Proceedings of the Edinburgh Mathematical Society (2013) **56**, 723–732
DOI:10.1017/S0013091513000400

LIMITING CASES OF BOARDMAN'S FIVE HALVES THEOREM

MICHAEL C. CRABB[1] AND PEDRO L. Q. PERGHER[2]

[1] *Institute of Mathematics, University of Aberdeen, Aberdeen AB24 3UE, UK* (m.crabb@abdn.ac.uk)

[2] *Departamento de Matemática, Universidade Federal de São Carlos, Caixa Postal 676, São Carlos, SP 13565-905, Brazil* (pergher@dm.ufscar.br)

(Received 21 June 2011)

Abstract The famous five halves theorem of Boardman states that, if $T\colon M^m \to M^m$ is a smooth involution defined on a non-bounding closed smooth m-dimensional manifold M^m $(m > 1)$ and if

$$F = \bigcup_{j=0}^{n} F^j \quad (n \leqslant m)$$

is the fixed-point set of T, where F^j denotes the union of those components of F having dimension j, then $2m \leqslant 5n$. If the dimension m is written as $m = 5k - c$, where $k \geqslant 1$ and $0 \leqslant c < 5$, the theorem states that the dimension n of the fixed submanifold is at least $\beta(m)$, where $\beta(m) = 2k$ if $c = 0, 1, 2$ and $\beta(m) = 2k - 1$ if $c = 3, 4$. In this paper, we give, for each $m > 1$, the equivariant cobordism classification of involutions (M^m, T), for which the fixed submanifold F attains the minimal dimension $\beta(m)$.

Keywords: five halves theorem; involution; fixed-point data; equivariant cobordism class; Stiefel–Whitney class; characteristic number

2010 *Mathematics subject classification:* Primary 57R85
Secondary 57R75

1. Introduction

Throughout this paper M^m denotes a closed smooth m-dimensional manifold and $T\colon M^m \to M^m$ is a smooth involution on M^m with fixed subset F expressed as a union of submanifolds

$$F = \bigcup_{j=0}^{m} F^j,$$

where F^j denotes the union of those components of F having dimension j. We write η_j for the $(m-j)$-dimensional normal bundle of F^j in M^m. The list $((F^j, \eta_j))_{j=0}^{m}$, in which we may omit the jth term if $F^j = \emptyset$, is referred to as the *fixed-point data* of (M^m, T).

The famous five halves theorem of Boardman, see [**1**], asserts that if M^m is non-bounding and F^j is empty for $j > n$, where $n \leqslant m$, then $m \leqslant \frac{5}{2}n$. For fixed n, this gives an upper bound on the dimension m, namely, if $n = 2k$ is even $(k \geqslant 1)$, then $m \leqslant 5k$, and if $n = 2k - 1$ is odd $(k \geqslant 1)$, then $m \leqslant 5k - 3$. Furthermore, these bounds are best possible: Boardman exhibited, for each $n \geqslant 1$, examples of involutions (M^m, T) with M^m non-bounding and m attaining the maximal value allowed by the theorem. A strengthened version of Boardman's result was obtained in [**4**] by Stong and Kosniowski, who established the same conclusion under the weaker hypothesis that (M^m, T) is a non-bounding involution. Since the equivariant cobordism class of (M^m, T) is determined by the cobordism class of the normal bundle of F in M^m (see [**3**]), this implies, in particular, that if at least one F^j is non-bounding, then $2m \leqslant 5n$.

Kosniowski and Stong also gave, in [**4**], an improvement of the theorem when $F = F^n$ has constant dimension n: if (M^m, T) is a non-bounding involution, then $m \leqslant 2n$. For each fixed n, with the exception of the dimensions $n = 1$ and $n = 3$, the maximal value $m = 2n$ is achieved by taking the involution $(F^n \times F^n, T)$, where F^n is any non-bounding n-dimensional manifold and T is the *twist* involution $T(x, y) = (y, x)$. Moreover, Kosniowski and Stong showed that every example is of this form up to $\mathbb{Z}_2$-equivariant cobordism: if $m = 2n$ and $F^j = \emptyset$ for $j \neq n$, then (M^m, T) is equivariantly cobordant to $(F^n \times F^n, \mathrm{twist})$. From a different perspective, we can fix the dimension m and look at the least value of n satisfying the condition $m \leqslant 2n$, that is, $n = k$ if m is written as $2k$ or $2k-1$, with $k \geqslant 1$. For even $m = 2k$, the result of Kosniowski and Stong gives the equivariant cobordism classification of involutions (M^m, T) with fixed-point set of constant dimension $n = k$ as the group $\{[(F^k \times F^k, \mathrm{twist})] \colon [F^k] \in \mathcal{N}_k\} \cong \mathcal{N}_k$, where, as usual, $\mathcal{N}_k$ is the k-dimensional unoriented cobordism group. For odd $m = 2k-1$, the corresponding, more complicated, classification was given by Stong in [**8**].

Motivated by these results, we obtain, for each $m \geqslant 1$, the cobordism classification of involutions (M^m, T) such that the top-dimensional component of the fixed subset F has the least value n satisfying Boardman's condition $2m \leqslant 5n$.

Definition 1.1. We denote by $\mathcal{N}_m^{\mathbb{Z}_2}$ the unoriented cobordism group of pairs (M^m, T), where M^m is a closed smooth m-dimensional manifold and T is a smooth involution defined on M^m. In terms of the notation for the fixed subset introduced above, we define $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$, for $0 \leqslant n \leqslant m$, to be the subgroup of $\mathcal{N}_m^{\mathbb{Z}_2}$ consisting of those cobordism classes $[(M^m, T)]$ such that η_j bounds as a bundle for $j > n$. (From the proof of the Conner–Floyd exact sequence of [**3**], every element of $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$ can be represented by a pair (M^m, T) such that F^j is empty for $j > n$.)

With this terminology, we can state a weak form of our main result.

Theorem 1.2. *Write $m = 5k - c$, where $k \geqslant 1$ and $0 \leqslant c \leqslant 4$, and set $\beta(m) = 2k$ if $c = 0, 1, 2$ and $\beta(m) = 2k - 1$ if $c = 3, 4$. Then, $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)} = 0$ if $n < \beta(m)$ and the dimension* $\dim (\mathcal{N}_m^{\mathbb{Z}_2})^{(\beta(m))}$ *is given, according to the values of c, by*

$$\begin{aligned}
c = 0&: \quad 1,\\
c = 1&: \quad 3 \text{ if } k = 1,\ 4 \text{ if } k \geqslant 2,\\
c = 2&: \quad 1 \text{ if } k = 1,\ 9 \text{ if } k = 2,\ 12 \text{ if } k = 3,\ 13 \text{ if } k \geqslant 4,\\
c = 3&: \quad 1,\\
c = 4&: \quad 0 \text{ if } k = 1,\ 4 \text{ if } k = 2,\ 6 \text{ if } k \geqslant 3.
\end{aligned}$$

https://doi.org/10.1017/S0013091513000400 Published online by Cambridge University Press

Moreover, multiplication by the generator b of $(\mathcal{N}_5^{\mathbb{Z}_2})^{(2)}$ *defines an injective map*

$$b\cdot\colon (\mathcal{N}_m^{\mathbb{Z}_2})^{(\beta(m))} \to (\mathcal{N}_{m+5}^{\mathbb{Z}_2})^{(\beta(m)+2)},$$

which is an isomorphism for all but finitely many dimensions m, namely, 1, 3, 4, 6, 8, 13.

Note that the cases $c = 0$ and $c = 3$ of the theorem say that the maximal examples of Boardman (for $(m, n) = (5k, 2k)$ and $(5k-3, 2k-1)$) are unique up to cobordism.

In § 3, we establish a more precise classification theorem, in which we give explicit bases for the vector spaces $(\mathcal{N}_m^{\mathbb{Z}_2})^{(\beta(m))}$. Our strategy consists in first showing that a suitable extension of the argument used by Kosniowski and Stong in [**4**] to prove the stronger Boardman Theorem can be used to show that, in the relevant dimensions ($n \leqslant \beta(m)$), few characteristic numbers can be non-zero. This will give bounds for the $\mathbb{Z}_2$-dimensions. The argument is then completed by constructing sets of linearly independent cobordism classes of involutions realizing these bounds.

2. Preliminaries

In this section, we review various standard results and notation that we need for the proof of the classification theorem. Unoriented bordism theory is denoted by $\mathcal{N}_*(-)$, with coefficient ring $\mathcal{N}_*$, so that, in particular, $\mathcal{N}_n(\mathrm{BO}(k))$ is the cobordism group of k-dimensional real vector bundles over closed n-dimensional manifolds.

The $\mathbb{Z}_2$-equivariant bordism group $\mathcal{N}_m^{\mathbb{Z}_2}$ is described in terms of non-equivariant bordism by the fundamental Conner–Floyd exact sequence [**3**]

$$0 \to \mathcal{N}_m^{\mathbb{Z}_2} \to \bigoplus_{0\leqslant j\leqslant m} \mathcal{N}_j(\mathrm{BO}(m-j)) \xrightarrow{\partial_m} \mathcal{N}_{m-1}(\mathrm{BO}(1)) \to 0,$$

which maps the cobordism class of the involution (M^m, T) to the cobordism class of its fixed-point data $([F^j, \eta_j])$. The boundary map ∂_m assigns to $[F^j, \eta_j]$ the class of the real projective space bundle $\mathbb{R}P(\eta_j)$ over F^j with the classifying map of the Hopf line bundle $\lambda \to \mathbb{R}P(\eta_j)$.

Lemma 2.1. *For* $0 \leqslant n \leqslant m$, *the group* $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$ *can be identified with the kernel of the restricted boundary map*

$$\partial_m|\colon \bigoplus_{0\leqslant j\leqslant n} \mathcal{N}_j(\mathrm{BO}(m-j)) \to \mathcal{N}_{m-1}(\mathrm{BO}(1)).$$

Proof. This follows at once from the Conner–Floyd sequence. □

If (M, T) and (M', T') are involutions, $(M, T) \times (M', T')$ means the involution on $M \times M'$ given by $(x, y) \mapsto (T(x), T'(y))$. This product induces on $\mathcal{N}_*^{\mathbb{Z}_2} = \bigoplus_{m\geqslant 0} \mathcal{N}_m^{\mathbb{Z}_2}$ the structure of a graded algebra over $\mathcal{N}_*$. If F^n is the top-dimensional component of the fixed-point set of (M, T), with normal bundle $\eta_n \to F^n$, and $(F')^{n'}$ is the top-dimensional component of the fixed-point set of (M', T'), with normal bundle $\eta'_{n'} \to (F')^{n'}$, then the top-dimensional component of the fixed-point set of $(M, T) \times (M', T')$ is $F^n \times (F')^{n'}$,

https://doi.org/10.1017/S0013091513000400 Published online by Cambridge University Press

with normal bundle $\eta_n \times \eta'_{n'}$. At the group level, the product maps $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)} \times (\mathcal{N}_{m'}^{\mathbb{Z}_2})^{(n')}$ into $(\mathcal{N}_{m+m'}^{\mathbb{Z}_2})^{(n+n')}$. In other words, the filtration of $\mathcal{N}_*^{\mathbb{Z}_2}$ is compatible with the ring structure.

Set

$$\mathcal{M}_m = \bigoplus_{j=0}^{m} \mathcal{N}_j(\mathrm{BO}(m-j)) \quad \text{and} \quad \mathcal{M}_* = \bigoplus_{m \geqslant 0} \mathcal{M}_m.$$

Then, $\mathcal{M}_*$ has the structure of a graded commutative algebra over $\mathcal{N}_*$ with identity the zero bundle over a point; the multiplication is induced by the usual product of bundles $(\xi \to N) \times (\xi' \to N') = (\xi \times \xi' \to N \times N')$. We filter $\mathcal{M}_*$ by setting

$$\mathcal{M}_m^{(n)} = \bigoplus_{j=0}^{n} \mathcal{N}_j(\mathrm{BO}(m-j))$$

for $0 \leqslant n \leqslant m$. Thus, $\mathcal{N}_*^{\mathbb{Z}_2}$ is included by the Conner–Floyd sequence as a subring of $\mathcal{M}_*$ and $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)} \subseteq \mathcal{M}_m^{(n)}$. The calculation of the ring $\mathcal{M}_*$ is recalled in the next lemma, in which the canonical line bundle over the n-dimensional real projective space $\mathbb{R}P^n$ is denoted by λ_n (with the convention that λ_0 is $\mathbb{R}$ over a point).

Proposition 2.2 (see [2, Lemma 25.1, § 25] and [7, Proposition 3.16]). *As an $\mathcal{N}_*$-algebra, $\mathcal{M}_*$ is a polynomial algebra with a generator in each $\mathcal{M}_m$, $m > 0$. For each $m > 0$, the generator can be chosen to be the class of $\lambda_{m-1} \to \mathbb{R}P^{m-1}$ in $\mathcal{N}_{m-1}(\mathrm{BO}(1)) \subseteq \mathcal{M}_m$.*

We next look at the detection of cobordism classes by characteristic numbers. Consider a decreasing list of positive integers $\omega = (i_1, i_2, \dots, i_s)$, $i_1 \geqslant i_2 \geqslant \cdots \geqslant i_s$. We set $|\omega| = i_1 + i_2 + \cdots + i_s$ and say that $\omega = (i_1, i_2, \dots, i_s)$ is *non-dyadic* if none of the i_t is of the form $2^p - 1$.

For $k \geqslant s$, let $s_\omega(X_1, X_2, \dots, X_k) \in \mathbb{Z}_2[X_1, \dots, X_k]$ be the smallest symmetric polynomial in variables $X_1, \dots, X_k$ containing the monomial $X_1^{i_1} X_2^{i_2} \cdots X_s^{i_s}$. More precisely, in terms of the action of the symmetric group $\mathfrak{S}_k$ on $\mathbb{Z}_2[X_1, \dots, X_k]$, we have that

$$s_\omega(X_1, \dots, X_k) = \sum_{\sigma \mathfrak{S}_k(\omega) \in \mathfrak{S}_k / \mathfrak{S}_k(\omega)} \sigma(X_1^{i_1} \cdots X_s^{i_s}),$$

where $\mathfrak{S}_k(\omega)$ is the stabilizer of $X_1^{i_1} \cdots X_s^{i_s}$. Given a k-dimensional real vector bundle ξ over a closed n-manifold N with tangent bundle TN, we denote by $s_\omega(\xi) \in H^{|\omega|}(N, \mathbb{Z}_2)$ the cohomology class obtained from $s_\omega(X_1, X_2, \dots, X_k)$ by replacing the rth elementary symmetric function in the variables X_j by the Stiefel–Whitney class $w_r(\xi)$. We allow the (non-dyadic) empty list $\omega_\emptyset$ ($s = 0$), with $|\omega_\emptyset| = 0$ and $s_{\omega_\emptyset}(\xi) = 1$. Then, the cobordism class of (N, ξ) in $\mathcal{N}_n(\mathrm{BO}(k))$ is determined by the modulo 2 integers obtained by evaluating the n-dimensional $\mathbb{Z}_2$-cohomology classes of the form $s_\omega(TN) s_{\omega'}(\xi)$, with $|\omega| + |\omega'| = n$ and ω non-dyadic, on the fundamental homology class $[N] \in H_n(N, \mathbb{Z}_2)$. We also need the following.

https://doi.org/10.1017/S0013091513000400 Published online by Cambridge University Press

Lemma 2.3 (see [4, p. 316]). *The map*

$$[N,\xi] \mapsto (s_\omega(TN)s_{\omega'}(\xi\oplus TN)[N])\colon \mathcal{N}_n(\mathrm{BO}(k)) \to \mathcal{N}_n(\mathrm{BO}(\infty)) \to \bigoplus_{(\omega,\omega')} \mathbb{Z}_2,$$

where the sum is over the pairs (ω,ω') with the decreasing lists ω, ω' satisfying $|\omega|+|\omega'| = n$ and ω non-dyadic, is injective.

Corollary 2.4. *Suppose that $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n-1)} = 0$. Then, the composition $[(M^m,T)] \mapsto (s_\omega(T(F^n))s_{\omega'}(\eta_n\oplus T(F^n))[F^n])$:*

$$(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)} \to \mathcal{N}_n(\mathrm{BO}(m-n)) \to \bigoplus_{(\omega,\omega')} \mathbb{Z}_2,$$

summed over lists with ω non-dyadic and $|\omega|+|\omega'| = n$, is injective.

Proof. This is immediate from Lemmas 2.1 and 2.3, since the map

$$\bigoplus_{0\leqslant j\leqslant n-1} \mathcal{N}_j(\mathrm{BO}(m-j)) \to \mathcal{N}_{m-1}(\mathrm{BO}(1))$$

is injective. □

We have the following key result of Kosniowski and Stong.

Proposition 2.5 (see [4]). *Consider an involution (M^m,T) with fixed-point data $((F^j,\eta_j), j=0,1,\dots)$, and suppose that $[(M^m,T)] \in (\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$. Let $f(X_1,\dots,X_m) \in \mathbb{Z}_2[X_1,\dots,X_m]$ be a symmetric polynomial in m variables, of degree at most m, and write $\phi(M) \in H^*(M;\mathbb{Z}_2)$ for the class obtained from $f(X_1,\dots,X_m)$ by substituting the Stiefel–Whitney class $w_r(TM)$ for the rth elementary symmetric function in the X_i. Then,*

$$\phi(M)[M] = \sum_{j=0}^{n} \psi_j(F^j,\eta_j)[F^j],$$

where $\psi_j(F^j,\eta_j)$ is obtained from the formal power series

$$g_j(Y_1,\dots,Y_{m-j},Z_1,\dots,Z_j) = \biggl(\prod_{i=1}^{m-j}(1+Y_i+Y_i^2+\cdots)\biggr) f(1+Y_1,\dots,1+Y_{m-j},Z_1,\dots,Z_j)$$

in $\mathbb{Z}_2[\![Y_1,\dots,Y_{m-j},Z_1,\dots,Z_j]\!]$ by replacing the rth symmetric polynomial in the Y_i by $w_r(\eta_j)$ and the rth symmetric function in the Z_i by $w_r(T(F^j))$.

Proof. This follows directly from the main theorem of Kosniowski and Stong [**4**, § 1]. We have just rewritten $(1+Y_i)^{-1}$ as $1+Y_i+Y_i^2+\cdots$ and omitted the terms for $j>n$, because (F^j,η_j) is a boundary for $j>n$. □

3. The classification theorem

Let (M^m, T) represent an element of $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$. Consider decreasing lists $\omega = (i_1, \ldots, i_s)$ and $\omega' = (j_1, \ldots, j_t)$, with $|\omega| + |\omega'| = n$ and ω non-dyadic. Following the proof of Boardman's theorem given by Kosniowski and Stong, we apply Proposition 2.5 to the polynomial

$$f(X_1, \ldots, X_m) = p_\omega(X_1, \ldots, X_m) \cdot q_{\omega'}(X_1, \ldots, X_m),$$

where

$$p_\omega(X_1, \ldots, X_m) = \sum_{\sigma\mathfrak{S}_m(\omega) \in \mathfrak{S}_m/\mathfrak{S}_m(\omega)} \sigma((1+X_1)^{i_1+1} X_1^{i_1} \cdots (1+X_s)^{i_s+1} X_s^{i_s}),$$

$$q_{\omega'}(X_1, \ldots, X_m) = \sum_{\sigma\mathfrak{S}_m(\omega') \in \mathfrak{S}_m/\mathfrak{S}_m(\omega')} \sigma((1+X_1)^{j_1} X_1^{j_1} \cdots (1+X_t)^{j_t} X_t^{j_t}).$$

We assume that the degree of $f(X_1, \ldots, X_m)$ satisfies the condition $s + 2|\omega| + 2|\omega'| < m$, so $\phi(M)[M] = 0$.

One checks that $g_j(Y_1, \ldots, Y_{m-j}, Z_1, \ldots, Z_j)$ has no homogeneous term of degree less than or equal to j if $j < n$ (because if we substitute either $1+Y$ or Z for X in $X(1+X)$, we get $Y(1+Y)$ or $Z(1+Z)$, so the degree of a homogeneous term is at least $|\omega| + |\omega'| = n$) and that

$$g_n(Y_1, \ldots, Y_{m-n}, Z_1, \ldots, Z_n) \\ = s_\omega(Z_1, \ldots, Z_n) \cdot s_{\omega'}(Y_1, \ldots, Y_{m-n}, Z_1, \ldots, Z_n) + \text{higher terms}$$

(because $p_\omega(1+Y_1, \ldots, 1+Y_{m-n}, Z_1, \ldots, Z_n)$ is equal to $s_\omega(Z_1, \ldots, Z_n) + $terms of degree greater than $|\omega|$ and $q_{\omega'}(1+Y_1, \ldots, 1+Y_{m-n}, Z_1, \ldots, Z_n)$ is $s_{\omega'}(Y_1, \ldots, Z_n) + $ terms of degree greater than $|\omega'|$). Hence, $\psi_j(F^j, \eta_j)[F^j] = 0$ if $j < n$ and $\psi_n(F^n, \eta_n)[F^n] = s_\omega(T(F^n)) s_{\omega'}(\eta_n \oplus T(F^n))[F^n]$. We have thus proved the following.

Lemma 3.1. *Suppose that* $[(M^m, T)] \in (\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$. *If* $\omega = (i_1, \ldots, i_s)$ *and* $\omega' = (j_1, \ldots, j_t)$ *are decreasing lists with* $n = |\omega| + |\omega'|$ *and* ω *non-dyadic, then*

$$s_\omega(T(F^n)) \cdot s_{\omega'}(\eta_n \oplus T(F^n))[F^n] = 0,$$

provided that $s + 2n < m$.

In other words, the possible non-zero characteristic numbers appearing in Corollary 2.4 are given by the condition $s + 2n \geqslant m$, which leaves few such numbers for values of m and n such that $n \leqslant \beta(m)$ (where $\beta(m)$ is defined as in Theorem 1.2). We consider the different congruence classes of m modulo 5, writing $m = 5k - c$, where $k \geqslant 1$. It is convenient to write 2_i for a string $2, \ldots, 2$ of length i with each entry equal to 2.

$\boldsymbol{c = 0}$. If $n < 2k$, then all (ω, ω') satisfy $s + 2n < m$. If $n = 2k$, then only $\omega = (2_k)$, $\omega' = \omega_\emptyset$ does not satisfy the condition.

https://doi.org/10.1017/S0013091513000400 Published online by Cambridge University Press

$c = 1$. If $n < 2k$, then all (ω, ω') satisfy $s + 2n < m$. If $n = 2k$, then several cases must be excluded, namely $(\omega, \omega') = ((2_k), \omega_\emptyset)$, $((2_{k-1}), (2))$, $((2_{k-1}), (1, 1))$ and, if $k \geqslant 2$, $((4, 2_{k-2}), \omega_\emptyset)$.

$c = 2$. If $n < 2k$, then all (ω, ω') satisfy $s + 2n < m$. If $n = 2k$, then the exclusions are $(\omega, \omega') = ((2_k), \omega_\emptyset)$, $((2_{k-1}), (2))$, $((2_{k-1}), (1, 1))$ and, if $k \geqslant 2$, $((2_{k-2}), (4))$, $((2_{k-2}), (3, 1))$, $((2_{k-2}), (2, 2))$, $((2_{k-2}), (2, 1, 1))$, $((2_{k-2}), (1, 1, 1, 1))$, $((4, 2_{k-2}), \omega_\emptyset)$ and, if $k \geqslant 3$, $((4, 2_{k-3}), (2))$, $((4, 2_{k-3}), (1, 1))$, $((5, 2_{k-3}), (1))$ and, if $k \geqslant 4$, $((4, 4, 2_{k-4}), \omega_\emptyset)$.

$c = 3$. If $n < 2k - 1$, then all (ω, ω') satisfy $s + 2n < m$. If $n = 2k - 1$, then only $\omega = (2_k)$, $\omega' = (1)$ does not.

$c = 4$. If $n < 2k - 1$, then all (ω, ω') satisfy $s + 2n < m$. If $n = 2k - 1$, then the excluded cases, if $k \geqslant 2$, are $(\omega, \omega') = ((2_{k-1}), (1))$, $((2_{k-2}), (3))$, $((2_{k-2}), (2, 1))$, $((2_{k-2}), (1, 1, 1))$ and, if $k \geqslant 3$, $((4, 2_{k-3}), (1))$, $((5, 2_{k-3}), \omega_\emptyset)$.

This establishes that $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)} = 0$ if $n < \beta(m)$ and that the dimension of $(\mathcal{N}_m^{\mathbb{Z}_2})^{(\beta(m))}$ is bounded above by the dimensions claimed in Theorem 1.2, with the exception of the elementary special cases $m = 1$, when $\mathcal{N}_1^{\mathbb{Z}_2} = 0$, and $m = 3$, when $(\mathcal{N}_3^{\mathbb{Z}_2})^{(1)} = 0$ and $\dim (\mathcal{N}_3^{\mathbb{Z}_2})^{(2)} = 1$.

The next task is to construct sets of linearly independent cobordism classes of involutions realizing the above bounds. The one-dimensional trivial vector bundle over a space N will be denoted by $\mathbb{R} \to N$. For a vector bundle $\xi \to N$ and a natural number $p \geqslant 1$, we write that $p\xi \to N$ for the Whitney sum of p copies of ξ.

We need the following construction of Conner (see [**2**]). For a given involution (M^m, T) with fixed-point data $(F^j, \eta_j)_{j=0}^n$, the involution

$$\Gamma(M, T) = ((S^1 \times M)/(z, x) \sim (-z, Tx), \tau),$$

where $S^1 \subseteq \mathbb{C}$ is the 1-sphere and τ is the involution induced by $(z, x) \mapsto (\bar{z}, x)$, has fixed-point data $((F^j, \eta_j \oplus \mathbb{R})_{j=0}^n, (M, \mathbb{R}))$. On cobordism classes this construction gives an operation $[(M, T)] \mapsto [\Gamma(M, T)]$, which we write as $\gamma\colon \mathcal{N}_m^{\mathbb{Z}_2} \to \mathcal{N}_{m+1}^{\mathbb{Z}_2}$. If M is a (non-equivariant) boundary, then $\gamma[M, T] \in (\mathcal{N}_{m+1}^{\mathbb{Z}_2})^{(n)}$. We can iterate this procedure. From the five halves theorem, if (M, T) is not a boundary, there will be a greatest natural number $r \geqslant 1$ (with $2r \leqslant 5n - 2m$) such that $\gamma^r[M, T] \in (\mathcal{N}_{m+r}^{\mathbb{Z}_2})^{(n)}$.

We begin the definition of the generating classes. The basic one-dimensional representation $\mathbb{R}$ of $\mathbb{Z}_2$ with the involution -1 will be written as L. Given a finite-dimensional $\mathbb{R}$-vector space V with a Euclidean inner product, we write λ_V for the Hopf bundle over the associated real projective space $P(V)$ and $\lambda_V^\perp$ for its orthogonal complement in the trivial bundle $P(V) \times V$. For any m, n, with $n \leqslant m \leqslant 2n + 1$, set $U = \mathbb{R}^{n+1}$ and $V = \mathbb{R}^{m-n}$. Then, $P(U \oplus L \otimes V)$ is a $\mathbb{Z}_2$-manifold with fixed-point data $((P(V), (n+1)\lambda_V)$, $(P(U), (m-n)\lambda_U))$. We set

$$x_m^{(n)} = [P(\mathbb{R}^{n+1} \oplus L \otimes \mathbb{R}^{m-n})] \in (\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}.$$

Further classes can be obtained by applying the operation γ.

Lemma 3.2 (see [6]). *Let m be odd and n be even, such that $n < m < 2n+1$, and let 2^p be the highest power of 2 dividing $2n+1-m$. Then, the greatest integer r such that $\gamma^r(x_m^{(n)})$ lies in $(\mathcal{N}_{m+r}^{\mathbb{Z}_2})^{(n)}$ is equal to 2 if $p = 1$ and to $2^p - 1$ if $p > 1$.*

In addition, we consider the involution $(\mathbb{R}P^n \times \mathbb{R}P^n, \text{twist})$, with fixed-point data $(\mathbb{R}P^n, T(\mathbb{R}P^n))$; we remark that, up to the Whitney sum with a trivial line bundle, $T(\mathbb{R}P^n)$ is equivalent to $(n+1)\lambda_n$. Write

$$y_{2n}^{(n)} = [(\mathbb{R}P^n \times \mathbb{R}P^n, \text{twist})] \in (\mathcal{N}_{2n}^{\mathbb{Z}_2})^{(n)}.$$

In [**1**], Boardman considered a family of $\mathbb{Z}_2$-manifolds $H_{2i,2j}$, $i < j$, defined as follows. Given four (finite-dimensional, Euclidean, non-zero) real vector spaces U, V, E and F, one can form the projective bundle $P(\lambda^{\perp}_{U\oplus L\otimes V} \oplus E \oplus L \otimes F)$ over the projective space $P(U \oplus L \otimes V)$. This is a $\mathbb{Z}_2$-manifold with fixed subspace the disjoint union of the projective bundles $P(\lambda_U^{\perp} \oplus E)$ over $P(U)$ and $P(\lambda_V^{\perp} \oplus F)$ over $P(V)$, $P(V \oplus F) \times P(U)$ and $P(U \oplus E) \times P(V)$. The $\mathbb{Z}_2$-manifold $H_{2i,2j}$, of dimension $2(i+j)-1$, is obtained by taking $U = \mathbb{R}^{i+1}$, $V = \mathbb{R}^i$, $E = F = \mathbb{R}^{j-i}$. We set

$$z_{11}^{(5)} = [H_{4,8}] \in (\mathcal{N}_{11}^{\mathbb{Z}_2})^{(5)}.$$

This completes the construction of the generators. One, which we now describe, has special importance.

Definition 3.3. We call the element $b = \gamma^2(x_3^{(2)}) \in (\mathcal{N}_5^{\mathbb{Z}_2})^{(2)}$ the *Boardman periodicity class*. It coincides with the class of the $\mathbb{Z}_2$-manifold $H_{2,4}$ and restricts, by forgetting the involution, to the generator of $\mathcal{N}_5$.

We can now state the classification theorem, from which Theorem 1.2 follows at once.

Theorem 3.4. *For $m > 1$, written as $m = 5k - c$, where $k \geqslant 1$ and $0 \leqslant c < 5$, the $\mathbb{Z}_2$-vector space $(\mathcal{N}_m^{\mathbb{Z}_2})^{(\beta(m))}$ has a basis consisting of the following elements.*

$c = 0$: *if* $k \geqslant 1$ $\quad b^k$,

$c = 1$: *if* $k \geqslant 1$ $\quad b^{k-1} \cdot \gamma(x_3^{(2)}), b^{k-1} \cdot x_4^{(2)}, b^{k-1} \cdot y_4^{(2)}$,
and, if $k \geqslant 2$, $\quad b^{k-2} \cdot \gamma^2(x_7^{(4)})$,

$c = 2$: *if* $k \geqslant 1$ $\quad b^{k-1} \cdot x_3^{(2)}$,
and, if $k \geqslant 2$, $\quad b^{k-2} \cdot (x_4^{(2)})^2, b^{k-2} \cdot (y_4^{(2)})^2, b^{k-2} \cdot \gamma(x_3^{(2)}) \cdot x_4^{(2)}, b^{k-2} \cdot x_6^{(3)} \cdot x_2^{(1)}$,
$b^{k-2} \cdot y_8^{(4)}, b^{k-2} \cdot \gamma^3(x_5^{(4)}), b^{k-2} \cdot x_8^{(4)}, b^{k-2} \cdot \gamma(x_7^{(4)})$,
and, if $k \geqslant 3$, $\quad b^{k-3} \cdot \gamma^2(x_7^{(4)}) \cdot x_4^{(2)}, b^{k-3} \cdot \gamma^2(x_7^{(4)}) \cdot y_4^{(2)}, b^{k-3} \cdot \gamma^2(x_{11}^{(6)})$,
and, if $k \geqslant 4$, $\quad b^{k-4} \cdot (\gamma^2(x_7^{(4)}))^2$,

$c = 3$: *if* $k \geqslant 1$ $\quad b^{k-1} \cdot x_2^{(1)}$,

$c = 4$: *if* $k \geqslant 2$ $\quad b^{k-2} \cdot \gamma(x_3^{(2)}) \cdot x_2^{(1)}, b^{k-2} \cdot x_4^{(2)} \cdot x_2^{(1)}, b^{k-2} \cdot y_4^{(2)} \cdot x_2^{(1)}, b^{k-2} \cdot x_6^{(3)}$,
and, if $k \geqslant 3$, $\quad b^{k-3} \cdot \gamma^2(x_7^{(4)}) \cdot x_2^{(1)}, b^{k-3} \cdot z_{11}^{(5)}$.

Thus, the basis when $c = 1$ and $k = 1$ has three elements, the basis when $c = 1$ and $k \geqslant 2$ has four elements, namely, the three listed for $k \geqslant 1$ and a fourth listed for $k \geqslant 2$, and similarly for the other values of c.

Proof. We first recall the classical result of Thom [**9**], that $\mathcal{N}_* = \bigoplus_{m \geqslant 0} \mathcal{N}_m$ is a graded polynomial algebra over $\mathbb{Z}_2$, with a generator in each dimension m that is not of the form $2^j - 1$. In even dimensions, the generator can be chosen to be the class of the real projective spaces $\mathbb{R}P^{2j}$; the generators in odd dimensions can be chosen to be the classes of certain Dold manifolds.

Note that the class b is not a zero-divisor in $\mathcal{N}_*^{\mathbb{Z}_2}$, because the ring $\mathcal{M}_*$ is polynomial (or simply because its zero-dimensional fixed-point component is a point).

It is clear, from the compatibility of the filtration with the product in $\mathcal{N}_*^{\mathbb{Z}_2}$, that the elements listed in each case belong to $(\mathcal{N}_m^{\mathbb{Z}_2})^{(n)}$. Given the dimensional bounds already obtained and the injectivity of multiplication by b, it remains to check linear independence in the three cases $m = 9$, 11 and 18. This will follow from the fact that, as required by Corollary 2.4, in each case the corresponding set of the cobordism classes of the top-dimensional components of the fixed-point data is linearly independent. In principle, this may be verified by a routine computation using characteristic classes, as in Corollary 2.4. This is most easily carried out by calculating in the ring $\mathcal{M}_*$ modulo terms of lower filtration. For all the generators, except $z_{11}^{(5)}$, the top component of the fixed-point set is a real projective space, and some simplification can be achieved by using Lemma 3.5. The classes $[(\mathbb{R}P^2, p\lambda_2)]$, $p = 1, 2, 3$, are linearly independent in $\mathcal{N}_2(\mathrm{BO})$ and $[(\mathbb{R}P^4, p\lambda_4)]$, $p = 1, 3, 4, 5$, are linearly independent in $\mathcal{N}_4(\mathrm{BO})$. We omit the details. □

Lemma 3.5 (see [10]). *Let $n > 1$ be even. Then, the subspace of $\mathcal{N}_n(\mathrm{BO})$ spanned by the classes $(\mathbb{R}P^n, \eta)$ as η ranges over all vector bundles on $\mathbb{R}P^n$ is isomorphic to $H^*(\mathbb{R}P^n, \mathbb{Z}_2)$ under a correspondence taking $(\mathbb{R}P^n, \eta)$ to the total Stiefel–Whitney class $(1, w_1(\eta), \dots, w_n(\eta))$ of η.*

Proof. This is established by calculating characteristic classes. Every bundle is stably equivalent to $q\lambda_n$ for some $q \geqslant 1$. Any characteristic number will be given by a numerical polynomial in q of degree at most n. Such a polynomial is an integral linear combination of the binomial coefficients $\binom{q}{r}$, $r = 0, \dots, n$, and these binomial coefficients arise from the characteristic numbers $(w_1^{n-r}(\mathbb{R}P^n) w_r(q\lambda_n))[\mathbb{R}P^n]$. □

Remark 3.6. The choice of specific generators in the classification theorem is fairly arbitrary.

The calculations show that

$$\begin{aligned} y_4^{(2)} &= \gamma(x_3^{(2)}) + (x_2^{(1)})^2, \\ y_8^{(4)} &= \gamma(x_7^{(4)}) + (x_2^{(1)})^2 \gamma(x_3^{(2)}) + \gamma(x_3^{(2)})^2 + (x_4^{(2)})^2, \\ \gamma^3(x_5^{(4)}) &= y_8^{(4)} + (x_2^{(1)})^4, \end{aligned}$$

so we could have avoided introducing the classes $y_{2n}^{(n)}$ and $\gamma^3(x_5^{(4)})$.

A (non-trivial) construction of Lü [**5**, § 2, Lemma 2.1] yields, as a special case, an involution defined on an 11-dimensional manifold, Z^{11}, whose fixed-point data is of the form $((\mathbb{R}P^1, \lambda_1 \oplus \mathbb{R}^9), (P(1,2), \eta_5))$, where $P(1,2)$ is the five-dimensional Dold manifold $(S^1 \times \mathbb{C}P^2)/(z,x) \sim (-z,\bar{x})$. (Here $\mathbb{C}P^2$ is the two-dimensional complex projective space and $\bar{x}$ is the complex conjugate of $x \in \mathbb{C}P^2$.) We might have chosen to take $z_{11}^{(5)}$ to be the class of Z^{11} instead of $H_{4,8}$.

Acknowledgements. P.L.Q.P. was partly supported by CNPq and FAPESP.

References

1. J. M. Boardman, On manifolds with involution, *Bull. Am. Math. Soc.* **73** (1967), 136–138.
2. P. E. Conner, *Differentiable periodic maps*, 2nd edn, Lecture Notes in Mathematics (Springer, 1979).
3. P. E. Conner and E. E. Floyd, *Differentiable periodic maps* (Springer, 1964).
4. C. Kosniowski and R. E. Stong, Involutions and characteristic numbers, *Topology* **17** (1978), 309–330.
5. Z. Lü, Involutions fixing $\mathbb{RP}^{\mathrm{odd}} \sqcup P(h,i)$, I, *Trans. Am. Math. Soc.* **354** (2002), 4539–4570.
6. R. de Oliveira, P. L. Q. Pergher and A. Ramos, Z_2^k-actions fixing $\mathbb{R}\mathrm{P}^2 \cup \mathbb{R}\mathrm{P}^{\mathrm{even}}$, *Alg. Geom. Topol.* **7** (2007), 29–45.
7. D. Sinha, Real equivariant bordism and stable transversality obstructions for $\mathbb{Z}/2$, *Proc. Am. Math. Soc.* **130** (2001), 271–281.
8. R. E. Stong, Involutions with n-dimensional fixed set, *Math. Z.* **178** (1981), 443–447.
9. R. Thom, Quelques propriétés globales des variétés différentiables, *Comment. Math. Helv.* **28** (1954), 18–88.
10. B. F. Torrence, Bordism classes of vector bundles over real projective spaces, *Proc. Am. Math. Soc.* **118** (1993), 963–969.

https://doi.org/10.1017/S0013091513000400 Published online by Cambridge University Press