosk}_n^D C_\bullet$ the skeleton and coskeleton functor for the category $(\mathcal{C} \downarrow D)^{\Delta^{\text{op}}}$. We denote by

$$M_n C = \lim_{(\Delta^{\text{op}} \downarrow n) \setminus \text{id}} C_\bullet$$

the n th matching object of C_\bullet , where $(\Delta^{\text{op}} \downarrow n) \setminus \text{id}$ is the category of maps to $[n]$ in Δ^{op} minus the identity map of $[n]$. There is an augmented version of the matching object. If $C_\bullet \rightarrow D$ is an augmented simplicial object of \mathcal{C} into a constant simplicial object D , then one can compute the limit seeing C as a functor from $(\Delta^{\text{op}} \downarrow n) \setminus \text{id}$ to the category $\mathcal{C} \downarrow D$ of maps to D in \mathcal{C} . We denote it by $M_n^D C_\bullet$. There are natural maps $C_n \rightarrow M_n C_\bullet$ and $C_n \rightarrow M_n^D C_\bullet$. Suppose \mathcal{C} is endowed with a Grothendieck topology so that we can talk about hypercoverings; for us it will be Top with the proper topology. A hypercovering $C_\bullet \rightarrow D$ is called *bounded* if there exists an integer $N \geq 0$ such that the maps $C_n \rightarrow M_n^D C_\bullet$ are isomorphisms for all $n > N$. The minimum N with this property is called the *dimension* of the hypercovering. A hypercovering is bounded of dimension $\leq N$ if and only if the unit map $C_\bullet \rightarrow \text{cosk}_N^D C_\bullet$ is an isomorphism.

If $f : C \rightarrow D$ is a map in \mathcal{C} , one can see it as a map of constant simplicial objects of \mathcal{C} . Then we can take the 0-coskeleton $\text{cosk}_0^D C \rightarrow D$. This augmented simplicial object is called the nerve of f . We have $(\text{cosk}_0^D C)_i = C \times_D \cdots \times_D C$, $n + 1$ times. The faces and degeneracies are the projections and diagonals, respectively. If $f : C \rightarrow D$ is a covering in \mathcal{C} then the nerve of f is a hypercovering of D of dimension zero, and these are the only hypercoverings of dimension zero.

To reduce to the case of bounded hypercoverings, we observe that for any $k \geq 0$, the hypercovering $\text{cosk}_{k+1}^A B_\bullet$ is bounded and that the unit map $B_\bullet \rightarrow \text{cosk}_{k+1}^A B_\bullet$ is an isomorphism on $(k + 1)$ -skeletons. By Lemma 3.25 below, this implies that the top map in the diagram

$$\begin{array}{ccc} \text{hocolim}_{\Delta^{\text{op}}} B_\bullet & \longrightarrow & \text{hocolim}_{\Delta^{\text{op}}} \text{cosk}_{k+1}^A B_\bullet \\ & \searrow & \downarrow \\ & & A \end{array}$$

induces an isomorphism on the π_k at any basepoint. Suppose the statement is proven for bounded hypercoverings. Then the right vertical induces an isomorphism on π_k because $\text{cosk}_{k+1}^A B_\bullet$ is bounded. Hence the last map $\text{hocolim}_{\Delta^{\text{op}}} B_\bullet \rightarrow A$ induces an isomorphism on π_k at any basepoint, hence is a weak equivalence because k is arbitrary.

LEMMA 3.25. *Let $C_\bullet \rightarrow D_\bullet$ be a map of simplicial spaces which induces an isomorphism on $(k + 1)$ -skeletons. Then the map*

$$\pi_i \operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet \rightarrow \pi_i \operatorname{hocolim}_{\Delta^{\text{op}}} D_\bullet$$

is an isomorphism for every $0 \leq i \leq k$ and any basepoint.

Proof. To prove this, we reduce to the case of bisimplicial sets using the singular functor, because for simplicial simplicial sets the homotopy colimit is weakly equivalent to the diagonal. We denote by

$$\text{SSet} \begin{array}{c} \xrightarrow{\text{Re}} \\ \xleftarrow{S} \end{array} \text{Top}$$

the standard adjunction with right adjoint the singular functor S . It induces an adjunction on the level of simplicial objects just by taking these functors levelwise. The counit map $\text{Re} \circ SC_\bullet \rightarrow C_\bullet$ is a levelwise weak equivalence in $\text{Top}^{\Delta^{\text{op}}}$, hence the induced map

$$\operatorname{hocolim}_{\Delta^{\text{op}}} \text{Re} \circ SC_\bullet \rightarrow \operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet$$

is a weak equivalence. But composing with the canonical weak equivalence $\operatorname{hocolim}_{\Delta^{\text{op}}} \text{Re} \circ SC_\bullet \simeq \text{Re}(\operatorname{hocolim}_{\Delta^{\text{op}}} \text{Sing } C_\bullet)$, we get a weak equivalence $\text{Re}(\operatorname{hocolim}_{\Delta^{\text{op}}} SC_\bullet) \simeq \operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet$. Then for every $i \geq 0$ we get a canonical isomorphism

$$\pi_i(\operatorname{hocolim}_{\Delta^{\text{op}}} SC_\bullet) \simeq \pi_i \text{Re}(\operatorname{hocolim}_{\Delta^{\text{op}}} SC_\bullet) \simeq \pi_i \operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet$$

at every basepoint. Therefore it suffices to prove the claim for $C_\bullet \rightarrow D_\bullet$ a map in $\text{SSet}^{\Delta^{\text{op}}}$. But in that case there is canonical weak equivalence $\operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet \simeq dC_\bullet$ in SSet where $d : \text{SSet}^{\Delta^{\text{op}}} \rightarrow \text{SSet}$ is the diagonal functor. Then if $C_\bullet \rightarrow D_\bullet$ is an isomorphism on $(k + 1)$ -skeletons, it is straightforward that $\pi_i dC_\bullet \rightarrow \pi_i dD_\bullet$ is an isomorphism for every $0 \leq i \leq k$. This finishes the proof of 3.25. \square

Returning to the proof of Theorem 3.24, we proceed by induction on the dimension of the hypercovering, reducing the proof to the dimension-zero case.

Let $n \geq 0$ be an integer. Suppose we have proven the statement for hypercoverings of dimension less than or equal to n , and let $B_\bullet \rightarrow A$ be a bounded proper hypercovering of dimension $n + 1$. Consider the unit map $B_\bullet \rightarrow \operatorname{cosk}_n^A B_\bullet =: C_\bullet$. Then C_\bullet is bounded of dimension less than or equal to n . Consider the bisimplicial space which is the nerve of the map $B_\bullet \rightarrow C_\bullet$,

$$E_{\bullet\bullet} := (B_\bullet \rightrightarrows B_\bullet \times_{C_\bullet} B_\bullet \rightleftarrows B_\bullet \times_{C_\bullet} B_\bullet \times_{C_\bullet} B_\bullet \cdots).$$

Considering C_\bullet as constant in one simplicial direction, we have a map $E_{\bullet\bullet} \rightarrow C_\bullet$. The k th row of $E_{\bullet\bullet} \rightarrow C_\bullet$ is the nerve of the map $B_k \rightarrow C_k$. Consider the diagonal $D_\bullet := dE_{\bullet\bullet}$. Then standard homotopy theory (e.g. [Hir09]) proves that $\operatorname{hocolim}_{\Delta^{\text{op}}} D_\bullet$ is weakly equivalent to the space obtained by taking the homotopy colimit of each rows of $E_{\bullet\bullet}$, and then taking the homotopy colimit of the resulting simplicial space. But by the induction hypothesis, the k th row being a dimension-zero hypercovering of C_k , its homotopy colimit is weakly equivalent to C_k . The resulting simplicial object is C_\bullet , which is of dimension less than or equal to n , so by the induction hypothesis $\operatorname{hocolim}_{\Delta^{\text{op}}} C_\bullet \simeq A$. Hence we prove that $\operatorname{hocolim}_{\Delta^{\text{op}}} D_\bullet \simeq A$.

Now we prove that B_\bullet is a retract of D_\bullet over A , hence that $\operatorname{hocolim}_{\Delta^{\text{op}}} D_\bullet \simeq \operatorname{hocolim}_{\Delta^{\text{op}}} B_\bullet \simeq A$. There is a natural map $B_\bullet \rightarrow D_\bullet$ given by the horizontal degeneracy $E_{0,k} \rightarrow E_{k,k}$.

Then we need a map $D_\bullet \rightarrow B_\bullet$. It is sufficient to find a map $sk_{n+1}^A D_\bullet \rightarrow sk_{n+1}^A B_\bullet$ because then the adjoint map $D_\bullet \rightarrow \text{cosk}_{n+1}^A sk_{n+1}^A B_\bullet \simeq B_\bullet$ is the desired map. Notice that because of the definition of C_\bullet the map $B_k \rightarrow C_k$ is an isomorphism for $k = 0, \dots, n$ and the map $sk_n^A B_\bullet \rightarrow sk_n^A D_\bullet$ is an isomorphism. Let $[0] \rightarrow [n+1]$ be any coface map, giving a face map $E_{n+1, n+1} \rightarrow E_{0, n+1}$ which gives the desired map $sk_{n+1}^A D_\bullet \rightarrow sk_{n+1}^A B_\bullet$. One can check that $B_\bullet \rightarrow D_\bullet \rightarrow B_\bullet$ is the identity which proves our claim.

It remains to prove the statement for a dimension-zero proper hypercovering $\pi : B_\bullet \rightarrow A$ with spaces satisfying the assumptions of Proposition 3.24. Such a hypercovering is the nerve of a proper surjective map $B_0 \rightarrow A$. Therefore $B_n \simeq B_0 \times_A \cdots \times_A B_0$, $n + 1$ times. We will use a proper base change argument. For this we will study simplicial presheaves on the simplicial space B_\bullet and their behavior with respect to π . Deligne defined in [Del74] a notion of sheaves on a simplicial space, constructing a site out of a simplicial space and taking sheaves on it. His construction can be directly used for simplicial presheaves. Indeed, let \tilde{B}_\bullet be the category with objects the pairs $([n], V)$ with $[n] \in \Delta$ and $V \subseteq B_n$ an open subset. A morphism between $([n], V)$ and $([m], V')$ is the data of a morphism $a : [n] \rightarrow [m]$ in Δ and a continuous map $V' \rightarrow V$ such that the square

$$\begin{array}{ccc} V' & \longrightarrow & V \\ \downarrow & & \downarrow \\ B_m & \xrightarrow{B(a)} & B_n \end{array}$$

commutes. Composition and identities are defined in the obvious way, and satisfy all the required conditions. The category \tilde{B}_\bullet is naturally endowed with the open covering topology induced by the topology of each B_n , and Δ is considered as discrete. Then we can consider the category $\text{SPr}(B_\bullet)$ of simplicial presheaves on the site \tilde{B}_\bullet . An object F in this category is equivalent to the data of simplicial presheaves F_n on B_n for every $n \geq 0$, and for every map $a : [n] \rightarrow [m]$ in Δ , a map of presheaves $u_a : F_n \rightarrow B(a)_* F_m$, such that $u_{\text{id}_{[n]}} = \text{id}_{F_n}$ and for every $a : [n] \rightarrow [m]$ and $b : [m] \rightarrow [k]$ in Δ , we have $u_{ba} = B(a)_* u_b u_a$.

If X is any space considered as a constant simplicial space, then \tilde{X} is nothing more than the Grothendieck site of opens subsets of X . The map of simplicial spaces $\pi : B_\bullet \rightarrow A$ gives a map of sites still denoted by $\pi : \tilde{B}_\bullet \rightarrow \tilde{A}$. Consider the diagram of categories

$$\begin{array}{ccc} \tilde{B}_\bullet & \xrightarrow{\pi} & \tilde{A} \\ & \searrow q & \swarrow p \\ & * & \end{array}$$

where $*$ is the punctual category. Then, taking simplicial presheaves, we get a set of adjoint functors

$$\begin{array}{ccc} \text{SPr}(B_\bullet) & \begin{array}{c} \xleftarrow{\pi_*} \\ \xrightarrow{\pi^{-1}} \end{array} & \text{SPr}(A) \\ & \begin{array}{c} \swarrow q_* \\ \searrow p_* \end{array} & \\ & \text{cst} & \text{cst} \\ & \text{SSet} & \end{array}$$

where $\text{cst}(K)$ is the constant simplicial presheaf with value K for any $K \in \text{SSet}$. The functors p_* and q_* are also famous under the name of global sections and are the right adjoints to cst . We endow $\text{SPr}(B_\bullet)$ and $\text{SPr}(A)$ with the local model structure (with respect to the open covering topology) obtained as a Bousfield localization of the injective model structure (what is really important is the weak equivalences which are local equivalences, but we will need below to

consider a homotopy limit, which explains why we need the injective one). Then the functors π_* , p_* and q_* are right Quillen. For any $K \in \mathbb{S}\text{Set}$ we have $\pi^{-1} \circ \text{cst}(K) \simeq \text{cst}(K)$ and there are isomorphisms $\mathbb{L}\pi^{-1} \simeq \pi^{-1}$ and $\mathbb{L}\text{cst} \simeq \text{cst}$. Therefore we have a canonical isomorphism $\mathbb{R}p_*\mathbb{R}\pi_* \simeq \mathbb{R}q_*$.

A direct consequence of Toën’s result [Toë02, Theorem 2.13] is that for any constant simplicial presheaf $K \in \mathbb{S}\text{Pr}(A)$ we have a canonical isomorphism $\mathbb{R}p_*(K) \simeq \mathbb{R}\text{Map}(SA, K)$ in $\text{Ho}(\mathbb{S}\text{Set})$. This uses the fact that our space A has the homotopy type of a CW complex. Next we want to calculate the derived global sections $\mathbb{R}q_*(K)$ of a constant simplicial presheaf K on B_\bullet . The site \tilde{B}_\bullet is naturally endowed with a functor $\alpha : \tilde{B}_\bullet \rightarrow \Delta$ where Δ is considered as a discrete site. The functor α is just the projection $\alpha([n], V) = [n]$. We denote by $\beta : \Delta \rightarrow *$ the unique functor. We have induced functors on simplicial presheaves

$$\begin{array}{ccc} \mathbb{S}\text{Pr}(B_\bullet) & \xrightarrow{\alpha_*} & \mathbb{S}\text{Pr}(\Delta) \\ & \searrow q_* & \downarrow \beta_* \\ & & \mathbb{S}\text{Set} \end{array}$$

These functors are right Quillen ($\mathbb{S}\text{Pr}(\Delta)$ is also endowed with the injective model structure). The derived functor $\mathbb{R}\beta_*$ is then isomorphic to holim_Δ . For any constant simplicial presheaf $K \in \mathbb{S}\text{Pr}(B_\bullet)$, we have $\mathbb{R}\alpha_*(K) = (\mathbb{R}q_{n*}K)_{n \geq 0}$ where $q_n : \tilde{B}_n \rightarrow *$. Using [Toë02, Theorem 2.13] (all spaces B_n having the homotopy type of a CW), we obtain an isomorphism $\mathbb{R}\alpha_*(K) \simeq (\mathbb{R}\text{Map}(SB_n, K))_{n \geq 0}$ in $\text{Ho}(\mathbb{S}\text{Pr}(\Delta))$. Therefore we have a canonical isomorphism in $\text{Ho}(\mathbb{S}\text{Set})$,

$$\mathbb{R}q_*(K) \simeq \text{holim}_{\Delta^{\text{op}}} \mathbb{R}\text{Map}(SB_\bullet, K) \simeq \mathbb{R}\text{Map}(\text{hocolim}_{\Delta^{\text{op}}} SB_\bullet, K).$$

Suppose for the time being that the following lemma is at our disposal.

LEMMA 3.26. *Let $K \in \mathbb{S}\text{Pr}(B_\bullet)$ be a constant simplicial presheaf, with K being a truncated simplicial set. Then the unit map*

$$K \longrightarrow \mathbb{R}\pi_*\pi^{-1}(K) \simeq \mathbb{R}\pi_*(K)$$

is an isomorphism in $\text{Ho}(\mathbb{S}\text{Pr}(A))$ (where K denotes the same constant presheaf on A).

We will give a proof below. Applying the isomorphism $\mathbb{R}p_*\mathbb{R}\pi_* \simeq \mathbb{R}q_*$ to a truncated constant simplicial presheaf K on B_\bullet , we obtain a canonical isomorphism for every truncated simplicial set K

$$\mathbb{R}\text{Map}(SA, K) \simeq \mathbb{R}\text{Map}(\text{hocolim}_{\Delta^{\text{op}}} SB_\bullet, K).$$

This implies that the map $\text{hocolim}_{\Delta^{\text{op}}} SB_\bullet \rightarrow SA$ is an isomorphism in $\text{Ho}(\mathbb{S}\text{Set})$. By taking the Quillen equivalence Re and the fact that Re commutes with homotopy colimits, this implies that the map $\text{hocolim}_{\Delta^{\text{op}}} B_\bullet \rightarrow A$ is an isomorphism in $\text{Ho}(\text{Top})$, proving our claim.

To sum up, it only remains to prove Lemma 3.26. To do so, it suffices to prove that the unit map is an isomorphism on the stalk at any point $a \in A$. We have a cartesian square of simplicial spaces

$$\begin{array}{ccc} B_\bullet^a & \xrightarrow{\pi^a} & * \\ \phi \downarrow & & \downarrow a \\ B_\bullet & \xrightarrow{\pi} & A \end{array}$$

We claim that the base change theorem holds for this square and for a truncated constant simplicial presheaf.

LEMMA 3.27. For any truncated constant simplicial presheaf $K \in \text{SPr}(B_\bullet)$, the canonical map

$$a^{-1}\mathbb{R}\pi_*(K) \longrightarrow \mathbb{R}\pi_*^a\phi^{-1}(K) \simeq \mathbb{R}\pi_*^a(K)$$

is an isomorphism in $\text{Ho}(\text{SSet})$.

Once the lemma is proven, it will just remain to prove that the map $K \longrightarrow \mathbb{R}\pi_*^a(K)$ is an isomorphism in $\text{Ho}(\text{SSet})$, which is exactly the statement of Proposition 3.24 for $A = *$ and B_\bullet a dimension-zero hypercovering. Indeed, we have an isomorphism

$$\mathbb{R}\pi_*^a(K) \simeq \text{holim}_{\Delta^{\text{op}}} \mathbb{R} \text{Map}(SB_\bullet^a, K).$$

To prove Lemma 3.27, we will calculate the stalk $a^{-1}\mathbb{R}\pi_*(K)$ and relate it to $\mathbb{R}\pi_*^a(K)$. For any open subset $U \subseteq A$ we have a cartesian square of simplicial spaces

$$\begin{array}{ccc} B_\bullet^U & \xrightarrow{\pi^U} & U \\ \phi^U \downarrow & & \downarrow i \\ B_\bullet & \xrightarrow{\pi} & A \end{array}$$

Then we have isomorphisms $\mathbb{R}\Gamma(U, \mathbb{R}\pi_*^U(K)) \simeq \mathbb{R}\Gamma(B_\bullet^U, K) \simeq \text{holim}_{\Delta^{\text{op}}} \mathbb{R} \text{Map}(SB_\bullet^U, K)$ in $\text{Ho}(\text{SSet})$. The stalk $a^{-1}\mathbb{R}\pi_*(K)$ is isomorphic to the usual filtered colimit

$$\text{colim}_{a \in U \subseteq A} \mathbb{R}\Gamma(U, \mathbb{R}\pi_*^U(K)) \simeq \text{colim}_{a \in U \subseteq A} \text{holim}_{\Delta^{\text{op}}} \mathbb{R} \text{Map}(SB_\bullet^U, K).$$

Now we use the assumption that K is truncated to deduce the fact this homotopy limit is isomorphic to a finite homotopy limit. Indeed, if K is n -truncated, then the simplicial set $\mathbb{R} \text{Map}(SB_\bullet^U, K)$ is also n -truncated and one can calculate this homotopy limit by restricting to the subcategory of Δ^{op} given by simplexes of dimension less than or equal to $n + 1$. Then we can make this filtered colimit and this finite homotopy limit commute to get

$$a^{-1}\mathbb{R}\pi_*(K) \simeq \text{holim}_{\Delta^{\text{op}}} \text{colim}_{a \in U \subseteq A} \mathbb{R} \text{Map}(SB_\bullet^U, K).$$

Now we wish to have for all $n \geq 0$ an isomorphism $\text{colim}_{a \in U \subseteq A} \mathbb{R} \text{Map}(SB_n^U, K) \simeq \mathbb{R} \text{Map}(SB_n^a, K)$. To do so we apply Lurie’s proper base change Theorem [Lur09, Corollary 7.3.1.18] to the cartesian square of locally compact Hausdorff spaces

$$\begin{array}{ccc} B_n^a & \xrightarrow{\pi_n^a} & * \\ \phi_n \downarrow & & \downarrow a \\ B_n & \xrightarrow{\pi_n} & A \end{array}$$

We can apply this result here because our simplicial presheaf K is truncated so that, according to [Lur09, Corollary 7.2.1.12], K satisfies hyperdescent if and only if K satisfies ordinary descent. We obtain an isomorphism $a^{-1}\mathbb{R}\pi_{n*}(K) \simeq \mathbb{R}\pi_{n*}^a(K)$. With the same argument as before, we have $a^{-1}\mathbb{R}\pi_{n*}(K) \simeq \text{colim}_{a \in U \subseteq A} \mathbb{R} \text{Map}(SB_n^U, K)$, which proves the expected isomorphism. This finishes the proof of Lemma 3.27.

Returning to the proof of Lemma 3.26, it remains to prove that the map $K \longrightarrow \mathbb{R}\pi_*^a(K)$ is an isomorphism in $\text{Ho}(\text{SSet})$ for all truncated K . In view of what has been said, it is equivalent to the statement that $\text{hocolim}_{\Delta^{\text{op}}} B_\bullet^a \longrightarrow *$ is a weak equivalence of spaces. It is treated by the following lemma.

LEMMA 3.28. *Let X be any nonempty topological space (respectively, a nonempty simplicial set). Then the nerve $X_\bullet \rightarrow *$ of the map $p : X \rightarrow *$ induces a weak equivalence $\text{hocolim}_{\Delta^{\text{op}}} X_\bullet \rightarrow *$ in Top (respectively, in SSet).*

The statement in SSet implies the statement in Top. Indeed, if $X \in \text{Top}$ we saw in the proof of Lemma 3.25 that $\text{hocolim}_{\Delta^{\text{op}}} X_\bullet$ and $\text{hocolim}_{\Delta^{\text{op}}} SX_\bullet$ have the same homotopy groups.

Let $X \in \text{SSet}$. We prove that the map $X_\bullet \rightarrow *$ is a simplicial homotopy equivalence in $\text{SSet}^{\Delta^{\text{op}}}$. Let $x : * \rightarrow X$ be a point. Then it suffices to find a homotopy $h : \Delta^1 \times X_\bullet \rightarrow X_\bullet$ between id_{X_\bullet} and xp . We define $h_n : \Delta([n], [1]) \times X_n \rightarrow X_n$ by the following formula. Let $a : [n] \rightarrow [1]$ be a map in Δ ; it is essentially given by an integer $0 \leq m \leq n$. We set $h_n(a, (x_0, \dots, x_n)) = (x_0, \dots, x_m, x, \dots, x)$. We then have a homotopy which satisfies $h(0, -) = \text{id}_{X_\bullet}$ and $h(1, -) = xp$.

Recall the realization functor

$$|-| : \text{SSet}^{\Delta^{\text{op}}} \rightarrow \text{SSet}$$

defined by the standard formula

$$|Y_\bullet| := \text{coeq} \left(\bigsqcup_{n \in \Delta} \Delta^n \times Y_n \rightleftarrows \bigsqcup_{p \rightarrow q \in \Delta} \Delta^p \times Y_q \right).$$

This functor sends simplicial homotopy equivalences to simplicial homotopy equivalences. This implies that $|X_\bullet|$ is contractible. Now we use the isomorphism $\text{hocolim}_{\Delta^{\text{op}}} X_\bullet \simeq |X_\bullet|$ in $\text{Ho}(\text{SSet})$ (see [Hir09]) to conclude that $\text{hocolim}_{\Delta^{\text{op}}} X_\bullet$ is contractible.

The proof of Proposition 3.24 is complete. □

4. Topological K-theory of noncommutative spaces

We now have almost all we need to define the semi-topological and topological K-theory of noncommutative spaces. The first part of this section is dedicated to the definitions of semi-topological and topological K-theory. The semi-topological K-theory is, roughly speaking, the spectral topological realization of algebraic K-theory. These definitions are possible modulo the calculation of the semi-topological K-theory of the point, which, roughly speaking, is the spectral topological realization of the stack of E_∞ -spaces of vector bundles, which is proved to be the usual connective spectrum **bu** in the second part. In the third part we give a convenient description of semi-topological K-theory in terms of the stack of perfect dg-modules. In the fourth part we prove that the Chern map descends to topological K-theory. Finally, we treat the examples of smooth schemes and of finite-dimensional algebras, with some surprising consequences for the relation between the periodic homology groups of an algebra and the homotopy groups of the stabilized topological realization of the stack of noncommutative vector bundles.

4.1 Definition of the (semi-)topological K-theory

In § 2.3 we defined a presheaf of symmetric ring spectra

$$\mathbf{k} : \text{Aff}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Sp}$$

such that, for all $\text{Spec}(A) \in \text{Aff}_{\mathbb{C}}$, we have a canonical isomorphism $\mathbf{k}(\text{Spec}(A)) \simeq \mathbf{K}(A)$ in $\text{Ho}(\text{Sp})$. For all \mathbb{C} -dg-categories $T \in \text{dgCat}_{\mathbb{C}}$, we defined a presheaf of symmetric **k**-module spectra

$$\underline{\mathbf{K}}(T) : \text{Aff}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Sp}$$

such that for all $\text{Spec}(A) \in \text{Aff}_{\mathbb{C}}$, we have a canonical isomorphism $\underline{\mathbf{K}}(T)(\text{Spec}(A)) \simeq \mathbf{K}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A)$ in $\text{Ho}(\text{Sp})$. Thus we have an isomorphism $\underline{\mathbf{K}}(\mathbb{1}) \simeq \mathbf{k}$ where $\mathbb{1}$ is the \mathbb{C} -dg-category with one object and the ring \mathbb{C} as endomorphisms. We recall that we denote by $\mathbf{k} - \text{Mod}_{\mathbb{S}}$ the category of \mathbf{k} -modules in the monoidal category $\text{Sp}(\text{Aff}_{\mathbb{C}})$. We therefore have $\underline{\mathbf{K}}(T) \in \mathbf{k} - \text{Mod}_{\mathbb{S}}$.

In §3.4 we mentioned that the spectral realization functor

$$|-|_{\mathbb{S}} : \text{Ho}(\text{Sp}(\text{Aff}_{\mathbb{C}})) \longrightarrow \text{Ho}(\text{Sp})$$

is a monoidal functor. Because of the existence of a homotopy category of monoids and modules (see [Hov98, Theorems 3.3 and 2.1]), $|\mathbf{k}|_{\mathbb{S}}$ is a ring spectrum and the topological realization extends to \mathbf{k} -modules with values in $|\mathbf{k}|_{\mathbb{S}}$ -modules,

$$|-|_{\mathbb{S}} : \text{Ho}(\mathbf{k} - \text{Mod}_{\mathbb{S}}) \longrightarrow \text{Ho}(|\mathbf{k}|_{\mathbb{S}} - \text{Mod}_{\mathbb{S}}).$$

We also have a connective version of K-theory $\tilde{\mathbf{k}}$ and $\tilde{\underline{\mathbf{K}}}(T)$. In the same way we obtain a topological realization for $\tilde{\mathbf{k}}$ -modules,

$$|-|_{\mathbb{S}} : \text{Ho}(\tilde{\mathbf{k}} - \text{Mod}_{\mathbb{S}}) \longrightarrow \text{Ho}(|\tilde{\mathbf{k}}|_{\mathbb{S}} - \text{Mod}_{\mathbb{S}}).$$

DEFINITION 4.1. The *semi-topological K-theory* (respectively, the *connective semi-topological K-theory*) of a \mathbb{C} -dg-category $T \in \text{dgCat}_{\mathbb{C}}$ is the symmetric $|\mathbf{k}|_{\mathbb{S}}$ -module spectrum (respectively, the symmetric $|\tilde{\mathbf{k}}|_{\mathbb{S}}$ -module spectrum),

$$\mathbf{K}^{\text{st}}(T) := |\underline{\mathbf{K}}(T)|_{\mathbb{S}} \quad (\text{respectively } \tilde{\mathbf{K}}^{\text{st}}(T) := |\tilde{\underline{\mathbf{K}}}(T)|_{\mathbb{S}}).$$

Because of the existence of a cofibrant replacement functor in model categories of modules, we have two functors,

$$\begin{aligned} \mathbf{K}^{\text{st}} &: \text{dgCat}_{\mathbb{C}} \longrightarrow |\mathbf{k}|_{\mathbb{S}} - \text{Mod}_{\mathbb{S}}, \\ \tilde{\mathbf{K}}^{\text{st}} &: \text{dgCat}_{\mathbb{C}} \longrightarrow |\tilde{\mathbf{k}}|_{\mathbb{S}} - \text{Mod}_{\mathbb{S}}. \end{aligned}$$

We denote the semi-topological K-groups by $\mathbf{K}_i^{\text{st}}(T) := \pi_i \mathbf{K}^{\text{st}}(T)$ for all $i \in \mathbb{Z}$.

Remark 4.2. It is *a priori* necessary to also consider the connective version $\tilde{\mathbf{K}}^{\text{st}}$ in our study, because we do not know if $\tilde{\mathbf{K}}^{\text{st}}(T)$ is the connective covering of $\mathbf{K}^{\text{st}}(T)$. Indeed, our first thought is to remark that the topological realization is a left adjoint while the connective cover is a right adjoint.

Remark 4.3. There exists a map from algebraic K-theory to semi-topological K-theory. If $T \in \text{dgCat}_{\mathbb{C}}$, we have the unit map of adjoint pair $(|-|_{\mathbb{S}}, H_{\mathbb{S}, \text{B}})$,

$$\underline{\mathbf{K}}(T) \longrightarrow H_{\mathbb{S}, \text{B}}(|\underline{\mathbf{K}}(T)|_{\mathbb{S}}).$$

Taking global sections, i.e. the value on $\text{Spec}(\mathbb{C})$, we obtain a map in $\text{Ho}(\text{Sp})$,

$$\eta_T : \mathbf{K}(T) \longrightarrow \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(\Sigma^{\infty}(\text{Spec}(\mathbb{C}))_+, \mathbf{K}^{\text{st}}(T)) \simeq \mathbf{K}^{\text{st}}(T).$$

This defines a map $\eta : \mathbf{K} \longrightarrow \mathbf{K}^{\text{st}}$ in $\text{Ho}(\text{Sp}^{\text{dgCat}_{\mathbb{C}}})$. There is also a connective version of it denoted by $\tilde{\eta} : \tilde{\mathbf{K}} \longrightarrow \tilde{\mathbf{K}}^{\text{st}}$. For all schemes $X \in \text{Sch}_{\mathbb{C}}$, taking π_0 , we obtain a map

$$\tilde{\mathbf{K}}_0(X) \longrightarrow \tilde{\mathbf{K}}_0^{\text{st}}(X).$$

Because of Proposition 3.10, this map is the quotient map of the equivalence relation on algebraic vector bundles on X which identifies two vector bundles when they can be related by a connected algebraic curve. We obtain then a map $\tilde{\mathbf{K}}_0^{\text{st}}(X) \longrightarrow K_{\text{top}}^0(\text{sp}(X))$ from our semi-topological K-group to the Grothendieck group of topological vector bundles on $\text{sp}(X)$. We will prove below that this map is an isomorphism in the case of a smooth and proper \mathbb{C} -scheme of finite type.

Remark 4.4. By applying the spectral version of Theorem 3.18, we see that we can calculate the topological realization of Definition 4.1 by first taking the restriction to smooth schemes.

The following two results are central in the definition of topological K-theory. We denote by bu the usual topological K-theory spectrum. This means that for a topological space $X \in \text{Top}$, if $K_{\text{top}}^0(X)$ is the Grothendieck group of complex topological vector bundles on X , we have an isomorphism $K_{\text{top}}^0(X) \simeq \pi_0 \text{Map}_{\text{Ho}(\text{Sp})}(\Sigma^\infty SX_+, bu)$. A model for bu as a symmetric spectrum will be given below.

THEOREM 4.5. *There exists a canonical isomorphism $\tilde{\mathbf{K}}^{\text{st}}(\mathbb{1}) \simeq bu$ in $\text{Ho}(\text{Sp})$.*

A proof will be given in §4.2. For the time being we use it in order to define topological K-theory.

THEOREM 4.6. *For all smooth commutative algebras $B \in \text{CAlg}_{\mathbb{C}}$, the canonical map*

$$\tilde{\mathbf{K}}^{\text{st}}(B) \longrightarrow \mathbf{K}^{\text{st}}(B)$$

is an isomorphism in $\text{Ho}(\text{Sp})$. In particular, by Theorem 4.5, we have an isomorphism $\mathbf{K}^{\text{st}}(\mathbb{1}) \simeq \mathbf{bu}$ in $\text{Ho}(\text{Sp})$.

Proof. It is a well-known fact that the negative algebraic K-theory of a smooth commutative algebra vanishes (see [Sch06, Remark 7]). Therefore the map of presheaves of spectra $\tilde{\mathbf{K}}(B) \rightarrow \mathbf{K}(B)$ is an equivalence on smooth affine schemes. By Theorem 3.18, we conclude that it induces an equivalence on the spectral topological realization $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq \mathbf{K}^{\text{st}}(B)$. \square

Notation 4.7. The last two theorems can be reformulated by saying we have isomorphisms

$$|\tilde{\mathbf{k}}|_{\mathbb{S}} \simeq |\mathbf{k}|_{\mathbb{S}} \simeq bu$$

in $\text{Ho}(\text{Sp})$. We denote by \mathbf{bu} the symmetric ring spectrum $|\mathbf{k}|_{\mathbb{S}}$. We have functors

$$\tilde{\mathbf{K}}^{\text{st}}, \mathbf{K}^{\text{st}} : \text{dgCat}_{\mathbb{C}} \longrightarrow \mathbf{bu} - \text{Mod}_{\mathbb{S}}.$$

Remark 4.8. It is a classical fact that bu admits a model as a strict commutative ring in symmetric spectra, with addition corresponding to the sum of vector bundles and multiplication to external tensor product. We can express bu as the spectrum associated to the special Γ -space which is the topological realization of the stack of algebraic vector bundles (see 4.2). This Γ -space has the structure of a Γ -ring with multiplication given by external tensor product. This commutative ring structure on bu is, moreover, *unique* by [BR08, Corollary 1.4], in the sense that for all commutative symmetric ring spectra A and for all ring maps $f : A \rightarrow bu$ inducing an isomorphism on all homotopy groups, there exists a map $g : A \rightarrow bu$ in the homotopy category of *commutative* ring spectra such that g is isomorphic to f .

Here we cannot consider \mathbf{bu} as a commutative ring spectrum, nor \mathbf{k} as a *commutative* ring spectrum because this ring structure is given by the ring structure of endomorphisms of unity in $\mathbb{M}_{\text{loc}}(\mathbb{C})$ (see §2.3). So we will just talk about associative unital ring spectra, still knowing at the same time that \mathbf{bu} is equivalent in Sp to a commutative ring spectrum and that this structure is unique. We feel here the limits imposed by the language of model categories and strict algebraic structures compared to the highly more flexible language of Lurie’s monoidal ∞ -categories.

Notation 4.9. By the Bott periodicity theorem, we know that the abelian group $\pi_2 \mathbf{bu}$ is rank-one free. To define topological K-theory, we choose a Bott generator $\beta \in \mathbf{K}_2^{\text{st}}(\mathbb{1}) = \pi_2 \mathbf{bu}$ from the two existing ones. It is more convenient to first choose a generator α of the nontrivial part of $\mathbf{K}_0(\mathbf{P}^1)$ and to take $\beta = \eta(\alpha)$ where $\eta : \mathbf{K}_0(\mathbf{P}^1) \rightarrow \mathbf{K}_0^{\text{st}}(\mathbf{P}^1)$ is the canonical map. Then β gives a generator of $\mathbf{K}_2^{\text{st}}(\mathbb{1})$ by the canonical map $\mathbf{K}_0^{\text{st}}(\mathbf{P}^1) \rightarrow K_{\text{top}}(S^2) \simeq K_{\text{top}}^0(*) \oplus \beta K_{\text{top}}^{-2}(*)$.

Notation 4.10. We recall that given a ring spectrum A , an integer k , and an element $a \in \pi_k A$, one can define the ring spectrum $A[a^{-1}]$ localized with respect to a . It is endowed with a map $i_a : A \rightarrow A[a^{-1}]$ and satisfies the following universal property. For all ring spectra B , and all ring maps $A \rightarrow B$, the simplicial set $\text{Map}_{A\text{-Alg}}(A[a^{-1}], B)$ is nonempty if and only if a is invertible in the $\pi_*(A)$ -module $\pi_*(B)$. This property characterizes the object $A[a^{-1}]$ up to equivalence. This is the noncommutative version of [TVe08, Proposition 1.2.9.1] applied to the monoidal model category Sp of symmetric spectra. We conclude by [TVe08, Corollary 1.2.9.3] that the functor induced by composition with i_a ,

$$i_a^* : \text{Ho}(A[a^{-1}] - \text{Mod}_{\mathbb{S}}) \rightarrow \text{Ho}(A - \text{Mod}_{\mathbb{S}}),$$

is fully faithful and its image consists of the A -modules M such that multiplication by a is invertible in the $\pi_* A$ -module $\pi_* M$. If M is a A -module and $a \in \pi_k A$ an element, the localized module $M[a^{-1}]$ is defined as the $A[a^{-1}]$ -module $M \wedge_A A[a^{-1}]$.

Remark 4.11. Let A be a ring spectrum and $a \in \pi_k A$. Denote by $A_{\text{ass}}[a^{-1}]$ the localization of A with respect to a in the sense of associative ring spectra, and by $A_{\text{com}}[a^{-1}]$ the localization of A with respect to a but in the sense of commutative ring spectra. Then by the universal property, there exists a map $c : A_{\text{ass}}[a^{-1}] \rightarrow A_{\text{com}}[a^{-1}]$ which is an equivalence. Indeed, there exists a commutative diagram

$$\begin{array}{ccc} \text{Ho}(A_{\text{ass}}[a^{-1}] - \text{Mod}_r) & \hookrightarrow & \text{Ho}(A - \text{Mod}_r) \\ \downarrow c^* & & \downarrow \wr \\ \text{Ho}(A_{\text{com}}[a^{-1}] - \text{Mod}) & \hookrightarrow & \text{Ho}(A - \text{Mod}) \end{array}$$

where the index r stands for the category of right modules. Since A is commutative, the map from right A -modules to two-sided A -modules is an equivalence. Then the two homotopy categories of modules $\text{Ho}(A_{\text{ass}}[a^{-1}] - \text{Mod}_r)$ and $\text{Ho}(A_{\text{com}}[a^{-1}] - \text{Mod})$ are equivalent to the subcategory of $\text{Ho}(A - \text{Mod})$ formed by the A -modules for which multiplication by a is an equivalence. We conclude that c^* is an equivalence of categories and that c is an equivalence.

Remark 4.12. We consider the ring spectrum \mathbf{bu} , the Bott generator we chose, $\beta \in \pi_2 \mathbf{bu}$, and its localization $\mathbf{bu}[\beta^{-1}]$. *A priori* the latter localization is calculated in the sense of associative rings. But after Remark 4.8, we know that there exists a ring map $\mathbf{bu} \xrightarrow{\sim} bu$ with bu a commutative ring model, which is an equivalence. From Remark 4.11 we deduce that there is no ambiguity on the ring $\mathbf{bu}[\beta^{-1}]$: it is equivalent to the commutative localization of bu and is therefore equivalent to the usual colimit

$$\mathbf{bu}[\beta^{-1}] \simeq \text{colim}(bu \xrightarrow{\cup\beta} bu \xrightarrow{\cup\beta} \dots)$$

where $\cup\beta$ is map multiplication by β . The ring structure of $\mathbf{bu}[\beta^{-1}]$ is therefore the usual structure, and there is an isomorphism $\mathbf{bu}[\beta^{-1}] \wedge_{\mathbb{S}} HC \simeq HC[u^{\pm 1}]$ (with u of degree two) in the homotopy category of HC -algebras. We adopt the notation $\mathbf{BU} := \mathbf{bu}[\beta^{-1}]$.

DEFINITION 4.13. The *topological K-theory of a \mathbb{C} -dg-category $T \in \text{dgCat}_{\mathbb{C}}$* is the symmetric spectrum

$$\mathbf{K}^{\text{top}}(T) := \mathbf{K}^{\text{st}}(T)[\beta^{-1}].$$

This defines a functor

$$\mathbf{K}^{\text{top}} : \text{dgCat}_{\mathbb{C}} \longrightarrow \mathbf{BU} - \text{Mod}_{\mathbb{S}}.$$

We denote the topological K-groups by $\mathbf{K}_i^{\text{top}}(T) := \pi_i \mathbf{K}^{\text{top}}(T)$ for all $i \in \mathbb{Z}$.

Remark 4.14. Following Remark 4.3, we compose the unit map $\mathbf{K} \longrightarrow \mathbf{K}^{\text{st}}$ with the structural map $\mathbf{K}^{\text{st}} \longrightarrow \mathbf{K}^{\text{top}}$ and obtain a map denoted by $\theta : \mathbf{K} \longrightarrow \mathbf{K}^{\text{top}}$ from algebraic K-theory to topological K-theory.

Topological K-theory inherits the properties of algebraic K-theory from Proposition 2.11.

PROPOSITION 4.15.

- (a) *Topological K-theory commutes with filtrant homotopy colimits of dg-categories.*
- (b) *Topological K-theory sends Morita equivalences to equivalences of spectra.*
- (c) *For all exact sequence of dg-categories $T' \rightarrow T \rightarrow T''$, the induced sequence*

$$\mathbf{K}^{\text{top}}(T') \longrightarrow \mathbf{K}^{\text{top}}(T) \longrightarrow \mathbf{K}^{\text{top}}(T'')$$

is a distinguished triangle in $\text{Ho}(\text{Sp})$.

Proof. (a) This follows from the fact that the topological realization $|-|_{\mathbb{S}}$ and the operation of localization by β are left adjoints and thus commute with homotopy colimits.

(b) By functoriality.

(c) This follows from the fact that $|-|_{\mathbb{S}}$ and the operation of localization by β are exact functors. □

4.2 The case of the point

We give a particular model for the spectrum bu , as a symmetric spectrum. For all commutative algebras $A \in \text{CAlg}_{\mathbb{C}}$, we denote by $\text{Proj}(A)$ the Waldhausen category of projective A -modules of finite type, i.e. of finite-rank vector bundles on $\text{Spec}(A)$ (see Remark 2.8). The equivalences in $\text{Proj}(A)$ are by definition the isomorphisms, and the cofibrations are admissible monomorphisms. We define a pseudo-functor $\text{Aff}_{\mathbb{C}}^{\text{op}} \longrightarrow \text{WCat}$ by setting for all maps of algebras $A \longrightarrow B$, the induced exact functor as the tensor product

$$\begin{aligned} \text{Proj}(A) &\longrightarrow \text{Proj}(B) \\ E &\longmapsto E \otimes_A B \end{aligned}$$

which satisfies the usual associativity conditions up to isomorphism. We denote by Proj the canonical strictification of this pseudo-functor. We denote by $\text{Vect} = \text{NwProj}$ the simplicial presheaf obtained by taking levelwise the nerve of equivalences in $\text{Proj}(A)$. The direct sum of modules induces a homotopy coherent commutative monoid structure on Vect . More precisely, using the construction B_W mentioned in the end of § 2.1, we have a Γ -simplicial presheaf

$$\text{Vect}_{\bullet} := \text{Nw}B_W \text{Vect} \in \Gamma - \text{SPr}(\text{Aff}_{\mathbb{C}})$$

such that for all integers $n \geq 0$, there is a levelwise equivalence $\text{Vect}_{(n)} \simeq \text{Vect}^n$. We define the connective symmetric spectrum bu as

$$bu := \mathcal{B}|\text{Vect}_{\bullet}|_{\Gamma}^+.$$

Remark 4.16. We use notation from § 3.4 concerning classifying spaces of groups. We denote by $\mathrm{Gl}_n : \mathrm{Aff}_{\mathbb{C}}^{\mathrm{op}} \rightarrow \mathrm{SSet}$ the discrete simplicial presheaf of linear groups. We denote by $\coprod_{n \geq 0} \mathrm{BGl}_n$ the Γ -simplicial presheaf whose commutative monoid structure is given by a block sum of matrices. It is well known that there exists an étale local equivalence of Γ -simplicial presheaves,

$$\mathrm{Vect}_{\bullet} \simeq \coprod_{n \geq 0} \mathrm{BGl}_n.$$

By Theorem 3.4, Propositions 3.7 and 3.11, we have equivalences

$$|\mathrm{Vect}_{\bullet}|^+ \simeq \left| \coprod_{n \geq 0} \mathrm{BGl}_n \right|^+ \simeq \left(\coprod_{n \geq 0} |\mathrm{BGl}_n| \right)^+ \simeq \left(\coprod_{n \geq 0} \mathrm{B}|\mathrm{Gl}_n| \right)^+ \simeq \left(\coprod_{n \geq 0} \mathrm{BGl}_n(\mathbb{C}) \right)^+$$

where $\mathrm{Gl}_n(\mathbb{C})$ stands the topological space of complex points, and we take its classifying space as a topological group. The topological group is homotopy equivalent to the unitary group $U_n(\mathbb{C})$, and we have an equivalence

$$|\mathrm{Vect}_{\bullet}|^+ \simeq \left(\coprod_{n \geq 0} \mathrm{BU}_n(\mathbb{C}) \right)^+.$$

But the group completion $\coprod_{n \geq 0} \mathrm{BU}_n(\mathbb{C})$ is known (see [FM94, Appendix Q]) to be equivalent to

$$\left(\coprod_{n \geq 0} \mathrm{BU}_n(\mathbb{C}) \right)^+ \simeq \mathrm{BU}_{\infty} \times \mathbb{Z},$$

where BU_{∞} is the colimit of the $\mathrm{BU}_n(\mathbb{C})$ with respect to the natural inclusion $\mathrm{BU}_n(\mathbb{C}) \hookrightarrow \mathrm{BU}_{n+1}(\mathbb{C})$, with the structure of Γ -objects still given by the block sum of matrices and the usual law for \mathbb{Z} . In consequence, by Theorem 2.5, we have an equivalence of spectra $\mathcal{B}|\mathrm{Vect}_{\bullet}|_{\Gamma}^+ \simeq \mathcal{B}(\mathrm{BU}_{\infty} \times \mathbb{Z})$, which is the common definition of bu .

Proof of Theorem 4.5. We have a chain of canonical isomorphisms in $\mathrm{Ho}(\mathrm{Sp})$,

$$\begin{aligned} \tilde{\mathbf{K}}^{\mathrm{st}}(*) &= |\tilde{\mathbf{K}}(*)|_{\mathbb{S}} \\ &\simeq |\tilde{\mathbf{K}}(\mathrm{Vect})|_{\mathbb{S}} \quad (\text{by Remark 2.8}) \\ &= |\mathcal{B}K^{\Gamma}(\mathrm{Vect})|_{\mathbb{S}} \quad (\text{by definition, see the end of § 2.1}) \\ &\simeq \mathcal{B}|K^{\Gamma}(\mathrm{Vect})|_{\Gamma} \quad (\text{by Proposition 3.17}) \\ &\simeq \mathcal{B}|\mathrm{Vect}_{\bullet}|_{\Gamma}^+ \quad (\text{[Bla13, Lemma 1.10], recalled at the end of § 2.1}) \\ &\simeq \mathcal{B}|\mathrm{Vect}_{\bullet}|_{\Gamma}^+ \quad (\text{by Proposition 3.16}) \\ &= bu. \end{aligned}$$

□

4.3 Topological K-theory via the stack of perfect modules

Semi-topological K-theory, as initiated by Toën (see [Toë10, Kal10, KKP08]), was first defined as the topological realization of the stack of pseudo-perfect dg-modules associated to a dg-category. We consider two stacks associated to a dg-category T : the stack \mathcal{M}_T of pseudo-perfect dg-modules and the stack \mathcal{M}^T of perfect dg-modules. The stack \mathcal{M}^T was studied by Toën and Vaquié in [TVa07]. We show below that the semi-topological K-theory of a dg-category can be recovered as the topological realization of the stack \mathcal{M}^T of perfect dg-modules. This result is based on the existence of an \mathbf{A}^1 -homotopy equivalence between \mathcal{M}^T and the S -construction of the category of perfect T -dg-modules. The stack \mathcal{M}_T of pseudo-perfect dg-modules gives

rise to a dual theory with respect to \mathbf{K}^{st} which can therefore be called *topological K-homology*. This \mathbf{A}^1 -homotopy equivalence also proves that semi-topological K-homology is recovered as the topological realization of \mathcal{M}_T .

For all \mathbb{C} -dg-categories T we define two presheaves of Waldhausen categories:

- $\underline{\text{Parf}}(T) : \text{Spec}(A) \mapsto \text{Perf}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A) = \text{Perf}(T, A)$.
- $\underline{\text{PsParf}}(T) : \text{Spec}(A) \mapsto \text{PsParf}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A) = \text{PsParf}(T, A)$, where the latter is the category of $T \otimes_{\mathbb{C}}^{\mathbb{L}} A$ -dg-modules which are perfect relative to A , which we call pseudo-perfect $T \otimes_{\mathbb{C}}^{\mathbb{L}} A$ -dg-modules (see [TVa07, Definition 2.7]).

These strict functors are obtained as canonical strictification of pseudo-functors for which functoriality is given by direct image. They give rise to two stacks:

- $\mathcal{M}^T = \text{Nw}\underline{\text{Parf}}(T) : \text{Spec}(A) \mapsto \text{Nw}\text{Perf}(T, A)$,
- $\mathcal{M}_T = \text{Nw}\underline{\text{PsParf}}(T) : \text{Spec}(A) \mapsto \text{Nw}\text{PsParf}(T, A)$,

where *Nw* stands for the nerve of the subcategory of equivalences. The direct sum of dg-modules induces a homotopy coherent commutative monoid structure on $\underline{\text{Parf}}(T)$ and on $\underline{\text{PsParf}}(T)$. We apply the functor B_W defined in § 2.1 and obtain special Γ -objects in $\text{SPr}(\text{Aff}_{\mathbb{C}})$,

$$\mathcal{M}_{\bullet}^T = \text{Nw}B_W\underline{\text{Parf}}(T), \quad \mathcal{M}_T^{\bullet} = \text{Nw}B_W\underline{\text{PsParf}}(T).$$

Let $T \in \text{dgCat}_{\mathbb{C}}$ be a dg-category. All statements in this subsection are also true for the stack \mathcal{M}_T if we replace the K-theory of perfect dg-modules by the K-theory of pseudo-perfect dg-modules. We choose to write the details just for the stack \mathcal{M}^T . We use the notation of Example 2.4; we have a Δ -simplicial presheaf given by

$$\mathcal{K}_{\bullet}^T := \mathcal{K}_{\bullet}(\underline{\text{Parf}}(T)) = \text{Nw}S_{\bullet}\underline{\text{Parf}}(T).$$

By an abuse of notation, we sometimes consider \mathcal{M}^T as a Δ -object applying the functor α^* defined in § 2.1. We define a map of Δ -objects

$$\lambda_{\bullet} : \mathcal{M}_{\bullet}^T \longrightarrow \mathcal{K}_{\bullet}^T$$

by letting

$$\lambda_n(a_1, \dots, a_n) = (a_1 \hookrightarrow a_1 \oplus a_2 \hookrightarrow \dots \hookrightarrow a_1 \oplus \dots \oplus a_n)$$

where we gloss over the choices of sum diagrams which are part of the data, and which plays a role in that λ_{\bullet} is indeed simplicial. The fact that λ_{\bullet} is a map of simplicial objects is justified by the formulas in [Bla13, § 1.2, p. 29].

PROPOSITION 4.17. *The map $\lambda_{\bullet} : \mathcal{M}_{\bullet}^T \longrightarrow \mathcal{K}_{\bullet}^T$ is a levelwise \mathbf{A}^1 -equivalence in $\Delta - \text{SPr}^{\text{ét}, \mathbf{A}^1}$.*

Remark 4.18. This result can be heuristically rephrased by saying that, in general, cofibrations are not split in the category $\text{Perf}(T)$, but if we look at the presheaf $\underline{\text{Parf}}(T)$ in the \mathbf{A}^1 -homotopy theory, then these cofibrations are all split up to \mathbf{A}^1 -homotopy equivalence. This explains the link with K-theory which is precisely the invariant through which cofibrations are split by the additivity theorem.

Proof of Proposition 4.17. We introduce some notation. Let $n \geq 1$ be an integer. We denote by $[n]$ the category associated to the ordered set $\{1 < 2 < \dots < n\}$. We set

$$\begin{aligned} M_n &: \text{Aff}_{\mathbb{C}}^{\text{op}} \longrightarrow \text{Cat} \\ \text{Spec}(A) &\longmapsto M_n(A) = \text{Perf}(T, A)^{[n-1]}. \end{aligned}$$

The latter object is the presheaf of sequences of length $n - 1$ of composable maps in $\underline{\text{Parf}}(T)$. A map from $a_1 \rightarrow \cdots \rightarrow a_n$ to $b_1 \rightarrow \cdots \rightarrow b_n$ in $\text{Perf}(T, A)^{[n-1]}$ is by definition the data of commutative squares in $\text{Perf}(T, A)$:

$$\begin{array}{ccccccc} a_1 & \longrightarrow & a_2 & \longrightarrow & \cdots & \longrightarrow & a_n \\ \downarrow & & \downarrow & & & & \downarrow \\ b_1 & \longrightarrow & b_2 & \longrightarrow & \cdots & \longrightarrow & b_n \end{array}$$

We have $M_1 = \underline{\text{Parf}}(T)$. For all $n \geq 1$ and all $A \in \text{CAlg}_{\mathbb{C}}$, the category $M_n(A)$ is endowed with the projective model structure. Let $X_n = NwM_n$ be the simplicial presheaf which classifies sequences of length $n - 1$ of composable maps in $\underline{\text{Parf}}(T)$. For all $n \geq 1$, we have a natural inclusion,

$$\mathcal{K}_n^T \hookrightarrow X_n.$$

Because every map in $\text{Perf}(T, A)$ factorizes as a cofibration followed by a quasi-isomorphism, this last map is a global equivalence in $\text{SPr}(\text{Aff}_{\mathbb{C}})$. Therefore, to prove our result, it suffices to prove that the map still denoted by $\lambda_n : \mathcal{M}_n^T \rightarrow X_n$ is an \mathbf{A}^1 -equivalence for all $n \geq 1$. We proceed by recurrence on n . We use level 2 and $n - 1$ to show level n . For level 1 we have natural isomorphisms $\mathcal{M}_1^T = X_1 = Nw \text{Perf}(T, -)$. For level 2 the map λ_2 acts on 0-simplexes by

$$\lambda_2(a, b) = (a \rightarrow a \oplus b).$$

We then define an explicit \mathbf{A}^1 -homotopy inverse to λ_2 denoted by μ_2 and defined on 0-simplexes by

$$\mu_2(i : x \rightarrow y) := (x, \text{Cone}(i)).$$

This naturally defines a map of simplicial presheaves $\mu_2 : X_2 \rightarrow \mathcal{M}_2^T$. We then have

$$\mu_2 \circ \lambda_2(a, b) = \mu_2(a \rightarrow a \oplus b) = (a, \text{Cone}(a \rightarrow a \oplus b)) \simeq (a, b),$$

where the last map is a quasi-isomorphism. For all $A \in \text{CAlg}_{\mathbb{C}}$, we have a homotopy $\mu_2 \circ \lambda_2 \Rightarrow \text{id}$ as endomorphisms of $\mathcal{M}_2^T(A)$. In the other direction we have

$$\lambda_2 \circ \mu_2(i : x \rightarrow y) = \lambda_2(x, \text{Cone}(i)) = (x \rightarrow x \oplus \text{Cone}(i)).$$

We then define an \mathbf{A}^1 -homotopy $h : \mathbf{A}^1 \times X_2 \rightarrow X_2$ for every $A \in \text{CAlg}_{\mathbb{C}}$ by

$$\begin{aligned} h_A : A \times X_2(A) &\longrightarrow X_2(A) \\ (f, i : x \rightarrow y) &\longmapsto (fi : x \rightarrow y). \end{aligned}$$

The map h is an \mathbf{A}^1 -homotopy between id_{X_2} and the endomorphism Z of X_2 defined by

$$Z(i : x \rightarrow y) = (0 : x \rightarrow y).$$

The endomorphism Z is conjugated by an autoequivalence of X_2 with the map $\lambda_2 \circ \mu_2$. This autoequivalence is given by the shift

$$\begin{aligned} t : X_2 &\longrightarrow X_2 \\ (i : x \rightarrow y) &\longmapsto (y \rightarrow \text{Cone}(i)). \end{aligned}$$

The map t satisfies $t^{(3)}(i) = i[1]$, and is therefore an autoequivalence of X_2 . The inverse of t is given by

$$t^{-1}(i : x \rightarrow y) = \text{Cocone}(i) \rightarrow x,$$

where the last map is given by the definition of the cocone. We have

$$\begin{aligned} tZt^{-1}(f) &= tZ(\text{Cocone}(i) \rightarrow x) \\ &= t(0 : \text{Cocone}(i) \rightarrow x) \\ &= x \rightarrow \text{Cone}(0 : \text{Cocone}(i) \rightarrow x). \end{aligned}$$

The module $\text{Cone}(0 : \text{Cocone}(i) \rightarrow x)$ is canonically quasi-isomorphic to $x \oplus \text{Cone}(i)$ with the sum differential. Thus we have a quasi-isomorphism $tZt^{-1} \simeq \lambda_2 \circ \mu_2$. To sum up, the map h is an \mathbf{A}^1 -homotopy $\text{id}_{X_2} \Rightarrow Z$, and we have $tZt^{-1} \simeq \lambda_2 \circ \mu_2$ and a homotopy $\mu_2 \circ \lambda_2 \Rightarrow \text{id}$ which implies that λ_2 is an \mathbf{A}^1 -equivalence.

Now let $n \geq 2$. We use the notion of pullback of model categories defined in [Toë06]. Consider the functor

$$\begin{aligned} F : M^{(n)} &\longrightarrow M^{(n-1)} \times_{M^{(1)}} M^{(2)} \\ (a_1 \rightarrow a_2 \rightarrow \cdots \rightarrow a_n) &\longmapsto ((a_1 \rightarrow \cdots \rightarrow a_{n-2} \rightarrow a_n), (a_{n-1}/a_{n-2} \rightarrow a_n/a_{n-2}), \\ &\quad a_{n-1}/a_{n-2}, \text{id}, \text{id}) \end{aligned}$$

where $a_{n-2} \rightarrow a_n$ is the composite map $a_{n-2} \rightarrow a_{n-1} \rightarrow a_n$, and the notation a/b stands for the cone of the map $a \rightarrow b$.

LEMMA 4.19. *The functor F satisfies the two assumptions of [Toë06, Lemma 4.2]. We deduce that the map induced by F ,*

$$q_n : X_n \longrightarrow X_n \times_{X_1} X_2,$$

is a global equivalence in $\text{SPr}(\text{Aff}_{\mathbb{C}})$ (the pullback being calculated in the global model category $\text{SPr}(\text{Aff}_{\mathbb{C}})$).

Proof. Toën’s proof that q_3 is an equivalence (immediately after the proof of [Toë06, Lemma 4.2]) generalizes to sequences of maps of arbitrary length.³ The major distinction comes from the fact that we work with a presheaf of categories of perfect objects and not with a stable model category. Nevertheless, the same proof makes sense for such perfect objects. Moreover, [Toë06, Lemma 4.2] can be apply levelwise, and in the global model structure on $\text{SPr}(\text{Aff}_{\mathbb{C}})$, a square is homotopy cartesian if and only if it is levelwise homotopy cartesian in SSet . \square

We therefore end up with a square of simplicial presheaves,

$$\begin{array}{ccc} \mathcal{M}_n^T & \xrightarrow{\lambda_n} & X_n \\ \downarrow p & & \downarrow q_n \\ \mathcal{M}_{n-1}^T \times \mathcal{M}_2^T & \xrightarrow{\lambda_n \times \lambda_2} & X_{n-1} \times_{X_1} X_2 \end{array}$$

where p is the map $p(a_1, \dots, a_n) = ((a_1, \dots, a_{n-2}, a_{n-1} \oplus a_n), (a_{n-1}, a_n))$. The latter is an equivalence by its very definition. We can check directly that the square is commutative up to global homotopy. The maps λ_{n-1} and λ_2 are \mathbf{A}^1 -homotopy equivalences by the recurrence hypothesis. The \mathbf{A}^1 -homotopy equivalences being stable by homotopy pullback, the map $\lambda_2 \times \lambda_2$

³ We remark that there is a shift of indexes between our notation and the paper [Toë06].

is an \mathbf{A}^1 -homotopy equivalence. We conclude by the two-out-of-three property that the map λ_n is an \mathbf{A}^1 -homotopy equivalence. The proof of Proposition 4.17 is then complete. \square

PROPOSITION 4.20. *Let $T \in \text{dgCat}_{\mathbb{C}}$ be a dg-category over \mathbb{C} . Then the special Γ -space $|\mathcal{M}_{\bullet}^T|_{\Gamma}$ is very special.*

Proof. We invoke Proposition 3.10. We have an isomorphism of sets $\pi_0|\mathcal{M}_{\Gamma}^T| \simeq \pi_0\mathcal{M}^T(\mathbb{C})/\sim$ where two class of dg-modules $[E]$ and $[E']$ are equivalent if there exists a connected algebraic curve which connects the two dg-modules in $\mathcal{M}^T(\mathbb{C})$. Let E be a perfect T^{op} -module. Let

$$\delta : \mathbf{A}^1 \longrightarrow \mathcal{M}^T$$

be the map such that for all $A \in \text{CAlg}_{\mathbb{C}}$,

$$\delta_A(f) = \text{Cone}(E \xrightarrow{\times f} E).$$

Then we have $\delta_A(0) = \text{Cone}(0 : E \longrightarrow E) = E \oplus E[1]$ and $\delta_A(1) = \text{Cone}(\text{id}_E)$ which is canonically isomorphic to zero. We have proved that the identity $[E \oplus E[1]] = [0]$ is valid in the monoid $\pi_0|\mathcal{M}_{\Gamma}^T|$, which is therefore a group. \square

THEOREM 4.21. *Let $T \in \text{dgCat}_{\mathbb{C}}$. Then there exists a canonical isomorphism,*

$$\tilde{\mathbf{K}}^{\text{st}}(T) \simeq \mathcal{B}|\mathcal{M}_{\bullet}^T|_{\Gamma},$$

in $\text{Ho}(\text{Sp})$.

Proof. We have a chain of equivalences

$$\tilde{\mathbf{K}}^{\text{st}}(T) = |\tilde{\mathbf{K}}(T)|_{\mathbb{S}} = |\mathcal{B}K^{\Gamma}(\text{Perf}(T, -))|_{\mathbb{S}} \simeq \mathcal{B}|K^{\Gamma}(\text{Perf}(T, -))|_{\Gamma}$$

where the latter equivalence comes from Proposition 3.17. Set $K^{\Gamma}(\text{Perf}(T, -)) =: K^{\Gamma}(T, -)$. We consider the map of Γ -simplicial presheaves,

$$\sigma : \mathcal{M}_{\bullet}^T \longrightarrow K^{\Gamma}(T, -),$$

defined as the map (1). We want to show that σ induces an equivalence on topological realizations. Since we are dealing with special Γ -objects, it suffices to prove that we have an equivalence at level 1. We have a commutative diagram in $\text{Ho}(\text{SSet})$, where we intentionally omit the indexes Δ and Γ from the notation,

$$\begin{array}{ccc} |\mathcal{M}_{\Gamma}^T| & \xrightarrow{|\sigma|_1} & |K^{\Gamma}(T, -)|_1 \\ \wr \downarrow & & \searrow \sim \\ & & |K(T, -)| \\ & & \nearrow \sim \\ |\mathcal{M}_{\bullet}^T|_1^+ & \xrightarrow{|\lambda|_1^+} & |\mathcal{K}_{\bullet}^T|_1^+ \end{array}$$

and where the map of Δ -objects λ is the one from Proposition 4.17. By this latter proposition and Theorem 3.4, the induced map $|\lambda|_1^+$ is an equivalence in SSet . The left vertical map is an equivalence by Proposition 4.20. On the other hand, the Δ -objects $|K^{\Gamma}(T, -)|$ and $|\mathcal{K}_{\bullet}^T|_1^+$ have the same level 1, which is equivalent to $|K(T, -)|$. We then deduce that $|\sigma|_1$ is an equivalence in SSet and therefore that $|\sigma|$ is an equivalence, which proves the expected formula. \square

THEOREM 4.22. *Let $T \in \text{dgCat}_{\mathbb{C}}$. Then the special Γ -space $|\mathcal{M}_T^\bullet|_\Gamma$ is very special and there exists a canonical isomorphism,*

$$|\tilde{\mathbf{K}}(\text{PsParf}(T))|_{\mathbb{S}} \simeq \mathcal{B}|\mathcal{M}_T^\bullet|_\Gamma,$$

in $\text{Ho}(\text{Sp})$.

Proof. The proofs of Propositions 4.17 and 4.20 and Theorem 4.21 work in the same way if we replace perfect dg-modules by pseudo-perfect ones. \square

4.4 Topological Chern character

In this part we give the construction of the topological Chern character or topological Chern map. Let $T \in \text{dgCat}_{\mathbb{C}}$ be a \mathbb{C} -dg-category. Recall that we defined in Definition 2.21 an algebraic Chern map which is a map of \mathbf{k} -modules spectra, functorial in T ,

$$\text{Ch}_T : \mathbf{K}(T) \longrightarrow \underline{\text{HN}}(T).$$

Composing this map with the map of \mathbf{k} -modules $\underline{\text{HN}}(T) \longrightarrow \underline{\text{HP}}(T)$, we obtain a map of \mathbf{k} -modules,

$$\mathbf{K}(T) \longrightarrow \underline{\text{HP}}(T).$$

We then apply the spectral topological realization $|-|_{\mathbb{S}}$ to obtain a map of \mathbf{bu} -modules,

$$\mathbf{K}^{\text{st}}(T) = |\mathbf{K}(T)|_{\mathbb{S}} \longrightarrow |\underline{\text{HP}}(T)|_{\mathbb{S}}.$$

We now use a Künneth type formula for periodic cyclic homology. The presheaf $\underline{\text{HP}}(T)$ is given by $\text{Spec}(A) \longmapsto \text{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A)$. By Kassel’s theorem [Kas87, Theorem 2.3] and [Kas87, Proposition 2.4], for any smooth commutative \mathbb{C} -algebra A , the natural map of spectra,

$$\text{HP}(T) \wedge_{HC[u^{\pm 1}]}^{\mathbb{L}} \text{HP}(A) \longrightarrow \text{HP}(T \otimes_{\mathbb{C}}^{\mathbb{L}} A),$$

is an equivalence in Sp . This implies that the map of presheaves of spectra

$$\text{HP}(T) \wedge_{HC[u^{\pm 1}]}^{\mathbb{L}} \underline{\text{HP}}(*) \longrightarrow \underline{\text{HP}}(T),$$

is an equivalence on smooth affine schemes in $\text{Sp}(\text{Aff}_{\mathbb{C}})$. Theorem 3.18 implies that the map induced on spectral topological realization,

$$\text{HP}(T) \wedge^{\mathbb{L}} |\underline{\text{HP}}(*)|_{\mathbb{S}} \simeq |\text{HP}(T) \wedge_{HC[u^{\pm 1}]}^{\mathbb{L}} \underline{\text{HP}}(*)|_{\mathbb{S}} \longrightarrow |\underline{\text{HP}}(T)|_{\mathbb{S}}$$

is an equivalence in Sp . Therefore we have an isomorphism

$$|\underline{\text{HP}}(T)|_{\mathbb{S}} \simeq \text{HP}(T) \wedge_{HC[u^{\pm 1}]}^{\mathbb{L}} |\underline{\text{HP}}(*)|_{\mathbb{S}}$$

in $\text{Ho}(\text{Sp})$. By composition we have map

$$\mathbf{K}^{\text{st}}(T) \longrightarrow \text{HP}(T) \wedge_{HC[u^{\pm 1}]}^{\mathbb{L}} |\underline{\text{HP}}(*)|_{\mathbb{S}} \tag{6}$$

which defines a natural transformation between objects of $\text{Ho}(\text{Sp}^{\text{dgCat}_{\mathbb{C}}})$. In consequence, to obtain a map with target $\text{HP}(T)$, we have to choose a map $|\underline{\text{HP}}(*)|_{\mathbb{S}} \longrightarrow HC[u^{\pm 1}]$. By adjunction, it remains to choose a map $\underline{\text{HP}}(*) \longrightarrow (HC[u^{\pm 1}])_{\mathbb{S},B}$. The presheaf of spectra $(HC[u^{\pm 1}])_{\mathbb{S},B}$ is given by

$$X \longmapsto \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(|X|_{\mathbb{S}}, HC[u^{\pm 1}]) \simeq \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(|X|_{\mathbb{S}}, HC) \wedge_{HC}^{\mathbb{L}} HC[u^{\pm 1}],$$

which is the 2-periodic Betti cohomology of the scheme X with coefficients in \mathbb{C} . We denote the latter presheaf by $H_B(-, \mathbb{C}) \wedge HC[u^{\pm 1}]$. We denote by $H_B(-, \mathbb{C})$ the usual Betti cohomology with coefficients in \mathbb{C} , i.e. the presheaf of HC -modules spectra

$$X \longmapsto \mathbb{R}\underline{\mathrm{Hom}}_{\mathrm{Ho}(\mathrm{Sp})}(|X|_{\mathbb{S}}, HC) =: H_B(X, \mathbb{C}),$$

whose homotopy groups are the Betti cohomology \mathbb{C} -vector spaces of X . We denote by $\mathrm{HP}^{\mathrm{alg}}$ the presheaf $\underline{\mathrm{HP}}(*) : \mathrm{Spec}(A) \mapsto \mathrm{HP}(A)$. We thus have to choose a map

$$\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_B(-, \mathbb{C}) \wedge HC[u^{\pm 1}].$$

We consider the standard antisymmetrization map,

$$\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}},$$

which goes from periodic cyclic homology to naive de Rham cohomology. By ‘naive de Rham cohomology’ we mean the presheaf of spectra $X \mapsto H_{\mathrm{DR}}^{\mathrm{naive}}(X)$, such that $H_{\mathrm{DR}}^{\mathrm{naive}}(X)$ is the spectrum associated to the algebraic de Rham complex of X , i.e. the complex of \mathbb{C} -vector spaces of algebraic differential forms everywhere defined on X . We denote by $H_{\mathrm{DR}}^{\mathrm{naive}, \mathrm{an}}$ the analytic analog of $H_{\mathrm{DR}}^{\mathrm{naive}}$ constructed out of analytic differential forms. The inclusion of algebraic differential forms into analytic differential forms induces a map $H_{\mathrm{DR}}^{\mathrm{naive}} \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}, \mathrm{an}}$. The evident map from \mathbb{C} to the analytic de Rham complex induces a map $H_B(-, \mathbb{C}) \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}, \mathrm{an}}$ which is an equivalence on smooth schemes.⁴ We then have maps of presheaves of spectra,

$$\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}} \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}, \mathrm{an}} \longleftarrow H_B(-, \mathbb{C}) \longrightarrow H_B(-, \mathbb{C}) \wedge HC[u^{\pm 1}], \tag{7}$$

and the map which goes from right to left is an equivalence on smooth schemes.⁵

Notation 4.23. We denote by $\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}}$ the proper local model structure on the category $\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})$ of presheaves of symmetric spectra on the category of separated schemes of finite type over \mathbb{C} , i.e. the local model structure with respect to the proper topology on the category $\mathrm{Sch}_{\mathbb{C}}$.

We naturally consider the maps (7) as maps in the homotopy category $\mathrm{Ho}(\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}})$ of presheaves of spectra on $\mathrm{Sch}_{\mathbb{C}}$ (with respect to the proper local model structure); recall Remark 2.16 for the cyclic homology of schemes. By Remark 3.19, the map $H_B(-, \mathbb{C}) \longrightarrow H_{\mathrm{DR}}^{\mathrm{naive}, \mathrm{an}}$ is a proper local equivalence, and is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}})$. We thus obtain a map $\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_B(-, \mathbb{C})$ in $\mathrm{Ho}(\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}})$ and a map

$$\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_B(-, \mathbb{C}) \wedge HC[u^{\pm 1}]$$

in $\mathrm{Ho}(\mathrm{Sp}(\mathrm{Sch}_{\mathbb{C}})^{\mathrm{pro}})$. By the spectral version of Proposition 3.21, the map

$$\mathrm{HP}^{\mathrm{alg}} \longrightarrow H_B(-, \mathbb{C}) \wedge HC[u^{\pm 1}] = H_{\mathbb{S}, B}(HC[u^{\pm 1}])$$

gives by adjunction the expected map

$$\mathcal{P} : |\mathrm{HP}^{\mathrm{alg}}|_{\mathbb{S}} = |\underline{\mathrm{HP}}(*)|_{\mathbb{S}} \longrightarrow HC[u^{\pm 1}]$$

⁴ This is true by the classical fact that for a smooth complex variety, the complex of sheaves of analytic differential forms is an injective resolution of the constant sheaf $\underline{\mathbb{C}}$.

⁵ In fact the first two maps are also equivalences on smooth affine schemes by the Hochschild–Konstant–Rosenberg theorem and the Grothendieck theorem respectively, but we will not need this fact.

in $\text{Ho}(\text{Sp})$. By composing the map (6) with what we have just found, we obtain a map

$$\text{Ch}_T^{\text{st}} : \mathbf{K}^{\text{st}}(T) \longrightarrow \text{HP}(T)$$

defined as the composite

$$\begin{array}{ccc} \mathbf{K}^{\text{st}}(T) & \xrightarrow{\text{Ch}_T^{\text{st}}} & \text{HP}(T) \\ \downarrow |\text{Ch}_T|_{\mathbb{S}} & & \uparrow \wr \\ |\underline{\text{HP}}(T)|_{\mathbb{S}} \xrightarrow{\sim} \text{HP}(T) \wedge_{\text{HC}[u^{\pm 1}]}^{\mathbb{L}} |\underline{\text{HP}}(*)|_{\mathbb{S}} & \xrightarrow{\text{id} \wedge^{\mathbb{L}} \mathcal{P}} & \text{HP}(T) \wedge_{\text{HC}[u^{\pm 1}]}^{\mathbb{L}} \text{HC}[u^{\pm 1}] \end{array}$$

By § 4.1, the spectral topological realization can be extended to categories of modules,

$$|-|_{\mathbb{S}} : \mathbf{k} - \text{Mod}_{\mathbb{S}} \longrightarrow \mathbf{bu} - \text{Mod}_{\mathbb{S}}.$$

We deduce from this that in the previous rectangle, all maps are maps of \mathbf{bu} -modules and we obtain in this way a map

$$\text{Ch}^{\text{st}} : \mathbf{K}^{\text{st}} \longrightarrow \text{HP}$$

in $\text{Ho}(\mathbf{bu} - \text{Mod}_{\mathbb{S}}^{\text{dgCat}_{\mathbb{C}}})$. Here it is convenient to suppress the T in $\text{HP}(T)$. We remark that for any \mathbb{C} -dg-category T we have a commutative square

$$\begin{array}{ccc} \mathbf{K}(T) & \xrightarrow{\text{Ch}} & \text{HP}(T) \\ \theta \downarrow & & \downarrow \text{id} \\ \mathbf{K}^{\text{st}}(T) & \xrightarrow{\text{Ch}^{\text{st}}} & \text{HP}(T) \end{array} \tag{8}$$

in $\text{Ho}(\mathbf{bu} - \text{Mod}_{\mathbb{S}})$, where θ is the natural map defined at Remark 4.3 and the composite

$$\text{HP}(T) \longrightarrow |\underline{\text{HP}}(T)|_{\mathbb{S}} \longrightarrow \text{HP}(T) \wedge_{\text{HC}[u^{\pm 1}]}^{\mathbb{L}} |\underline{\text{HP}}(*)|_{\mathbb{S}} \xrightarrow{\text{id} \wedge^{\mathbb{L}} \mathcal{P}} \text{HP}(T)$$

is equal to the identity in $\text{End}_{\text{Ho}(\text{Sp})}(\text{HP}(T))$. Now it remains to verify that the image $\text{Ch}^{\text{st}}(\beta)$ of the Bott generator is invertible in the ring $\text{HP}(*) = \text{HC}[u^{\pm 1}]$. We use Notation 4.9. We follow the Bott generator in the K-theory of \mathbf{P}^1 . As a particular case of the square (8), we have a commutative square of abelian groups

$$\begin{array}{ccc} \mathbf{K}_0(\mathbf{P}^1) & \xrightarrow{\text{Ch}} & \text{HP}_0(\mathbf{P}^1) \\ \eta \downarrow & & \downarrow \text{id} \\ \mathbf{K}_0^{\text{st}}(\mathbf{P}^1) & \xrightarrow{\text{Ch}^{\text{st}}} & \text{HP}_0(\mathbf{P}^1) \end{array}$$

We recall that $\eta(\alpha) = \beta$ and we choose for example a generator $u \in \text{HP}_0(\mathbf{P}^1)$ such that $\text{Ch}(\alpha) = u$. We then verify that $\text{Ch}^{\text{st}}(\beta) = \text{Ch}^{\text{st}}(\eta(\alpha)) = \text{Ch}(\alpha) = u$. By the universal property we obtain for any \mathbb{C} -dg-category T a map

$$\text{Ch}_T^{\text{top}} : \mathbf{K}^{\text{top}}(T) \longrightarrow \text{HP}(T).$$

This defines a map

$$\text{Ch}^{\text{top}} : \mathbf{K}^{\text{top}} \longrightarrow \text{HP}$$

in $\text{Ho}(\mathbf{BU} - \text{Mod}_{\mathbb{S}}^{\text{dgCat}_{\mathbb{C}}})$.

THEOREM 4.24. *There exists a map $\text{Ch}^{\text{top}} : \mathbf{K}^{\text{top}} \longrightarrow \text{HP}$ called the topological Chern map such that the square*

$$\begin{array}{ccc}
 \mathbf{K} & \xrightarrow{\text{Ch}} & \text{HN} \\
 \downarrow & & \downarrow \\
 \mathbf{K}^{\text{top}} & \xrightarrow{\text{Ch}^{\text{top}}} & \text{HP}
 \end{array} \tag{9}$$

is commutative in $\text{Ho}(\text{Sp}^{\text{dgCat}_{\mathbb{C}}})$.

Proof. This follows immediately from the commutativity of the square (8). □

4.5 Conjectures

The following conjectures are all analogs of known facts for smooth proper algebraic varieties. The first one concerns the rational part of the hypothetical noncommutative Hodge structure for a smooth proper dg-category over \mathbb{C} . We recall that a dg-category T is said to be proper if its complexes of morphisms are perfect complexes and if the triangulated category $[\widehat{T}]$ has a compact generator. A dg-category T is said to be smooth if the $T^{\text{op}} \otimes^{\mathbb{L}} T$ -module $(x, y) \mapsto T(x, y)$ is perfect. It is proved in [TVa07, Corollary 2.13] that a smooth proper dg-category is of finite type, and therefore is equivalent to a homotopically finitely presented, smooth and proper dg-algebra. To prove the following conjectures, it suffices to prove them for smooth and proper dg-algebras.

CONJECTURE 4.25 (The lattice conjecture). Let T be a smooth proper dg-category over \mathbb{C} . Then the map

$$\text{Ch}^{\text{top}} \wedge_{\mathbb{S}} \text{HC} : \mathbf{K}^{\text{top}}(T) \wedge_{\mathbb{S}} \text{HC} \longrightarrow \text{HP}(T)$$

is an equivalence.

We remark that the class of dg-categories satisfying this lattice conjecture is stable by the operations of filtrant colimits, retracts, quotient, and extension in the Morita homotopy category $\text{Ho}(\text{dgMor}_{\mathbb{C}})$. The following conjecture is a generalization of Theorem 4.6 for smooth and proper dg-categories.

CONJECTURE 4.26. Let T be a smooth proper dg-category over \mathbb{C} . Then $\mathbf{K}_i^{\text{st}}(T) = 0$ for all $i < 0$.

Cisinski has recently proved that the vanishing of the negative algebraic K-groups of smooth proper dg-algebras whose cohomology is concentrated in positive degrees. This proof is based on Schlichting’s proof of the vanishing the negative algebraic K-groups of a noetherian abelian category.

The following conjecture is inspired by Thomason’s result cited in the introduction [Tho85, Theorem 4.11] and by Friedlander and Walker’s analogous result for certain type of projective varieties (see [FW01, Corollary 3.8]). We recall that given a spectrum E and an integer $n \in \mathbb{Z}$, we can, because the stable homotopy category $\text{Ho}(\text{Sp})$ is additive, give a meaning to the reduction modulo n of the spectrum E denoted by E/n . We state this conjecture with any dg-category – we at least expect very general assumptions on the type of dg-categories satisfying this conjecture (as I was told by DC Cisinski), because it would only involve the \mathbf{A}^1 -invariance of algebraic K-theory with torsion coefficients and Gabber rigidity theorem.

CONJECTURE 4.27. Let T be a dg-category over \mathbb{C} and $n > 0$ an integer. Then the map $\mathbf{K}(T)/n \longrightarrow \mathbf{K}^{\text{st}}(T)/n$ is an equivalence.

4.6 Schemes and Deligne cohomology

The aim here is to compare the topological K-theory of the dg-category of perfect complexes on a \mathbb{C} -scheme with the usual topological K-theory of its complex points together with their Chern maps, thus completing point (b) in Theorem 1.1. The proof is based on Riou’s Spanier–Whitehead duality in Morel and Voevodsky’s stable homotopy theory of smooth schemes over \mathbb{C} .

At the end we give a comparison result for Deligne cohomology of the dg-category of perfect complexes, showing that we actually recover the commutative Deligne cohomology.

Notation 4.28. For any \mathbb{C} -scheme X , we denote by $\mathbf{K}^{\text{top}}(X) := \mathbf{K}^{\text{top}}(\mathcal{L}_{\text{perf}}(X))$ the topological K-theory of its dg-category of perfect complexes.

Notation 4.29. For a topological space $Y \in \text{Top}$, we denote by $K_{\text{top}}(Y) := \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(\Sigma^\infty SY_+, \mathbf{BU})$ its topological K-theory spectrum (nonconnective). If $X \in \text{Sch}_{\mathbb{C}}$ is a \mathbb{C} -scheme, we can consider the topological K-theory of its complex points $K_{\text{top}}(\text{sp}(X))$.

Notation 4.30. For a scheme $X \in \text{Sch}_{\mathbb{C}}$, and any \mathbb{Z} -module A , we denote by $H_{\mathbb{B}}(X, A) = \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(|X|_{\mathbb{S}}, HA)$ its Betti cohomology with coefficients in A . For smooth X , we have an isomorphism $\text{HP}(X) \rightarrow H_{\mathbb{B}}(X, \mathbb{C}[u^{\pm 1}])$ given by the composite of the antisymmetrization map and the usual isomorphism between de Rham and Betti cohomology.

Notation 4.31. For every topological space Y , the usual topological Chern map

$$\text{Ch}_{\text{utop}} : K_{\text{top}}(Y) \rightarrow H_{\mathbb{B}}(Y, \mathbb{C}[u^{\pm 1}])$$

is defined by $\text{Ch}_{\text{utop}} = \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp})}(\Sigma^\infty SY_+, \text{Ch}_{\mathbb{1}}^{\text{top}})$, where $\text{Ch}_{\mathbb{1}}^{\text{top}} : \mathbf{BU} \rightarrow H\mathbb{C}[u^{\pm 1}]$ is the topological Chern map of the point (Theorem 4.24) which is a map of ring spectra. The identity $\text{Ch}_{\mathbb{1}}^{\text{top}}(\beta) = u$ characterizes uniquely the Chern map in the set of homotopy classes of ring maps $[\mathbf{BU}, H\mathbb{C}[u^{\pm 1}]]$. The latter set is isomorphic to the set $[\mathbf{BU} \wedge H\mathbb{C}, H\mathbb{C}[u^{\pm 1}]]$. The standard equivalence between ring spectra and dg-algebras maps the ring spectrum $\mathbf{BU} \wedge H\mathbb{C}$ to the polynomial dg-algebra $\mathbb{C}[\beta, \beta^{-1}]$, where the degree of β is two. The image of β is nothing more than a nonzero multiple of u ; the usual Chern map corresponds to $\text{Ch}_{\text{utop}}(\beta) = u$.

PROPOSITION 4.32. *Let X be a separated \mathbb{C} -scheme of finite type. Then there exists a canonical isomorphism*

$$\mathbf{K}^{\text{top}}(X) \xrightarrow{\sim} K_{\text{top}}(\text{sp}(X))$$

in $\text{Ho}(\text{Sp})$. Moreover, the square

$$\begin{array}{ccc} \mathbf{K}^{\text{top}}(X) & \xrightarrow{\text{Ch}^{\text{top}}} & \text{HP}(X) \\ \downarrow \wr & & \downarrow \\ K_{\text{top}}(\text{sp}(X)) & \xrightarrow{\text{Ch}_{\text{utop}}} & H_{\mathbb{B}}(X, \mathbb{C}) \wedge H\mathbb{C}[u^{\pm 1}] \end{array} \tag{10}$$

is commutative in $\text{Ho}(\text{Sp})$. This implies that the lattice conjecture (Conjecture 4.25) is valid for dg-categories of the form $T = \mathcal{L}_{\text{perf}}(X)$ with X any separated \mathbb{C} -scheme of finite type.

The proof is presented later in this subsection. We begin with some notation and reminders. Recall from Notation 2.12 that we have an ‘algebraic K-theory of schemes’ functor,

$$\mathbf{K} : \text{Sch}_{\mathbb{C}}^{\text{op}} \rightarrow \text{Sp}.$$

For a scheme X we define a presheaf of K-theory by the formula

$$\underline{\mathbf{K}}(X) := (\text{Spec}(A)) \mapsto \mathbf{K}(X \times_{\mathbb{C}} \text{Spec}(A)).$$

This defines a functor

$$\mathbf{K} : \text{Sch}_{\mathbb{C}}^{\text{op}} \longrightarrow \text{Sp}(\text{Aff}_{\mathbb{C}}).$$

Recall from Notation 2.12 the ‘perfect complexes’ functor,

$$\mathcal{L}_{\text{perf}} : \text{Sch}_{\mathbb{C}}^{\text{op}} \longrightarrow \text{dgCat}_{\mathbb{C}}.$$

For every $X \in \text{Sch}_{\mathbb{C}}$ and every $\text{Spec}(A) \in \text{Aff}_{\mathbb{C}}$, there exists a map

$$\mathcal{L}_{\text{perf}}(X) \otimes_{\mathbb{C}}^{\mathbb{L}} A \longrightarrow \mathcal{L}_{\text{perf}}(X \times_{\mathbb{C}} \text{Spec}(A)) \tag{11}$$

in $\text{dgCat}_{\mathbb{C}}$ given by pulling back perfect complexes along the projection map $X \times_{\mathbb{C}} \text{Spec}(A) \longrightarrow X$. The map (11) is known to be a Morita equivalence of dg-categories (this uses the fact that we work over a field, and hence that at least one of the two schemes is flat over the base). It can be proved by reducing the assumption to the affine case (using the descent property of $\mathcal{L}_{\text{perf}}$) for which it is immediate. We deduce that by Proposition 2.11, when $\text{Spec}(A)$ is smooth, there exists an equivalence

$$\mathbf{K}(X \times_{\mathbb{C}} \text{Spec}(A)) \simeq \mathbf{K}(\mathcal{L}_{\text{perf}}(X) \otimes_{\mathbb{C}}^{\mathbb{L}} A)$$

in Sp . This implies the existence of an isomorphism of presheaves

$$\underline{\mathbf{K}}(X) \simeq \underline{\mathbf{K}}(\mathcal{L}_{\text{perf}}(X))$$

in $\text{Ho}(\text{Sp}(\text{Aff}_{\mathbb{C}}^{\text{liss}}))$.

We denote by $\text{Sp}_{\text{Liss}}^{\text{Nis}, \mathbf{A}^1}$ the \mathbf{A}^1 -Nisnevich local model structure on $\text{Sp}(\text{Aff}_{\mathbb{C}}^{\text{liss}})$. This is a monoidal model category and we denote by \wedge its monoidal product. We denote by $\mathcal{SH}_{\mathbb{C}}$ the Morel–Voevodsky stable homotopy category of smooth \mathbb{C} -schemes. It can be defined in the following way. The symbol $T := S^1 \wedge \mathbf{G}_m$ stands for the Tate sphere in $\text{Sp}_{\text{Liss}}^{\text{Nis}, \mathbf{A}^1}$; the category $\mathcal{SH}_{\mathbb{C}}$ is defined here as the homotopy category of symmetric T -spectra in $\text{Sp}_{\text{Liss}}^{\text{Nis}, \mathbf{A}^1}$,

$$\mathcal{SH}_{\mathbb{C}} := \text{Ho}(\text{Sp}_T \text{Sp}_{\text{Liss}}^{\text{Nis}, \mathbf{A}^1}).$$

This is a closed symmetric monoidal category. Riou’s statement about Spanier–Whitehead duality says that every object of $\mathcal{SH}_{\mathbb{C}}$ of the form $\Sigma_{T, S^1}^{\infty} X_+$ for X a smooth \mathbb{C} -scheme of finite type is (strongly) dualizable in the monoidal category $\mathcal{SH}_{\mathbb{C}}$ (see [Rio05]).

We choose a presentation of the scheme $\mathbf{G}_m = \mathbb{C} \otimes_{\mathbb{Z}} \mathbb{Z}[t, t^{-1}]$. The invertible function t gives a class b in the group $\mathbf{K}_1(\mathbf{G}_m)$ and we choose t such that the canonical map

$$\mathbf{K}_1(\mathbf{G}_m) \longrightarrow \mathbf{K}_0(\mathbf{P}^1) \simeq \mathbb{Z} \oplus \alpha\mathbb{Z}$$

sends the class b to α (which was chosen in Notation 4.9). We denote by KH the object of $\mathcal{SH}_{\mathbb{C}}$ whose underlying S^1 -spectrum is the presheaf $\mathbf{K} \in \text{Sp}(\text{Aff}_{\mathbb{C}}^{\text{liss}})$, endowed with the T -spectrum structure given by the map $T = S^1 \wedge \mathbf{G}_m \longrightarrow \mathbf{K}$ which corresponds to the class $b \in \mathbf{K}_1(\mathbf{G}_m)$. More precisely, for all $n \geq 0$ the class b gives rise to a map

$$T \wedge^{\mathbb{L}} \mathbf{K} \xrightarrow{b \wedge^{\mathbb{L}} \text{id}} \mathbf{K} \wedge^{\mathbb{L}} \mathbf{K} \longrightarrow \mathbf{K}$$

where the last map is given by the ring structure of \mathbf{K} in $\text{Ho}(\text{Sp}(\text{Aff}_{\mathbb{C}}^{\text{liss}}))$. Taking cofibrant-fibrant replacement and applying [Cis13, Proposition 2.3], we obtain a T -spectrum KH with $KH(Y)_n = \mathbf{K}(Y)$ for all $n \geq 0$. A fibrant model of KH in $\mathcal{SH}_{\mathbb{C}}$ represents Weibel’s homotopy invariant algebraic K-theory, as proved by Cisinski [Cis13, Theorem 2.20]. For all smooth

\mathbb{C} -schemes of finite type X , we define

$$\underline{KH}(X) := \mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{SH}_{\mathbb{C}}}(\Sigma_{T,S^1}^{\infty} X_+, KH).$$

By Yoneda’s lemma in $\mathcal{SH}_{\mathbb{C}}$, this latter is globally equivalent to the presheaf of T -spectra

$$Y \longmapsto KH(X \times Y)$$

in $\mathrm{Sp}(\mathrm{Aff}_{\mathbb{C}}^{\mathrm{liiss}})$. We have an equivalence $\underline{KH}(\mathbb{1}) \simeq KH$. The topological realization of T is given by

$$\mathrm{ssp}(T) \simeq \mathrm{ssp}(S^1 \wedge \mathbf{G}_m) \simeq S^1 \wedge \mathrm{ssp}(\mathbf{G}_m) \simeq S^1 \wedge S^1 = S^2.$$

We locally adopt the more precise notation $\mathrm{Sp} = \mathrm{Sp}_{S^1}$ for symmetric S^1 -spectra and we denote by $\mathrm{Sp}_{S^2} \mathrm{Sp}_{S^1}$ the model category of symmetric S^2 -spectra inside symmetric S^1 -spectra. The infinite loop space functor $\Omega_{S^2}^{\infty} : \mathrm{Sp}_{S^2} \mathrm{Sp}_{S^1} \rightarrow \mathrm{Sp}_{S^1}$ is a Quillen equivalence. There exists a topological realization functor on the level of the category of T -spectra $\mathcal{SH}_{\mathbb{C}}$; It is simply defined levelwise. We denote it by

$$|-|_{S^2} : \mathcal{SH}_{\mathbb{C}} \rightarrow \mathrm{Ho}(\mathrm{Sp}_{S^2} \mathrm{Sp}_{S^1}).$$

We have a noncommutative square of categories

$$\begin{array}{ccc} \mathcal{SH}_{\mathbb{C}} & \xrightarrow{|-|_{S^2}} & \mathrm{Ho}(\mathrm{Sp}_{S^2} \mathrm{Sp}_{S^1}) \\ \mathbb{R}\Omega_T^{\infty} \downarrow & & \downarrow \mathbb{R}\Omega_{S^2}^{\infty} \\ \mathrm{Ho}(\mathrm{Sp}_{\mathrm{Liss}}^{\mathrm{Nis}, \mathbf{A}^1}) & \xrightarrow{|-|_{\mathbb{S}}} & \mathrm{Ho}(\mathrm{Sp}_{S^1}) \end{array}$$

where $\mathbb{R}\Omega_T^{\infty}$ stands for the derived loop space functor for T -spectra. The S^2 -spectrum structure of $|\underline{KH}(X)|_{S^2}$ corresponds to the multiplication by the Bott generator $\beta \in \pi_2 |\underline{\mathbf{K}}(\mathbb{1})|_{\mathbb{S}}$ inside the S^1 -spectrum $|\underline{\mathbf{K}}(X)|_{\mathbb{S}}$. This is because the T -spectrum structure of $\underline{KH}(X)$ is given by multiplication by the class b and because we also have an equivalence $\mathrm{ssp}(T) \simeq S^2$ such that the triangle

$$\begin{array}{ccc} \Sigma^{\infty}(S^2)_+ & & \\ \downarrow \wr & \searrow \beta & \\ |T|_{\mathbb{S}} & \xrightarrow{|b|} & |\underline{\mathbf{K}}(\mathbb{1})|_{\mathbb{S}} \simeq \mathbf{bu} \end{array}$$

is commutative by the choice of b . In consequence, applying the functor $\mathbb{R}\Omega_{S^2}^{\infty}$ to $|\underline{KH}(X)|_{S^2}$ corresponds to inverting β in $|\underline{\mathbf{K}}(X)|_{\mathbb{S}}$. We therefore have a canonical map

$$|\underline{\mathbf{K}}(X)|_{\mathbb{S}}[\beta^{-1}] \rightarrow \mathbb{R}\Omega_{S^2}^{\infty} |\underline{KH}(X)|_{S^2},$$

which is an isomorphism in $\mathrm{Ho}(\mathrm{Sp}_{S^1})$ because the underlying S^1 -spectrum of $\underline{KH}(X)$ is equivalent to $\underline{\mathbf{K}}(X)$.

Proof of Proposition 4.32. We first deal with the smooth case by using Riou’s Spanier–Whitehead duality in the homotopy category of smooth schemes, and then we prove it for possible singular schemes by using cdh descent. Let X be a smooth separated \mathbb{C} -scheme of finite type. We have canonical isomorphisms in $\mathrm{Ho}(\mathrm{Sp}_{S^1})$,

$$\begin{aligned} \mathbf{K}^{\mathrm{top}}(X) &= |\underline{\mathbf{K}}(X)|_{\mathbb{S}}[\beta^{-1}] \\ &\simeq \mathbb{R}\Omega_{S^2}^{\infty} |\underline{KH}(X)|_{S^2} \\ &= \mathbb{R}\Omega_{S^2}^{\infty} |\mathbb{R}\underline{\mathrm{Hom}}_{\mathcal{SH}_{\mathbb{C}}}(\Sigma_{T,S^1}^{\infty} X_+, KH)|_{S^2}. \end{aligned}$$

By Riou’s theorems [Rio05, Theorems 1.4 and 2.2], the object $\Sigma_{T,S^1}^\infty X_+$ is strongly dualizable in $\mathcal{SH}_{\mathbb{C}}$. The functor $|-|_{S^2}$ being monoidal, and because a monoidal functor commutes with the duality functor, we have canonical isomorphisms in $\text{Ho}(\text{Sp}_{S^1})$,

$$\begin{aligned} \mathbf{K}^{\text{top}}(X) &\simeq \mathbb{R}\Omega_{S^2}^\infty \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp}_{S^2} \text{Sp}_{S^1})}(|\Sigma_{T,S^1}^\infty X_+|_{S^2}, |KH|_{S^2}) \\ &\simeq \mathbb{R}\Omega_{S^2}^\infty \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp}_{S^2} \text{Sp}_{S^1})}(\Sigma_{S^2,S^1}^\infty |X|_+, |KH|_{S^2}) \\ &\simeq \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp}_{S^1})}(\Sigma_{S^1}^\infty |X|_+, \mathbb{R}\Omega_{S^2}^\infty |KH|_{S^2}). \end{aligned}$$

We also have isomorphisms in $\text{Ho}(\text{Sp}_{S^1})$,

$$\mathbb{R}\Omega_{S^2}^\infty |KH|_{S^2} \simeq \mathbf{K}^{\text{top}}(\mathbb{1}) = \mathbf{BU}.$$

This proves the existence of the expected isomorphism

$$\mathbf{K}^{\text{top}}(X) \simeq \mathbb{R}\underline{\text{Hom}}_{\text{Ho}(\text{Sp}_{S^1})}(\Sigma_{S^1}^\infty |X|_+, \mathbf{BU}) = K_{\text{top}}(\text{sp}(X)). \tag{12}$$

It remains to compare the two Chern maps up to homotopy. The square (10) decomposes into

$$\begin{array}{ccc} \mathbf{K}^{\text{top}}(X) \simeq \mathbb{R}\underline{\text{Hom}}(|X, \mathbf{K}|_{\mathbb{S}}[\beta^{-1}]) & \xrightarrow{\mathbb{R}\underline{\text{Hom}}(|X, \text{Ch}|_{\mathbb{S}})} & \mathbb{R}\underline{\text{Hom}}(|X, \text{HP}^{\text{alg}}|_{\mathbb{S}}) \\ \downarrow & & \downarrow \\ K_{\text{top}}(\text{sp}(X)) = \mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, |\mathbf{K}|_{\mathbb{S}}[\beta^{-1}]) & \xrightarrow{\mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, |\text{Ch}|_{\mathbb{S}})} & \mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, |\text{HP}^{\text{alg}}|_{\mathbb{S}}) \\ & \searrow \mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, \text{Ch}^{\text{top}}) & \downarrow \mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, \mathcal{P}) \\ & & \mathbb{R}\underline{\text{Hom}}(|X|_{\mathbb{S}}, \text{HC}[u^{\pm 1}]) = \text{H}_B(X, \mathbb{C}[u^{\pm 1}]) \end{array}$$

In this diagram, the top square is commutative by functoriality of $|-|_{\mathbb{S}}$. Moreover, the bottom triangle is commutative because it is commutative for $X = \text{Spec}(\mathbb{C})$ by definition of Ch^{top} .

Now let X be a not necessarily smooth \mathbb{C} -scheme of finite type. We consider the cdh topology on the category $\text{Sch}_{\mathbb{C}}$ and the corresponding homotopy category $\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}})^{\text{cdh}})$ defined using cdh-local equivalences. We denote by $\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}}^{\text{liiss}})^{\text{cdh}})$ the smooth version. By using the same argument as in the proof of Proposition 3.22 (mainly the resolution of singularities) we obtain that the restriction and extension functors induce an equivalence of categories

$$\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}})^{\text{cdh}}) \simeq \text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}}^{\text{liiss}})^{\text{cdh}}). \tag{13}$$

Haesemeyer [Hae04, Theorem 6.4] has proven that \mathbf{A}^1 -invariant algebraic K-theory has cdh descent on singular schemes of finite type over \mathbb{C} . Since the topological K-theory $\mathbf{K}^{\text{top}}(\mathcal{L}_{\text{perf}}(X))$ is the Bott inverted topological realization of the \mathbf{A}^1 -invariant algebraic K-theory defined on singular schemes, we have that topological K-theory has cdh descent too. Moreover, $K_{\text{top}}(\text{sp}(-))$ has also cdh descent because of Theorems 3.4 and 3.24 which imply that it has étale descent and proper descent, respectively. Thus we can consider the isomorphism $\mathbf{K}^{\text{top}}(\mathcal{L}_{\text{perf}}(-)) \rightarrow K_{\text{top}}(\text{sp}(-))$ defined above in (12) as a well-defined isomorphism in $\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}}^{\text{liiss}})^{\text{cdh}})$. By the

equivalence (13), it extends uniquely up to an isomorphism in $\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}})^{\text{cdh}})$. This proves the comparison result, and with the same argument we obtain a commutative square (10) of Chern characters in $\text{Ho}(\text{Sp}(\text{Sch}_{\mathbb{C}})^{\text{cdh}})$ (using the fact that periodic cyclic homology has cdh descent by [CHSW08, Corollary 3.13]). The lattice conjecture is then true for dg-categories $T = \mathcal{L}_{\text{perf}}(X)$ with X separated of finite type over \mathbb{C} because it is true for the usual topological Chern character. \square

We can now justify the definition of Deligne cohomology of smooth and proper dg-categories given in the introduction.

DEFINITION 4.33. The *Deligne cohomology of a smooth proper \mathbb{C} -dg-category T* is the symmetric spectrum

$$H_{\mathcal{D}}(T) := \mathbf{K}^{\text{top}}(T) \times_{\text{HP}(T)}^h \text{HN}(T)$$

where the maps defining the homotopy pullback are those of the square (9).

Notation 4.34. We recall the definition of the Deligne cohomology of a smooth and proper \mathbb{C} -scheme X . If A is a \mathbb{Z} -module, for each integer $p \in \mathbb{N}$ we adopt the standard notation $A(p) := (2i\pi)^p A \subseteq \mathbb{C}$. We denote by \underline{A}_X the constant sheaf associated to A on X . The p th Deligne complex of X is by definition the complex of \mathbb{Z} -sheaves

$$\mathbb{Z}(p)_{\mathcal{D}}(X) = (\underline{\mathbb{Z}(p)}_X \longrightarrow \mathcal{O}_X \xrightarrow{d} \Omega_X^1 \xrightarrow{d} \Omega_X^2 \xrightarrow{d} \dots)$$

where Ω_X^q is the sheaf of Kähler q -forms on X , and d is the de Rham differential. The *Deligne cohomology of X* is by definition the complex of \mathbb{Z} -modules

$$H_{\mathcal{D}}^*(X, \mathbb{Z}) := \prod_{p \geq 0} \mathbb{H}^*(X, \mathbb{Z}(p)_{\mathcal{D}}(X))[2p],$$

i.e. the hypercohomology of X with coefficients in the total Deligne complex $\prod_{p \geq 0} \mathbb{Z}(p)_{\mathcal{D}}(X)[2p]$;⁶ see, for example, [EV87, § 1]. This is a reindexed version of the usual Deligne cohomology. The $H^0(H_{\mathcal{D}}^*(X, \mathbb{Z}))$ gives the group $\prod_{p \geq 0} \mathbb{H}^{2p}(X, \mathbb{Z}(p)_{\mathcal{D}}(X))$. We denote by $H_{\mathcal{D}}(X, \mathbb{Z}) := H(H_{\mathcal{D}}^*(X, \mathbb{Z}))$ the associated symmetric spectrum (see § 2.3 for the functor $H : C(\mathbb{Z}) \rightarrow \text{Sp}$).

We recall that the p th Deligne complex of sheaves $\mathbb{Z}(p)_{\mathcal{D}}(X)$ is given by the homotopy quotient

$$\mathbb{Z}(p)_{\mathcal{D}}(X) = \text{Cone}(\underline{\mathbb{Z}(p)}_X \oplus F^p \Omega_X^* \longrightarrow \Omega_X^*)$$

in the derived category $D(X, \mathbb{Z})$ of complex of \mathbb{Z} -sheaves on X , where $F^p \Omega_X^* = \Omega_X^{\geq p}$ means the p th layer of the Hodge filtration on the de Rham complex Ω_X^* , which is just the truncation (see [EV87, § 2.7]). If $E = E^* \in D(X, \mathbb{Z})$ is a cochain complex of \mathbb{Z} -sheaves on X , we adopt the notation $E^*[u^{\pm 1}] = \prod_{i \in \mathbb{Z}} E^{*+2i} = \prod_{i \in \mathbb{Z}} E^*[-2i]$ for the 2-periodic complex associated to E . We also note $H(E[u^{\pm 1}]) =: H(E)[u^{\pm 1}]$.

Because the topological K-theory and the Betti cohomology become isomorphic only with rational coefficients, we will work with Deligne cohomology with rational coefficients which is denoted by $H_{\mathcal{D}}^*(X, \mathbb{Q}) \in C(\mathbb{Q})$ and $H_{\mathcal{D}}(X, \mathbb{Q}) \in \text{Sp}$.

Remark 4.35. Let X be a separated smooth \mathbb{C} -scheme of finite type. We denote by $\text{HP}_*(X)$ (respectively, $\text{HN}_*(X)$) the complex \mathbb{Z} -module which calculates the periodic cyclic homology of the scheme X (respectively, the negative cyclic homology of X), i.e. such that $H(\text{HP}_*(X)) = \text{HP}(X)$ and $H(\text{HN}_*(X)) = \text{HN}(X)$. Recall Remark 2.16 for the Hochschild homology of schemes.

⁶ The use of the shift comes from the comparison with negative cyclic homology.

By the Hochschild–Konstant–Rosenberg theorem (see [HKR62] or [Lod98, Theorem 3.4.4, p. 102], and [Wei97, Example 2.7] for the case of not necessarily affine schemes), there exists an isomorphism

$$\mathrm{HP}_*(X) \simeq \mathbb{H}^{-*}(X, \Omega_X^*)[u^{\pm 1}]$$

in $D(\mathbb{C})$ with the 2-periodic hypercohomology of Ω_X^* which is nothing more than the 2-periodic de Rham cohomology of X . The negative theory is itself related to the Hodge filtration on the de Rham complex. That is, by [Wei97, Theorem 3.3], we have an isomorphism

$$\mathrm{HN}_*(X) \simeq \prod_{p \geq 0} \mathrm{HN}_*^{(p)}(X)$$

in $D(\mathbb{C})$, where for each integer $p \geq 0$, $\mathrm{HN}_*^{(p)}(X)$ is the p th component of the Hodge type decomposition on $\mathrm{HN}_*(X)$, i.e.

$$\mathrm{HN}_*^{(p)}(X) \simeq \mathbb{H}^{-*}(X, F^p \Omega_X^*)[-2p]$$

in $D(\mathbb{C})$.

PROPOSITION 4.36. *Let X be a separated, smooth, and proper \mathbb{C} -scheme of finite type. Then there exists a canonical isomorphism*

$$\mathrm{H}_{\mathcal{D}}(\mathcal{L}_{\mathrm{perf}}(X)) \wedge_{\mathbb{S}} H\mathbb{Q} \simeq \mathrm{H}_{\mathcal{D}}(X, \mathbb{Q})$$

in $\mathrm{Ho}(\mathrm{Sp})$ between the rational Deligne cohomology of the dg-category $\mathcal{L}_{\mathrm{perf}}(X)$ and the Deligne cohomology spectrum of X .

Proof. Because of the expression of the Deligne complexes as cones, for each $p \in \mathbb{Z}$ there exists a pullback/pushout square in the derived category of \mathbb{Q} -sheaves on X denoted by $D(X, \mathbb{Q})$:

$$\begin{array}{ccc} \mathbb{Q}(p)_{\mathcal{D}}(X) & \longrightarrow & F^p \Omega_X^* \\ \downarrow & & \downarrow \\ \underline{\mathbb{Q}}_X & \longrightarrow & \Omega_X^* \end{array}$$

By taking the hypercohomology on X , applying the shift $[-2p]$, and taking the product on all integer p , we obtain a pullback/pushout square

$$\begin{array}{ccc} \mathrm{H}_{\mathcal{D}}^{-*}(X, \mathbb{Q}) & \longrightarrow & \mathrm{HN}_*(X) \\ \downarrow & & \downarrow \\ \mathrm{H}_{\mathbb{B}}^{-*}(X, \mathbb{Q})[u^{\pm 1}] & \longrightarrow & \mathrm{HP}_*(X) \end{array}$$

in $D(\mathbb{Q})$, where we have used Remark 4.35 to identify hypercohomology with negative and periodic cyclic homology, and where $\mathrm{H}_{\mathbb{B}}^{-*}(X, \mathbb{Q})$ is the rational Betti cohomology complex. Using the isomorphism $H(\mathrm{H}_{\mathbb{B}}^{-*}(X, \mathbb{Q})[u^{\pm 1}]) \simeq \mathbf{K}^{\mathrm{top}}(X) \wedge_{\mathbb{S}} H\mathbb{Q}$ in $\mathrm{Ho}(\mathrm{Sp})$ given by Proposition 4.32 and the classical Chern isomorphism, we have a pullback/pushout square

$$\begin{array}{ccc} \mathrm{H}_{\mathcal{D}}(X, \mathbb{Q}) & \longrightarrow & \mathrm{HN}(X) \\ \downarrow & & \downarrow \\ \mathbf{K}^{\mathrm{top}}(X) \wedge_{\mathbb{S}} H\mathbb{Q} & \longrightarrow & \mathrm{HP}(X) \end{array}$$

in $\mathrm{Ho}(\mathrm{Sp})$, which gives the expected isomorphism. □

4.7 Finite-dimensional associative algebras

Throughout this subsection, the base ring is still the complex field \mathbb{C} . By default, all algebras are associative unital \mathbb{C} -algebras. We say that an algebra is finite dimensional if it is so as a \mathbb{C} -vector space. The periodic cyclic homology of an algebra is by convention the periodic cyclic homology relative to the base field \mathbb{C} . Our result on finite-dimensional algebras does not concern the whole spectrum of topological K-theory but what we call ‘pseudo-connective topological K-theory’. Recall that for any dg-category T over \mathbb{C} , the connective semi-topological K-theory (Definition 4.1) $\tilde{\mathbf{K}}^{\text{st}}(T)$ is a **bu**-module. We can therefore invert the Bott generator,

$$\tilde{\mathbf{K}}^{\text{top}}(T) := \tilde{\mathbf{K}}^{\text{st}}(T)[\beta^{-1}].$$

We call this invariant the *pseudo-connective topological K-theory of T* . We will give below a proof of the following proposition concerning the pseudo-connective topological K-theory of a finite-dimensional \mathbb{C} -algebra.

PROPOSITION 4.37. *Let B be a finite-dimensional \mathbb{C} -algebra. Then Conjecture 4.25 is true for B . More precisely, the canonical map*

$$\text{Ch}_{\mathbb{C}}^{\text{top}} \wedge_{\mathbb{S}} \text{HC} : \tilde{\mathbf{K}}^{\text{top}}(B) \wedge_{\mathbb{S}} \text{HC} \longrightarrow \text{HP}(B)$$

is an isomorphism in $\text{Ho}(\text{Sp})$.

Notation 4.38. We work over the étale site $\text{Aff}_{\mathbb{C}}$ of affine \mathbb{C} -schemes of finite type. We have localization functors,

$$\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})) \longrightarrow \text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}}) \longrightarrow \text{Ho}(\text{SPr}^{\text{ét}, \mathbf{A}^1}).$$

The expression ‘étale stack’ stands for an object of the category $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}})$. This category is called the homotopy category of stacks. In §3.4, we defined a classifying space functor

$$\mathbf{B} : \text{Gr}(\text{SPr}(\text{Aff}_{\mathbb{C}})) \longrightarrow \text{SPr}(\text{Aff}_{\mathbb{C}})$$

for strict group objects in $\text{SPr}(\text{Aff}_{\mathbb{C}})$. The classifying stack functor

$$\mathbf{B} : \text{Gr}(\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}}) \longrightarrow \text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}}$$

is defined by $\mathbf{B}(G) = \underline{a}BG$ where \underline{a} is a fibrant replacement functor in $\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}}$. For all presheaves of groups G (simplicially constant) and all $X \in \text{Aff}_{\mathbb{C}}$, the space $\mathbf{B}G(X)$ is equivalent to the nerve of the groupoid of G -torsors over X . In what follows we will call an algebraic stack a 1-geometric stack in the sense of [TVe08, Definition 1.3.3.1] for the standard context of the étale site $\text{Aff}_{\mathbb{C}}$ and the class of morphisms is the class of smooth morphisms of schemes. Therefore we deal with Artin algebraic stacks. We continue to denote by $\text{Aff}_{\mathbb{C}}^{\text{liiss}} \hookrightarrow \text{Aff}_{\mathbb{C}}$ the inclusion of smooth schemes, $l^* : \text{SPr}(\text{Aff}_{\mathbb{C}}) \longrightarrow \text{SPr}(\text{Aff}_{\mathbb{C}}^{\text{liiss}})$ the restriction and $l^* : \text{Sp}(\text{Aff}_{\mathbb{C}}) \longrightarrow \text{Sp}(\text{Aff}_{\mathbb{C}}^{\text{liiss}})$ the restriction for presheaves of spectra.

Different stacks associated to a finite-dimensional algebra. Let B be a finite-dimensional \mathbb{C} -algebra. We consider four stacks associated to B , which are organized in the following square of stacks:

$$\begin{array}{ccc} \text{Vect}^B \hookrightarrow \mathcal{M}^B & & \\ \downarrow & & \downarrow \\ \text{Vect}_B \hookrightarrow \mathcal{M}_B & & \end{array}$$

These stacks are defined in the following way. We first define four presheaves of Waldhausen categories.

- $\underline{\text{Proj}}(B) : \text{Spec}(A) \rightarrow \text{Proj}(B \otimes_{\mathbb{C}} A)$, where $\text{Proj}(B \otimes_{\mathbb{C}} A)$ is the Waldhausen category of right finitely projective $B \otimes_{\mathbb{C}} A$ -modules of finite type.
- $\underline{\text{PsProj}}(B) : \text{Spec}(A) \rightarrow \text{PsProj}(B \otimes_{\mathbb{C}} A)$, where $\text{PsProj}(B \otimes_{\mathbb{C}} A)$ is the Waldhausen category of right $B \otimes_{\mathbb{C}} A$ -modules which are projective of finite type relative to A (i.e. as right A -modules).
- $\underline{\text{Parf}}(B) : \text{Spec}(A) \rightarrow \text{Perf}(B \otimes_{\mathbb{C}} A)$, where $\text{Perf}(B \otimes_{\mathbb{C}} A)$ is the Waldhausen category of cofibrant perfect complexes of right $B \otimes_{\mathbb{C}} A$ -modules (perfect means homotopically finitely presented in the model category of complexes).
- $\underline{\text{PsParf}}(B) : \text{Spec}(A) \rightarrow \text{PsParf}(B \otimes_{\mathbb{C}} A)$, where $\text{PsParf}(B \otimes_{\mathbb{C}} A)$ is the Waldhausen category of cofibrant complexes of right $B \otimes_{\mathbb{C}} A$ -modules which are perfect relative to A (i.e. as complexes of right A -modules).

Taking the nerve of weak equivalences we have by definition:

- $\text{Vect}^B = Nw\underline{\text{Proj}}(B) : \text{Spec}(A) \rightarrow Nw \text{Proj}(B \otimes_{\mathbb{C}} A)$;
- $\text{Vect}_B = Nw\underline{\text{PsProj}}(B) : \text{Spec}(A) \rightarrow Nw \text{PsProj}(B \otimes_{\mathbb{C}} A)$;
- $\mathcal{M}^B = Nw\underline{\text{Parf}}(B) : \text{Spec}(A) \rightarrow Nw \text{Perf}(B \otimes_{\mathbb{C}} A)$;
- $\mathcal{M}_B = Nw\underline{\text{PsParf}}(B) : \text{Spec}(A) \rightarrow Nw \text{PsParf}(B \otimes_{\mathbb{C}} A)$.

As we are working over a field, B is locally cofibrant as a dg-category, and therefore the tensor product written previously are in fact derived. The stack \mathcal{M}_B is studied by Toën and Vaquié in [TVa07]. We remark that since B is finite dimensional, it is proper as a dg-category (in the sense of [TVa07, Definition 2.4]). By [TVa07, Lemma 2.8], a perfect complex of right $B \otimes_{\mathbb{C}} A$ -modules is therefore perfect relative to A . Hence we have monomorphisms $\text{Vect}^B \hookrightarrow \text{Vect}_B$ and $\mathcal{M}^B \hookrightarrow \mathcal{M}_B$. The monomorphism $\text{Vect}^B \hookrightarrow \mathcal{M}^B$ and $\text{Vect}_B \hookrightarrow \mathcal{M}_B$ are inclusion of degree-zero concentrated complexes. These four stacks admit a homotopy coherent commutative monoid structure given by direct sum of modules and dg-modules, respectively. We apply the functor B_W (defined in § 2.1) and we obtain special Γ -objects in $\text{SPr}(\text{Aff}_{\mathbb{C}})$, Vect_{\bullet}^B , Vect_{\bullet}^B , \mathcal{M}_{\bullet}^B , \mathcal{M}_{\bullet}^B whose level 1 are respectively Vect^B , Vect_B , \mathcal{M}^B , \mathcal{M}_B .

The stack Vect^B is an algebraic stack which is locally finitely presented and smooth (see, for example, [TVa07, § 1]). It admits the following description in terms of residual gerbes on global points; there exists an isomorphism in $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}})^{\text{ét}})$,

$$\text{Vect}^B \simeq \coprod_{M \in \chi} \underline{\text{BAut}}(M), \tag{14}$$

where $\chi = \pi_0 \text{Vect}^B(\mathbb{C})$ is the set of isomorphism classes of projective right B -modules of finite type, and for all $M \in \chi$, $\underline{\text{Aut}}(M)$ is the group scheme of B -automorphisms of M . This formula can be derived from a more general formula valid for all algebraic stacks locally finitely presented over \mathbb{C} , whose tangent complex at every point satisfies a finiteness property.

PROPOSITION 4.39. *Let F be a locally finitely presented smooth algebraic stack over \mathbb{C} such that for all global points $E \in \pi_0 F(\mathbb{C})$ the tangent complex $\mathbb{T}_E F$ satisfies $H^0(\mathbb{T}_E F) = 0$. Then the canonical morphism of algebraic stacks,*

$$\coprod_{E \in \pi_0 F(\mathbb{C})} \mathcal{G}_E \rightarrow F,$$

is an equivalence of stacks, where \mathcal{G}_E stands for the residual gerbe of F at E .

Proof. The morphism is immediately seen to be locally finitely presented. It is a monomorphism of algebraic stacks and therefore a representable morphism. The assumption on the tangent complexes implies that the morphism induces an equivalence on tangent complexes and is therefore an étale morphism of algebraic stacks. Hence it is an open immersion of algebraic stacks. The morphism is an epimorphism in the π_0 because it is surjective on the complex points of the π_0 . It is therefore an epimorphism of stacks. We deduce that it is an equivalence. \square

The stacks Vect^B satisfy the assumption of Proposition 4.39 because for all $M \in \chi$, we have a quasi-isomorphism $\mathbb{T}_M \text{Vect}^B \simeq \underline{\text{End}}(M)[1]$ where $\underline{\text{End}}(M)$ stands for the \mathbb{C} -vector space of endomorphisms of M . This gives formula (14).

Topological K-theory of a finite-dimensional algebra. Recall that in §4.3 we proved that the connective semi-topological K-theory of a dg-category T can be described in terms of the topological realization of the stack \mathcal{M}^T endowed with its homotopy coherent commutative monoid structure. In other words, we have a natural equivalence in T ,

$$\tilde{\mathbf{K}}^{\text{st}}(T) \simeq \mathcal{B}|\mathcal{M}_{\bullet}^T|_{\Gamma}.$$

Now if $T = B$ is an algebra, after Remark 2.8, the connective algebraic K-theory of B can be calculated with the presheaf of categories $\text{Proj}(B)$ endowed with its homotopy coherent commutative monoid structure. Using similar arguments to those in the proof of Proposition 4.5, we have canonical isomorphisms

$$\tilde{\mathbf{K}}^{\text{st}}(B) \simeq |\underline{\mathbf{K}}(B)|_{\mathbb{S}} \simeq |\mathcal{B}K^{\Gamma}(\text{Proj}(B))|_{\mathbb{S}} \simeq \mathcal{B}|K^{\Gamma}(\text{Proj}(B))|_{\Gamma} \simeq \mathcal{B}|\text{Vect}_{\bullet}^B|_{\Gamma}^+ \tag{15}$$

in $\text{Ho}(\text{Sp})$. Therefore we have canonical isomorphisms

$$\tilde{\mathbf{K}}^{\text{st}}(B) \simeq \mathcal{B}|\mathcal{M}_{\bullet}^B|_{\Gamma} \simeq \mathcal{B}|\text{Vect}_{\bullet}^B|_{\Gamma}^+ \tag{16}$$

in $\text{Ho}(\text{Sp})$, and

$$|\mathcal{M}_{\bullet}^B|_{\Gamma} \simeq |\text{Vect}_{\bullet}^B|_{\Gamma}^+ \tag{17}$$

in $\text{Ho}(\Gamma - \text{SSet})$.

Proof of Proposition 4.37. If B is semi-simple, then B satisfies Proposition 4.37 for the following reasons. By the theory of representations of semi-simple algebras, B is Morita equivalent to a finite product of copies of the unit algebra $\mathbb{1}$ (i.e. of copies of \mathbb{C}). Therefore there exists a Morita equivalence $B \xrightarrow{\sim} \prod_J \mathbb{1}$ in $\text{dgCat}_{\mathbb{C}}$ with J a finite set. Pseudo-connective topological K-theory and periodic homology commute both with finite products, and we thus have canonical isomorphisms $\tilde{\mathbf{K}}^{\text{top}}(B) \xrightarrow{\sim} \prod_J \tilde{\mathbf{K}}^{\text{top}}(\mathbb{1}) \simeq \prod_J \mathbf{BU}$ and $\text{HP}(B) \xrightarrow{\sim} \prod_J \text{HP}(\mathbb{1}) \simeq \prod_J \text{HC}[u^{\pm 1}]$ in $\text{Ho}(\text{Sp})$. We then have a commutative square

$$\begin{array}{ccc} \tilde{\mathbf{K}}^{\text{top}}(B) & \xrightarrow{\text{Ch}_c^{\text{top}}} & \text{HP}(B) \\ \downarrow \wr & & \downarrow \wr \\ \prod_J \mathbf{BU} & \xrightarrow{\prod_J \text{Ch}_c^{\text{top}}} & \prod_J \text{HC}[u^{\pm 1}] \end{array}$$

in $\text{Ho}(\text{Sp})$. The bottom map is an isomorphism after $\wedge_{\mathbb{S}} \text{HC}$. We deduce that the top map is an isomorphism after $\wedge_{\mathbb{S}} \text{HC}$.

Now if B is not semi-simple, its radical $\text{rad}(B)$ is a nilpotent two-sided ideal of B . We denote by $B_0 = B/\text{rad}(B)$ the smallest semi-simple quotient of B . To prove that B satisfies Proposition 4.37 it suffices to show that the map induced by base change $\text{Vect}^B \rightarrow \text{Vect}^{B_0}$ is an \mathbf{A}^1 -equivalence in $\text{SPr}^{\text{ét}, \mathbf{A}^1}$. Indeed, such an \mathbf{A}^1 -equivalence implies that the map of special Γ -objects $\text{Vect}_{\bullet}^B \rightarrow \text{Vect}_{\bullet}^{B_0}$ is an isomorphism in $\text{Ho}(\Gamma - \text{SPr}^{\text{ét}, \mathbf{A}^1})$. Taking topological realization, we deduce that the map $B \rightarrow B_0$ induces an isomorphism $\tilde{\mathbf{K}}^{\text{top}}(B) \simeq \tilde{\mathbf{K}}^{\text{top}}(B_0)$ in $\text{Ho}(\text{Sp})$. On the other hand, Goodwillie proved the invariance of periodic cyclic homology under infinitesimal extension [Goo85, Theorem 2.5.1]); indeed, the map $B \rightarrow B_0$ induces an isomorphism $\text{HP}(B) \simeq \text{HP}(B_0)$ in $\text{Ho}(\text{Sp})$. We therefore reduce the proof to the semi-simple case.

It suffices thus to show that for all finite-dimensional \mathbb{C} -algebras B and all nilpotent two-sided ideals $I \subseteq B$ the projection map $B \rightarrow B/I$ induces an \mathbf{A}^1 -equivalence $\text{Vect}^B \rightarrow \text{Vect}^{B/I}$. By a classical recurrence argument on the nilpotency degree of I it suffices to show the assertion for a square-zero two-sided ideal. Indeed, suppose that for all algebras C and all nilpotent two-sided ideals J of C , the projection map $C \rightarrow C/J$ induces an \mathbf{A}^1 -equivalence $\text{Vect}^C \rightarrow \text{Vect}^{C/J}$. If I is a two-sided ideal of B with nilpotency degree $n \geq 2$, we consider the following commutative square of algebras:

$$\begin{array}{ccc} B & \longrightarrow & B/I^2 = C \\ & \searrow & \downarrow \\ & & B/I \simeq C/I \end{array}$$

Since I^2 is a two-sided ideal of B with nilpotency degree $\leq n - 1$ and the two-sided ideal $I \cdot C \subseteq C$ generated by the image of I in C satisfies $I \cdot C^2 = 0$, the arrows $B \rightarrow C$ and $C \rightarrow C/I$ induce \mathbf{A}^1 -equivalences on Vect . We deduce that $B \rightarrow B/I$ also induces an \mathbf{A}^1 -equivalence on Vect .

Now let $I \subseteq B$ be a square-zero two-sided ideal in a finite-dimensional \mathbb{C} -algebra B . We have to show that $B \rightarrow B/I = B_0$ induces an \mathbf{A}^1 -equivalence on Vect . Using formula (14), we deduce that the map $\text{Vect}^B \rightarrow \text{Vect}^{B_0}$ is equivalent to the map

$$\coprod_{M \in \chi} \mathbf{BAut}(M) \rightarrow \coprod_{M \in \chi_0} \mathbf{BAut}(M_0),$$

where $\chi = \pi_0 \text{Vect}^B(\mathbb{C})$, $\chi_0 = \pi_0 \text{Vect}^{B_0}(\mathbb{C})$, $M_0 = M \otimes_B B_0$ for all $M \in \chi$. First, we observe that the sets χ and χ_0 are isomorphic as shown in [Bas68, Proposition 2.12] (completeness assumptions are satisfied because $I^2 = 0$). Since the \mathbf{A}^1 -equivalences are stable by arbitrary sums in $\text{SPr}^{\text{ét}, \mathbf{A}^1}$, it remains to show that for all $M \in \chi$, the map $\mathbf{BAut}(M) \rightarrow \mathbf{BAut}(M_0)$ is an \mathbf{A}^1 -equivalence. This latter map is the image by the functor \mathbf{B} of the map of group schemes $\underline{\text{Aut}}(M) \rightarrow \underline{\text{Aut}}(M_0)$ whose kernel is denoted by K . We write M as a direct factor of a free B -module $B^r = M \oplus N$, where N is a sub-right- B -module of B^r and r a positive integer. We have $B_0^r = M_0 \oplus N_0$. If M is free, the automorphism group is the invertible matrix group $\underline{\text{Aut}}(B^r) = \underline{\text{GL}}_r(B)$. The kernel of the map $\underline{\text{GL}}_r(B) \rightarrow \underline{\text{GL}}_r(B_0)$ is given by the multiplicative group $I_r + \underline{\mathbf{M}}_r(I)$ where I_r the identity matrix of rank r and $\underline{\mathbf{M}}_r(I)$ the additive group scheme of matrices with coefficients in I . The scheme $I_r + \underline{\mathbf{M}}_r(I)$ is isomorphic to an affine space (of dimension $\dim_{\mathbb{C}}(I) \cdot r^2$) and is therefore \mathbf{A}^1 -contractible. If M is just projective, we have a diagram with exact rows

$$\begin{array}{ccccccccc} 1 & \longrightarrow & K & \longrightarrow & \underline{\text{Aut}}(M) & \xrightarrow{g} & \underline{\text{Aut}}(M_0) & \longrightarrow & 1 \\ & & \downarrow k & & \downarrow i & & \downarrow j & & \\ 1 & \longrightarrow & I_r + \underline{\mathbf{M}}_r(I) & \longrightarrow & \underline{\text{GL}}_r(B) & \xrightarrow{f} & \underline{\text{GL}}_r(B_0) & \longrightarrow & 1 \end{array}$$

The map i sends an automorphism of M to the automorphism of B^r which is the identity on N . The map j is defined in the same way. These two maps are closed immersions, and the corresponding square is commutative, i.e. $fi = jg$. We deduce the existence of the map k , which is also a closed immersion. The kernel K corresponds to the sub-group scheme of \mathbb{C} -automorphisms of B^r which are B -linear, restrict to the identity on N , and which are in $I_r + \underline{M}_r(I)$. These three conditions can be translated into affine equations in the affine space $I_r + \underline{M}_r(I)$, and we deduce that K is itself isomorphic to an affine space of a certain dimension, and is therefore \mathbf{A}^1 -contractible. This implies that the classifying stack \mathbf{BK} is also \mathbf{A}^1 -contractible, i.e. isomorphic to the point $*$ in $\mathrm{Ho}(\mathrm{SPr}^{\acute{e}t, \mathbf{A}^1})$. The functor classifying stack \mathbf{B} sends exact sequences of group schemes to fibration sequences. We therefore have a fibration sequence in $\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})^{\acute{e}t}_*)$,

$$\mathbf{BK} \longrightarrow \mathbf{BAut}(M) \longrightarrow \mathbf{BAut}(M_0).$$

Given a group stack G , it is known that there exists a Quillen equivalence between the model category of G -equivariant stacks and the model category of stacks over \mathbf{BG} (see [KPT09, Lemma 3.20]). The latter result implies the existence of an isomorphism of stacks over $\mathbf{BAut}(M_0)$,

$$\begin{array}{ccc} \mathbf{BAut}(M) & \xrightarrow{\sim} & [\mathbf{BK}/\mathbf{Aut}(M_0)] \\ & \searrow & \swarrow \\ & \mathbf{BAut}(M_0) & \end{array}$$

in $\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})^{\acute{e}t}/\mathbf{BAut}(M_0))$, for a certain action of $\mathbf{Aut}(M_0)$ on \mathbf{BK} , and the notation $[-/-]$ stands for the quotient stack. Because \mathbf{BK} is contractible in $\mathrm{Ho}(\mathrm{SPr}^{\acute{e}t, \mathbf{A}^1})$ and because the localization functor $\mathrm{Ho}(\mathrm{SPr}(\mathrm{Aff}_{\mathbb{C}})^{\acute{e}t}) \rightarrow \mathrm{Ho}(\mathrm{SPr}^{\acute{e}t, \mathbf{A}^1})$ commutes with the operation of quotient, the latter triangle is isomorphic to

$$\begin{array}{ccc} \mathbf{BAut}(M) & \xrightarrow{\sim} & [*/\mathbf{Aut}(M_0)] = \mathbf{BAut}(M_0) \\ & \searrow & \swarrow \text{id} \\ & \mathbf{BAut}(M_0) & \end{array}$$

in $\mathrm{Ho}(\mathrm{SPr}^{\acute{e}t, \mathbf{A}^1})$. The map $\mathbf{BAut}(M) \rightarrow \mathbf{BAut}(M_0)$ is therefore an isomorphism in $\mathrm{Ho}(\mathrm{SPr}^{\acute{e}t, \mathbf{A}^1})$. This completes the proof that $\mathrm{Vect}^B \rightarrow \mathrm{Vect}^{B/I}$ is an \mathbf{A}^1 -equivalence for all square-zero two-sided ideals I and thus of Proposition 4.37. \square

Remark that in the proof of Proposition 4.37, we proved the invariance of connective semi-topological K-theory by infinitesimal extension, which is expressed by the following result.

PROPOSITION 4.40. *Let B be a finite-dimensional associative algebra over \mathbb{C} and I a right nilpotent ideal of B . Then the projection map $B \rightarrow B/I$ induces an isomorphism $\tilde{\mathbf{K}}^{\mathrm{st}}(B) \simeq \tilde{\mathbf{K}}^{\mathrm{st}}(B/I)$ in $\mathrm{Ho}(\mathrm{Sp})$.*

Remark 4.41. Proposition 4.37 and formula (16) allow us to express the periodic cyclic homology of a finite-dimensional algebra in terms of the infinite loop space $(|\mathrm{Vect}_{\Gamma}^B|_+^{\bullet})_1 =: |\mathrm{Vect}^B|_+^{\bullet}$. This group completion has the following description. The monoid $\pi_0|\mathrm{Vect}^B|$ being in general not isomorphic to \mathbb{N} with the usual addition (as it is for B commutative for example), the calculation of this group completion is a bit more complicated than in the commutative case, because we cannot apply the Quillen result on the homology of the group completion. Since every projective

right B -module of finite type is a direct factor of a free B -module of finite type, in order to ‘group complete’ the direct sum in $|\mathbf{Vect}^B|$, it suffices to invert the action of the regular B -module B . Hence we have an isomorphism

$$|\mathbf{Vect}^B|^+ \simeq |\mathbf{Vect}^B|[-B]$$

in $\mathrm{Ho}(\mathbf{SSet})$, where the latter object is level 1 of the localization in the sense of Γ -spaces of the special Γ -space $|\mathbf{Vect}^B|$ with respect to the B -module B . We now want to calculate this localization in terms of the standard colimit

$$\mathrm{hocolim}(|\mathbf{Vect}^B| \xrightarrow{\oplus B} |\mathbf{Vect}^B| \xrightarrow{\oplus B} |\mathbf{Vect}^B| \xrightarrow{\oplus B} \dots) =: |\mathbf{Vect}^B|^{ST},$$

where the map $\oplus B$ is induced by the endomorphism of $\mathbf{Parf}(B)$ which sends a B -dg-module E on $E \oplus B$ and the notation ‘stable’ is in reference to Loday–Quillen stable homology of Lie algebras of matrices. To achieve this formula, we need the language of Lurie’s monoidal ∞ -groupoids rather than Γ -spaces; because of this, we will willingly stay vague with respect to the definitions we use. Hence we consider $|\mathbf{Vect}^B|$ as a symmetric monoidal ∞ -groupoid (whose monoidal law is given by the direct sum of modules) and we want to invert the object B with respect to the sum. For this we apply [Rob12, Corollary 4.24] (which works more generally for all presentable symmetric monoidal ∞ -categories; see [Rob12, Remark 4.7] and [Rob12, Remark 4.26] for the ∞ -groupoid case). Then there exists a canonical isomorphism

$$|\mathbf{Vect}^B|[-B] \simeq |\mathbf{Vect}^B|^{ST}$$

in $\mathrm{Ho}(\mathbf{SSet})$, provided we show that B is a symmetric object in the sense of [Rob12, Definition 4.18]. We have to show that there exists a homotopy between the map

$$B \oplus B \oplus B \xrightarrow{(123)} B \oplus B \oplus B$$

and the identity of $B \oplus B \oplus B$ in the space of complex points of $\mathbf{GL}_3(B)$, where (123) is the automorphism induced by the cyclic permutation (123). Such a homotopy is given by the composite of a homotopy $(123) \Rightarrow \mathrm{id}$ in $|\mathbf{GL}_3(\mathbb{C})|$ with the canonical map $|\mathbf{GL}_3(\mathbb{C})| \rightarrow |\mathbf{GL}_3(B)|$ induced by the structural morphism $\mathbb{C} \rightarrow B$.

In conclusion, there exists an isomorphism

$$|\mathbf{Vect}^B|^+ \simeq |\mathbf{Vect}^B|^{ST}$$

in $\mathrm{Ho}(\mathbf{SSet})$ between the group completion of $|\mathbf{Vect}^B|$ and the stabilization of $|\mathbf{Vect}^B|$ (where the colimit is a homotopy one). Taking homotopy groups of the formula of Proposition 4.37, we obtain the following corollary.

COROLLARY 4.42. *Let B be a finite-dimensional associative \mathbb{C} -algebra. Then for all $i \geq 0$ the Chern map $\tilde{\mathbf{K}}^{\mathrm{top}}(B) \rightarrow \mathrm{HP}(B)$ induces an isomorphism of \mathbb{C} -vector spaces,*

$$\mathrm{colim}_{k \geq 0} \pi_{i+2k} |\mathbf{Vect}^B|^{ST} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \mathrm{HP}_i(B),$$

where the colimit is induced by the action of the Bott generator β on homotopy groups, $\pi_i |\mathbf{Vect}^B|^{ST} \xrightarrow{\times \beta} \pi_{i+2} |\mathbf{Vect}^B|^{ST}$.

Consequences in the smooth case. Let B be a finite-dimensional associative \mathbb{C} -algebra which is furthermore of finite global dimension. This extra assumption means precisely that B is smooth in the sense of dg-categories [TVa07, Definition 2.4]. If $\text{Spec}(A) \in \text{Aff}_{\mathbb{C}}$ is *smooth*, then by the Quillen resolution theorem [Qui, §4, Corollary 1], the connective algebraic K-theory of $\text{Proj}(B \otimes_{\mathbb{C}} A)$ is the same as the connective algebraic K-theory of $\text{PsProj}(B \otimes_{\mathbb{C}} A)$ and also the same as the connective algebraic K-theory of all right $B \otimes_{\mathbb{C}} A$ -modules of finite type. Indeed, since $B \otimes_{\mathbb{C}} A$ is smooth, all right $B \otimes_{\mathbb{C}} A$ -modules of finite type have a finite projective resolution relative to $B \otimes_{\mathbb{C}} A$. We therefore have a global equivalence of presheaves of spectra restricted to $\text{Aff}_{\mathbb{C}}^{\text{liiss}}$,

$$l^* \tilde{\mathbf{K}}(\text{Proj}(B)) \simeq l^* \tilde{\mathbf{K}}(\text{PsProj}(B)). \tag{18}$$

By Theorem 3.18 we thus have a canonical isomorphism in $\text{Ho}(\text{Sp})$,

$$\tilde{\mathbf{K}}^{\text{st}}(B) \simeq |\tilde{\mathbf{K}}(\text{Proj}(B))|_{\mathbb{S}} \simeq |\tilde{\mathbf{K}}(\text{PsProj}(B))|_{\mathbb{S}}. \tag{19}$$

By the foregoing, we have the following proposition.

PROPOSITION 4.43. *Let B be a finite-dimensional associative \mathbb{C} -algebra of finite global dimension. Then we have canonical isomorphisms in $\text{Ho}(\text{Sp})$,*

$$\tilde{\mathbf{K}}^{\text{st}}(B) \simeq \mathcal{B}|\text{Vect}_{\bullet}^B|_{\Gamma}^+ \simeq \mathcal{B}|\text{Vect}_B^{\bullet}|_{\Gamma}^+ \simeq \mathcal{B}|\mathcal{M}_{\bullet}^B|_{\Gamma} \simeq \mathcal{B}|\mathcal{M}_B^{\bullet}|_{\Gamma},$$

and therefore canonical isomorphisms in $\text{Ho}(\text{SSet})$,

$$|\text{Vect}^B|^+ \simeq |\text{Vect}_B|^+ \simeq |\mathcal{M}^B| \simeq |\mathcal{M}_B|,$$

where the first two objects are the level 1 of the corresponding group completion.

Proof. We have already proved the isomorphism $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq \mathcal{B}|\text{Vect}_{\bullet}^B|_{\Gamma}^+$ in $\text{Ho}(\text{Sp})$ with formula (15). By formula (19), there is an isomorphism $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq |\tilde{\mathbf{K}}(\text{PsProj}(B))|_{\mathbb{S}}$ in $\text{Ho}(\text{Sp})$. Proceeding as in the proof of 4.5, with formula (15) and by the end of Remark 2.6 applied to the Waldhausen category of pseudo-projective modules which has split cofibrations, we have an isomorphism $|\tilde{\mathbf{K}}(\text{PsProj}(B))|_{\mathbb{S}} \simeq \mathcal{B}|\text{Vect}_B^{\bullet}|_{\Gamma}^+$. We thus have an isomorphism $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq |\text{Vect}_B^{\bullet}|_{\Gamma}^+$. The isomorphism $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq \mathcal{B}|\mathcal{M}_{\bullet}^B|_{\Gamma}$ is Theorem 4.21. The Gillet and Waldhausen theorem ([TT90, Theorem 1.11.7]; see Remark 2.8 above) gives an isomorphism $\tilde{\mathbf{K}}(\text{PsProj}(B)) \simeq \tilde{\mathbf{K}}(\text{PsParf}(B))$ in $\text{Ho}(\text{SPr}(\text{Aff}_{\mathbb{C}}))$. By Theorem 4.22, we obtain the isomorphisms $\tilde{\mathbf{K}}^{\text{st}}(B) \simeq |\tilde{\mathbf{K}}(\text{PsProj}(B))|_{\mathbb{S}} \simeq |\tilde{\mathbf{K}}(\text{PsParf}(B))|_{\mathbb{S}} \simeq \mathcal{B}|\mathcal{M}_B^{\bullet}|_{\Gamma}$. The second part of the theorem follows directly because \mathcal{B} is an equivalence. \square

As in the case of not necessarily smooth associative algebras, we have isomorphisms

$$|\text{Vect}_B|^+ \simeq \text{hocolim}(|\text{Vect}_B| \xrightarrow{\oplus B} |\text{Vect}_B| \xrightarrow{\oplus B} |\text{Vect}_B| \xrightarrow{\oplus B} \dots) =: |\text{Vect}_B|^{\text{ST}}$$

in $\text{Ho}(\text{SSet})$ (see Remark 4.41). We deduce from this the following corollary of Proposition 4.37.

COROLLARY 4.44. *Let B be a finite-dimensional associative \mathbb{C} -algebra of finite global dimension. Then for all $i \geq 0$ the Chern map $\tilde{\mathbf{K}}^{\text{top}}(B) \rightarrow \text{HP}(B)$ induces an isomorphism of \mathbb{C} -vector spaces,*

$$\text{colim}_{k \geq 0} \pi_{i+2k} |\text{Vect}_B|^{\text{ST}} \otimes_{\mathbb{Z}} \mathbb{C} \simeq \text{HP}_i(B)$$

where the colimit is induced by the action of the Bott generator β on homotopy groups, $\pi_i |\text{Vect}_B|^{\text{ST}} \xrightarrow{\times \beta} \pi_{i+2} |\text{Vect}_B|^{\text{ST}}$.

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