Mathematical Structures in Computer Science (2024), **34**, pp. 81–97
doi:10.1017/S0960129523000385

PAPER

Fixed point logics and definable topological properties

David Fernández-Duque[1,2] and Quentin Gougeon[3]

[1]Department of Philosophy, University of Barcelona, Barcelona, Spain
[2]Czech Academy of Sciences, Institute of Computer Science, Prague, Czech Republic
[3]CNRS-INPT-UT3, Toulouse University, Toulouse, France
Corresponding author: Quentin Gougeon; Email: quentin.gougeon@irit.fr

(Received 12 March 2023; revised 5 September 2023; accepted 10 November 2023; first published online 13 December 2023)

Abstract

Modal logic enjoys topological semantics that may be traced back to McKinsey and Tarski, and the classification of topological spaces via modal axioms is a lively area of research. In the past two decades, there has been interest in extending topological modal logic to the language of the mu-calculus, but previously no class of topological spaces was known to be mu-calculus definable that was not already modally definable. In this paper, we show that the full mu-calculus is indeed more expressive than standard modal logic, in the sense that there are classes of topological spaces (and weakly transitive Kripke frames), which are mu-definable but not modally definable. The classes we exhibit satisfy a modally definable property outside of their perfect core, and thus we dub them *imperfect spaces.* We show that the mu-calculus is sound and complete for these classes. Our examples are minimal in the sense that they use a single instance of a greatest fixed point, and we show that least fixed points alone do not suffice to define any class of spaces that is not already modally definable.

Keywords: Mu-calculus; expressivity; topological semantics

1. Introduction

Topological semantics for modal logic originated with McKinsey and Tarski (1944) in the 1940s but saw a more recent revival due to the work of Esakia (2001), Shehtman (1999), and others. In what we call the *closure semantics*, the modal $\Diamond$ is interpreted as the topological closure and $\Box$ as the interior. The logic of all topological spaces in this semantics is **S4**, and we refer to van Benthem and Bezhanishvili (2007) for an overview of topological completeness of modal logics above **S4**. The more expressive *derivational semantics* – Kudinov and Shehtman (2014) – has gained traction in recent years but was already considered by McKinsey and Tarski. It is obtained by interpreting the modal $\Diamond$ as the Cantor derivative.[1] Esakia (2001, 2004) showed that the derivative logic of all topological spaces is the modal logic $\mathbf{wK4} = \mathbf{K} + (\Diamond\Diamond p \to p \vee \Diamond p)$. This is also the modal logic of all *weakly transitive* frames, that is, those for which the reflexive closure of the accessibility relation is transitive. It is well known that the modal logic of transitive frames is **K4** – Blackburn et al. (2001), Chagrov and Zakharyaschev (1997) – which moreover corresponds to a natural class of topological spaces denoted by T_d. Many familiar topological spaces are T_d, such as Euclidean spaces.

Even more recently, topological semantics have been extended to the language of the μ-calculus – see Baltag et al. (2021), Fernández-Duque (2011*a*,b), and Goldblatt and Hodkinson (2017). The relational μ-calculus is notoriously challenging from a theoretical perspective, with difficult completeness and decidability proofs – see, respectively, Walukiewicz (2000) and Kozen (1983). See also Afshari and Leigh (2017), Santocanale and Venema (2010), and Santocanale (2008) for more recent work exhibiting various modifications to these results and their proofs. Since a transitive modality is already definable in the basic μ-calculus, Goldblatt and Hodkinson (2018) obtained completeness and decidability as a corollary for transitive frames, and thus for T_d spaces. This does not work for weakly transitive frames, but surprisingly, Baltag et al. (2021) showed that the combination of the μ-calculus with topological semantics is much more manageable than the original μ-calculus, with natural and transparent proofs of decidability and completeness involving only classical tools from modal logic (albeit intricately combined).

Thus, the topological μ-calculus is decidable and complete, potentially placing it as a powerful yet technically manageable framework for reasoning about topologically defined fixed points. The Achilles' heel of this proposal is that despite the sophisticated machinery, no class of topological spaces was formerly known to be μ-definable but not modally definable. Our goal is to exhibit such classes of spaces. Here, it is convenient to recall the notion of reducibility of formal languages, following Kudinov and Shehtman (2014). If $\mathscr{L}$ and $\mathscr{L}'$ are sub-languages of the μ-calculus, then $\mathscr{L}$ *reduces to* $\mathscr{L}'$ if every class of spaces definable in $\mathscr{L}$ is also definable in $\mathscr{L}'$ (see Section 2). If $\mathscr{L}$ reduces to $\mathscr{L}'$, we may also say that $\mathscr{L}'$ is *at least as expressive as* $\mathscr{L}$, and if moreover $\mathscr{L}'$ does not reduce to $\mathscr{L}$, we say that $\mathscr{L}'$ is *more expressive than* $\mathscr{L}$.[2] We first show that least fixed points do not yield any additional expressivity. We then manage to exhibit infinitely many topologically complete logics in the language of the μ-calculus whose classes of spaces are not modally definable. These axioms separate spaces into two parts, a perfect part (i.e., without isolated points) and a complement satisfying some property definable by a modal formula φ; we call these spaces *φ-imperfect spaces*. The perfect part is defined via a greatest fixed point operator. The paper is structured as follows: in Section 2, we present the relevant material regarding derivative spaces, the μ-calculus, and axiomatic expressivity. In Section 3 we show that the full μ-calculus is not more expressive than the μ-free language (with greatest fixed points only). In Section 4, we use greatest fixed points to construct classes of spaces that are not modally definable. Completeness results for some of these classes are then laid out in Section 5. We end with some concluding remarks in Section 6.

2. Background

In this section, we review the syntax and semantics of the topological μ-calculus. Following Baltag et al. (2021) and Fernández-Duque and Iliev (2018), we present our semantics in the general setting of *derivative spaces* and work in a language with ν (rather than μ) as primitive.

Definition 1. *We fix a countable set* Prop *of* atomic propositions *(also called* variables*). The language $\mathscr{L}_\mu$ of the modal μ-calculus is defined by the following grammar:*

$$\varphi ::= p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \Diamond\varphi \mid \nu p.\varphi$$

where $p \in$ Prop *and in the construct $\nu p.\varphi$, the formula φ is* positive in *p, that is, every occurrence of p lies under the scope of an even number of negations. The abbreviations $\varphi \vee \psi$, $\varphi \to \psi$, $\varphi \leftrightarrow \psi$, $\Box\varphi$, $\bot$, and $\top$ are defined as usual. We also assume that every formula φ is* clean*, that is, no bound variable is also a free variable, and for every variable p there is at most one subformula of φ of the form $\nu p.\psi$. We denote by $\varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n]$ the formula φ where each formula ψ_i is substituted for every free occurrence of the variable p_i. Some implicit renaming may be carried out to ensure that the resulting formula is clean. We then introduce the abbreviation $\mu p.\varphi := \neg\nu p.\neg\varphi[\neg p/p]$. Finally, the* basic modal language *$\mathscr{L}_\Diamond$ is the fragment of $\mathscr{L}_\mu$ without occurrences of ν.*

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

Definition 2. *A* derivative space *is a pair* $\mathscr{X} = (X, \mathrm{d})$*, where* X *is a set of* points *and* $\mathrm{d}\colon \mathscr{P}(X) \to \mathscr{P}(X)$ *is an operator on subsets of* X*, satisfying for all* $A, B \subseteq X$*:*

- $\mathrm{d}(\varnothing) = \varnothing$,
- $\mathrm{d}(A \cup B) = \mathrm{d}(A) \cup \mathrm{d}(B)$,
- $\mathrm{d}(\mathrm{d}(A)) \subseteq A \cup \mathrm{d}(A)$.

A derivative model *based on* $\mathscr{X}$ *is a tuple of the form* $\mathfrak{M} = (X, \mathrm{d}, V)$ *with* $V\colon \mathsf{Prop} \to \mathscr{P}(X)$ *a* valuation. *Given* $x \in X$*, we then call* $(\mathfrak{M}, x)$ *a* pointed derivative model. *If* $p \in \mathsf{Prop}$ *and* $A \subseteq X$*, we define the valuation* $V[p := A]$ *by* $V[p := A](p) := A$ *and* $V[p := A](q) := V(q)$ *if* $q \neq p$*. We then write* $\mathfrak{M}[p := A] := (X, \mathrm{d}, V[p := A])$.

Definition 3. *Given a derivative model* $\mathfrak{M} = (X, \mathrm{d}, V)$*, we define by induction on a formula* $\varphi \in \mathscr{L}_\mu$ *the* extension $[\![\varphi]\!]_{\mathfrak{M}}$ *of* φ *in* $\mathfrak{M}$ *as follows:*

- $[\![p]\!]_{\mathfrak{M}} := V(p)$,
- $[\![\neg\varphi]\!]_{\mathfrak{M}} := X \setminus [\![\varphi]\!]_{\mathfrak{M}}$,
- $[\![\varphi \wedge \psi]\!]_{\mathfrak{M}} := [\![\varphi]\!]_{\mathfrak{M}} \cap [\![\psi]\!]_{\mathfrak{M}}$,
- $[\![\Diamond\varphi]\!]_{\mathfrak{M}} := \mathrm{d}([\![\varphi]\!]_{\mathfrak{M}})$,
- $[\![\nu p.\varphi]\!]_{\mathfrak{M}} := \bigcup\{A \subseteq W \mid A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}\}$.

We then write $\mathfrak{M}, x \vDash \varphi$ *whenever* $x \in [\![\varphi]\!]_{\mathfrak{M}}$ *and we say that* φ *is* true *at the point* x*. If* $\mathfrak{M}$ *is based on* $\mathscr{X}$ *and* $\mathfrak{M}, x \vDash \varphi$*, we say that* φ *is* satisfiable *on* $\mathfrak{M}$*, or on* $\mathscr{X}$*, or on* $\mathscr{X}, x$ *(depending on what is deemed relevant).*

If $[\![\varphi]\!]_{\mathfrak{M}} = X$*, we write* $\mathfrak{M} \vDash \varphi$*. If* $\mathfrak{M} \vDash \varphi$ *for all models* $\mathfrak{M}$ *based on* $\mathscr{X}$ *we write* $\mathscr{X} \vDash \varphi$ *and we say that* φ *is* valid *on* $\mathscr{X}$*. We also have a notion of pointwise validity, that is, if* $\mathfrak{M}, x \vDash \varphi$ *for every model* $\mathfrak{M}$ *based on* $\mathscr{X}$*, then we write* $\mathscr{X}, x \vDash \varphi$*. If* $\mathscr{X} \vDash \varphi$ *for all derivative spaces* $\mathscr{X}$*, we write* $\vDash \varphi$*. Given a class* $\mathscr{C}$ *of derivative spaces, we write* $\mathscr{C} \vDash \varphi$ *whenever* $\mathscr{X} \vDash \varphi$ *for all* $\mathscr{X} \in \mathscr{C}$*. If* Γ *is a set of formulas, we write* $\mathfrak{M}, x \vDash \Gamma$ *whenever* $\mathfrak{M}, x \vDash \varphi$ *for all* $\varphi \in \Gamma$*, and all of the other notations are adapted accordingly.*

Definition 4. *Let* $\mathscr{X} = (X, \mathrm{d})$ *be a derivative space. A* subspace *of* $\mathscr{X}$ *is any derivative space* $\mathscr{X}' = (X', \mathrm{d}')$ *such that* $X' \subseteq X$ *and* $\mathrm{d}'(A) = \mathrm{d}(A) \cap X'$ *for all* $A \subseteq X'$*. If* $\mathfrak{M} = (X, \mathrm{d}, V)$ *is a derivative model based on* $\mathscr{X}$*, then a* submodel *of* $\mathfrak{M}$ *is any model* $\mathfrak{M}' = (X', \mathrm{d}', V')$ *based on a subspace of* $\mathscr{X}$*, and such that* $V'(p) = V(p) \cap X'$ *for all* $p \in \mathsf{Prop}$*. Note that* d' *and* V' *are entirely characterized by* X', d, *and* V*. Hence, we will often abuse notations and let* (X', d, V) *stand for* (X', d', V').

In modal logic, it is customary to study morphisms that preserve validity. In the context of derivative spaces, these are known as *d-morphisms* – see, e.g., Kudinov and Shehtman (2014).

Definition 5. *Let* $\mathscr{X} = (X, \mathrm{d})$ *and* $\mathscr{X}' = (X', \mathrm{d}')$ *be two derivative spaces. A map* $f\colon X \to X'$ *is called a* d-morphism *from* $\mathscr{X}$ *to* $\mathscr{X}'$ *if it satisfies* $f^{-1}[\mathrm{d}'(A')] = \mathrm{d}(f^{-1}[A'])$ *for all* $A' \subseteq X'$.

Proposition 6. *Let* $\mathscr{X} = (X, \mathrm{d})$ *and* $\mathscr{X}' = (X', \mathrm{d}')$ *be two derivative spaces and* $f\colon X \to X'$ *a d-morphism. If* $\varphi \in \mathscr{L}_\mu$ *and* $\mathscr{X} \vDash \varphi$*, then* $\mathscr{X}' \vDash \varphi$.

Presenting our semantics in terms of derivative spaces is useful, as both weakly transitive Kripke frames and topological spaces (either with the closure or the d operator) can be viewed as special cases of derivative spaces. While our "intended" semantics is topological, Kripke semantics will be useful in establishing many of our main results.

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

Definition 7. *A Kripke frame is a pair* $\mathfrak{F} = (W, R)$*, with* W *a set of* possible worlds *and* $R \subseteq W^2$*. We denote by* $R^+ := R \cup \{(w, w) \mid w \in W\}$ *the* reflexive closure *of* R*. The frame* $\mathfrak{F}$ *is said to be* rooted *in* r *if for all* $w \in W$ *we have* rR^+w*. We say that* $\mathfrak{F}$ *is* weakly transitive *if* wRu *and* uRv *implies* wR^+v*. In this case,* $\mathfrak{F}$ *is also called a* **wK4** frame*, and it induces a derivative space* (W, d) *with* d *defined by* $\mathrm{d}(A) := \{w \mid wRu \text{ and } u \in A\}$*.*

Slightly abusing terminology, we will identify $\mathfrak{F}$ *and* (W, d) *(since one can be constructed from the other). Then, (pointed) derivative models based on* **wK4** *frames will be called (pointed)* Kripke models*, while d-morphisms between* **wK4** *frames will be called* bounded morphisms*.*

Useful will be the notion of path. Recall that ω denotes the smallest infinite ordinal.

Definition 8. *Let* $\mathfrak{F} = (W, R)$ *be a Kripke frame. A* path *in* $\mathfrak{F}$ *is a sequence* $\overline{w} = (w_i)_{1 \leq i \leq n} \in W^n$*, where* $n \leq \omega$*, and such that we have* $w_i R w_{i+1}$ *whenever* $1 \leq i < n$*. We also say that* $\overline{w}$ begins *on* w_i*. If* $n = \omega$*, then* $\overline{w}$ *is called an* infinite path*; otherwise,* $\overline{w}$ *is said to be of* size n*.*

Now we turn our attention to the "official" semantics of the topological μ-calculus.

Definition 9. *Let* X *be a set of* points*. A* topology *on* X *is a set* $\tau \subseteq \mathscr{P}(X)$ *containing* $\varnothing$ *and* X*, closed under arbitrary unions, and closed under finite intersections. The pair* (X, τ) *is then called a* topological space*. The elements of* τ *are called the* open *sets of* X*. The complement of an open set is called a* closed *set. If* $x \in U \in \tau$*, then* U *is called an* open neighborhood *of* x*. Slightly abusing notation, we will often keep* τ *implicit and let* X *refer to the space* (X, τ)*.*

Definition 10. *Let* X *be a topological space,* $A \subseteq X$ *and* $x \in X$*. The point* x *is said to be a* limit point *of* A *if for all open neighborhoods* U *of* x*, we have* $U \cap A \setminus \{x\} \neq \varnothing$*. We denote by* $\mathrm{d}(A)$ *the set of all limit points of* A *and call it the* derived set *of* A*. The dual of* d *is defined by* $\widehat{\mathrm{d}}(A) := X \setminus \mathrm{d}(X \setminus A)$*.*

Given a topological space X, it is easily observed that the pair (X, d) is a derivative space. Conversely, the topology τ can be recovered from d since for all $A \subseteq X$, the set A is closed if and only if $\mathrm{d}(A) \subseteq A$. For this reason, we choose, again, to identify (X, τ) and (X, d). Then, (pointed) derivative models based on topological spaces will be called (pointed) *topological models*. Observe that the familiar *closure* and *interior* operators can be defined by $\mathrm{Cl}(A) := A \cup \mathrm{d}(A)$ and $\mathrm{Int}(A) := A \cap \widehat{\mathrm{d}}(A)$. Writing $\Box^+\varphi := \varphi \wedge \Box\varphi$ and $\Diamond^+\varphi := \varphi \vee \Diamond\varphi$, we then have $[\![\Box^+\varphi]\!]_{\mathfrak{M}} = \mathrm{Int}([\![\varphi]\!]_{\mathfrak{M}})$ and $[\![\Diamond^+\varphi]\!]_{\mathfrak{M}} = \mathrm{Cl}([\![\varphi]\!]_{\mathfrak{M}})$ for all topological models $\mathfrak{M}$. We recall some important classes of topological spaces that will be useful throughout the text.

Definition 11. *Let* X *be a topological space. A point* $x \in X$ *is said to be* isolated *if* $\{x\}$ *is open. Given* $x \in A \subseteq X$ *we say that* x *is* isolated in A *if there exists* U *open such that* $\{x\} = U \cap A$*. The space* X *is called* dense-in-itself *if it contains no isolated point. The space* X *is called* scattered *if any subspace of* X *contains an isolated point. We say that* X *is* T_d *if every* $x \in X$ *is isolated in* $\mathrm{Cl}(\{x\})$*. We say that* X *is* extremally disconnected *if* $\mathrm{Cl}(U)$ *is open for every open set* U*, and* Aleksandroff *if arbitrary intersections of open sets are open.*

Aleksandroff spaces are closely connected to Kripke frames, via the following construction.

Definition 12. *Let* $\mathfrak{F} := (W, R)$ *be a* **wK4** *frame. A set* $U \subseteq W$ *is called an* upset *if* $w \in U$ *and* wRu *implies* $u \in U$*. The collection* τ_R *of all upsets over* W *is then a topology, and* (W, τ_R) *is called the topological space* induced *by* $\mathfrak{F}$*. If* $\mathfrak{M} = (W, R, V)$ *is a Kripke model based on* $\mathfrak{F}$*, then* $((W, \tau_R), V)$ *is the topological model* induced *by* $\mathfrak{M}$*.*

It is not hard to check that a space of the form (W, τ_R) is always Aleksandroff – and, indeed, every Aleksandroff space is of this form, see Aleksandroff (1937). In fact, we will simply not distinguish a weakly transitive Kripke frame from the topological space induced by it. This is partly motivated by the following proposition.

Proposition 13. *Let $\mathfrak{M} = (W, R, V)$ be an irreflexive and weakly transitive Kripke model, and let $\mathfrak{M}' := ((W, \tau_R), V)$ be the space induced by it. For all $w \in W$ and $\varphi \in \mathcal{L}_\mu$, we have*

$$\mathfrak{M}, w \vDash \varphi \iff \mathfrak{M}', w \vDash \varphi.$$

The modal logic of all topological spaces is known as **wK4** and consists of the following inference rules and axioms:

Name	Axiom/inference rule
	All propositional tautologies
Uniform substitution	From φ infer $\varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n]$
N	$\neg\Diamond\bot$
K (Distribution)	$\Box(p \to q) \to (\Box p \to \Box q)$
Weak transitivity	$\Diamond\Diamond p \to p \vee \Diamond p$
Modus Ponens	From φ and $\varphi \to \psi$ infer ψ
Monotonicity	From $\varphi \to \psi$ infer $\Diamond\varphi \to \Diamond\psi$

Note that this axiomatization differs from the usual presentation, as it is adapted to a language where $\Diamond$ (instead of $\Box$) is taken as primitive. The axiomatic system **K4** is the extension of **wK4** with the axiom $4 := \Diamond p \to \Diamond\Diamond p$. The axiomatic system μ**wK4** is the extension of **wK4** with the *fixed point axiom* $\nu p.\varphi \to \varphi[\nu p.\varphi/p]$ and the *induction rule*:

$$\text{from } \varphi \to \psi[\varphi/p] \text{ infer } \varphi \to \nu p.\psi.$$

Definition 14. *Let* **L** *be a logic in a sub-language of $\mathcal{L}_\mu$. If φ is a formula, the statement* $\mathbf{L} \vdash \varphi$ *says that φ is derivable in* **L**. *We say that* **L** *is* sound and complete *with respect to a class $\mathscr{C}$ of derivative spaces if for all formulas φ we have* $\mathbf{L} \vdash \varphi$ *iff $\mathscr{C} \vDash \varphi$. We call* **L** Kripke complete *if it is sound and complete with respect to some class of Kripke frames, and* topologically complete *if it is sound and complete with respect to some class of topological spaces.*

Theorem 15. (Baltag et al. 2021). *The logic μ***wK4** *is sound and complete with respect to the class of all* **wK4** *frames, with respect to the class of all topological spaces, and with respect to the class of all derivative spaces.*

In order to compare the expressivity of different languages, we need to introduce the notion of definable classes.

Definition 16. *Given a formula φ, we let $\mathscr{C}(\varphi)$ be the class of derivative spaces $\mathscr{X}$ such that $\mathscr{X} \vDash \varphi$. Let $\mathscr{C}_0$ be a class of derivative spaces, and let $\mathscr{L} \subseteq \mathcal{L}_\mu$. We say that $\mathscr{C}$ is $\mathscr{L}$-*definable within *$\mathscr{C}_0$ if there exists $\varphi \in \mathscr{L}$ such that $\mathscr{C}(\varphi) \cap \mathscr{C}_0 = \mathscr{C} \cap \mathscr{C}_0$.*

If $\mathscr{L}, \mathscr{L}' \subseteq \mathcal{L}_\mu$, we say that $\mathscr{L}'$ is at least as expressive as *$\mathscr{L}$* over *$\mathscr{C}_0$ if every class definable in $\mathscr{L}$ within $\mathscr{C}_0$ is also definable in $\mathscr{L}'$ within $\mathscr{C}_0$. If $\mathscr{L}'$ is at least as expressive as $\mathscr{L}$ but $\mathscr{L}$ is not at least as expressive as $\mathscr{L}'$, we say that $\mathscr{L}'$* is more expressive than *$\mathscr{L}$* over *$\mathscr{C}_0$.*

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

In particular, a $\mathcal{L}_{\Diamond}$-definable class will be called *modally definable*, and a $\mathcal{L}_{\mu}$-definable class will be called *μ-definable*. As discussed in Footnote 2, this notion of expressivity is also known as *reducibility* or *axiomatic expressivity*. The choice to compare expressivity relatively to a class of derivative spaces is convenient as it allows to derive all kinds of auxiliary results. We will consider the following classes of interest:

$$
\begin{aligned}
\mathscr{C}_{\mathsf{all}} &:= \{\mathscr{X} \mid \mathscr{X} \text{ is a derivative space}\}\\
\mathscr{C}_{\mathsf{fin}} &:= \{(X, \mathrm{d}) \in \mathscr{C}_{\mathsf{all}} \mid X \text{ is finite}\}\\
\mathscr{C}_{\mathsf{Kripke}} &:= \{\mathfrak{F} \mid \mathfrak{F} \text{ is a } \mathbf{wK4} \text{ frame}\}\\
\mathscr{C}_{\mathsf{irrefl}} &:= \{\mathfrak{F} \in \mathscr{C}_{\mathsf{Kripke}} \mid \mathfrak{F} \text{ is irreflexive}\}\\
\mathscr{C}_{\mathsf{topo}} &:= \{X \mid X \text{ is a topological space}\}\\
\mathscr{C}_{\mathbf{K4}} &:= \{\mathscr{X} \in \mathscr{C}_{\mathsf{all}} \mid \mathscr{X} \vDash \mathbf{K4}\}
\end{aligned}
$$

It is well established that $\mathscr{C}_{\mathsf{Kripke}} \cap \mathscr{C}_{\mathbf{K4}}$ is the class of transitive Kripke frames – see Blackburn et al. (2001), while $\mathscr{C}_{\mathsf{topo}} \cap \mathscr{C}_{\mathbf{K4}}$ is the class of T_d spaces – see van Benthem and Bezhanishvili (2007).

3. Classes Defined by Least Fixed Points

Our primary goal is to prove that the μ-calculus is more axiomatically expressive than basic modal logic; however, as we will see in this section, least (as opposed to greatest) fixed points alone do not yield additional expressive power. Of course, least and greatest fixed points are interdefinable using negation, but this is not the case for formulas expressed in *negation normal form* (or NNF for short), defined by the following grammar:

$$\varphi ::= p \mid \neg p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \nu p.\varphi \mid \mu p.\varphi$$

It is well known that for every formula in $\mathcal{L}_{\mu}$, there is an equivalent formula in NNF. Then, when omitting the μ operator in the above grammar, one obtain the *μ-free language* $\mathcal{L}_{\mu}^{0}$.

We are thus going to show that the full language does not define more classes of spaces than the μ-free language. In other words, one can recover all the axioms of the μ-calculus (up to equivalence) by enumerating only the μ-free formulas. This result is not only interesting in itself: by providing a simpler syntactic form for axioms, it will simplify the process of finding one that is not reducible to a basic modal axiom.

We recall that the extension of $\mu p.\varphi$ in a derivative model $\mathfrak{M} = (X, \mathrm{d}, V)$ is defined as

$$[\![\mu p.\varphi]\!]_{\mathfrak{M}} := \bigcap\{A \subseteq X \mid [\![\varphi]\!]_{\mathfrak{M}[p:=A]} \subseteq A\}.$$

So, given $x \in X$, we have

$$\mathfrak{M}, x \vDash \mu p.\varphi \text{ iff } \forall A \subseteq X, ([\![\varphi]\!]_{\mathfrak{M}[p:=A]} \subseteq A) \implies x \in A.$$

We can then observe that the universal quantification over the subsets of X is, implicitly, nothing more than a quantification over the possible valuations of p – and this is precisely the kind of quantification that validity of formulas is able to capture. As a result, one can rewrite axioms in a way that rids them of their least fixed points. Here, the textbook example would be the formula $\mu p.\Box p$, which defines the same class of spaces as the well-known Löb axiom $\Box(\Box p \to p) \to \Box p$ – see van Benthem (2006). Drawing inspiration from this result, we arrive at a uniform translation $\mathsf{tr} : \mathcal{L}_{\mu} \to \mathcal{L}_{\mu}^{0}$, defined by induction as follows:

- $\mathsf{tr}(p) := p$,
- $\mathsf{tr}(\neg p) := \neg p$,

- $\mathsf{tr}(\varphi \wedge \psi) := \mathsf{tr}(\varphi) \wedge \mathsf{tr}(\psi)$,
- $\mathsf{tr}(\varphi \vee \psi) := \mathsf{tr}(\varphi) \vee \mathsf{tr}(\psi)$,
- $\mathsf{tr}(\Box\varphi) := \Box\mathsf{tr}(\varphi)$,
- $\mathsf{tr}(\Diamond\varphi) := \Diamond\mathsf{tr}(\varphi)$,
- $\mathsf{tr}(\nu p.\varphi) := \nu p.\mathsf{tr}(\varphi)$,
- $\mathsf{tr}(\mu p.\varphi) := \Box^+(\mathsf{tr}(\varphi) \to p) \to p$.

Recall that formulas of the μ-calculus are assumed to be clean, so each formula of the form $\nu p.\varphi$ or $\mu p.\varphi$ comes with its own variable p. Our goal is then to prove that $\mu\mathbf{wK4} + \varphi = \mu\mathbf{wK4} + \mathsf{tr}(\varphi)$. One direction is obtained by a stronger claim.

Lemma 17. *For all $\varphi \in \mathscr{L}_\mu$ we have $\mu\mathbf{wK4} \vdash \varphi \to \mathsf{tr}(\varphi)$.*

Proof. By induction on φ. This is straightforward for Boolean and modal formulas, so we only address the fixed point operators. Applying Theorem 15, we reason by Kripke completeness. Let $(\mathfrak{M}, w)$ be a pointed **wK4** model and suppose that $\mathfrak{M}, w \vDash \nu p.\varphi$. We write $\mathfrak{M} = (W, R, V)$. Up to taking the submodel of $\mathfrak{M}$ generated by w, we can assume that $\mathfrak{M}$ is rooted in w – see Blackburn et al. (2001, Section 2.1) and Baltag et al. (2021, Lemma V.10). Then there exists $A \subseteq W$ such that $w \in A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$. By the induction hypothesis, we have $[\![\varphi]\!]_{\mathfrak{M}} \subseteq [\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}}$, whence $A \subseteq [\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}}$. It follows that $\mathfrak{M}, w \vDash \nu p.\mathsf{tr}(\varphi)$, as desired.

Now assume that $\mathfrak{M}, w \vDash \mu p.\varphi$ and $\mathfrak{M}, w \vDash \Box^+(\mathsf{tr}(\varphi) \to p)$. Since $\mathfrak{M}$ is rooted in w, this implies $[\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}} \subseteq [\![p]\!]_{\mathfrak{M}}$. By the induction hypothesis, we also have $[\![\varphi]\!]_{\mathfrak{M}} \subseteq [\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}}$, and thus $[\![\varphi]\!]_{\mathfrak{M}} \subseteq [\![p]\!]_{\mathfrak{M}}$. If we set $A := [\![p]\!]_{\mathfrak{M}}$, we obtain $[\![\varphi]\!]_{\mathfrak{M}[p:=A]} = [\![\varphi]\!]_{\mathfrak{M}} \subseteq A$, and thus $[\![\mu p.\varphi]\!]_{\mathfrak{M}} \subseteq A$. Since $\mathfrak{M}, w \vDash \mu p.\varphi$, it follows that $w \in A$, i.e., $\mathfrak{M}, w \vDash p$. Therefore, $\mathfrak{M}, w \vDash \Box^+(\mathsf{tr}(\varphi) \to p) \to p$, as desired. □

For the other direction, we will need to transform a model of $\mathsf{tr}(\varphi)$ into a model of φ. This is obtained by tweaking a valuation in a way that makes any formula of the form $\mu p.\psi$ coextensive with p.

Definition 18. *Let $\mathfrak{M} = (W, R, V)$ be a* **wK4** *model, and let $\varphi \in \mathscr{L}_\mu$. We define a valuation V^φ as follows: for any subformula of φ of the form $\mu p.\psi$, we set $V^\varphi(p) := [\![\mu p.\psi]\!]_{\mathfrak{M}}$, and for any other $q \in \mathsf{Prop}$ we set $V^\varphi(q) := V(q)$. We then define $\mathfrak{M}^\varphi := (W, R, V^\varphi)$.*

Note that $\mathfrak{M}^\varphi$ is well-defined precisely because the formula φ is clean.

Lemma 19. *Let $\mathfrak{M} = (W, R, V)$ be a* **wK4** *model, and let $w \in W$ and $\varphi \in \mathscr{L}_\mu$. If $\mathfrak{M}^\varphi, w \vDash \mathsf{tr}(\varphi)$ then $\mathfrak{M}, w \vDash \varphi$.*

Proof. By induction on φ. Again, this is straightforward for Boolean and modal formulas. Suppose that $\mathfrak{M}^{\nu p.\varphi}, w \vDash \nu p.\mathsf{tr}(\varphi)$. Then, there exists $A \subseteq W$ such that $w \in A \subseteq [\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}^{\nu p.\varphi}[p:=A]}$. By the induction hypothesis, we have

$$[\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}[p:=A]^\varphi} \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$$

and by construction we also have $\mathfrak{M}^{\nu p.\varphi}[p := A] = \mathfrak{M}[p := A]^\varphi$. It follows that $A \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$, and therefore $\mathfrak{M}^{\nu p.\varphi}, w \vDash \nu p.\varphi$.

Now suppose that $\mathfrak{M}^{\mu p.\varphi}, w \vDash \Box^+(\mathsf{tr}(\varphi) \to p) \to p$. We write $A := [\![\mu p.\varphi]\!]_{\mathfrak{M}}$ and then the fixed point equation gives $A = [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$. By the induction hypothesis, we also have $[\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}[p:=A]^\varphi} \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]}$, and $\mathfrak{M}[p := A]^\varphi = \mathfrak{M}^{\mu p.\varphi}$ by construction, so

$$[\![\mathsf{tr}(\varphi)]\!]_{\mathfrak{M}^{\mu p.\varphi}} \subseteq [\![\varphi]\!]_{\mathfrak{M}[p:=A]} = A = [\![\mu p.\varphi]\!]_{\mathfrak{M}} = [\![p]\!]_{\mathfrak{M}^{\mu p.\varphi}}.$$

Hence, $\mathfrak{M}^{\mu p.\varphi} \vDash \mathsf{tr}(\varphi) \to p$, and in particular $\mathfrak{M}^{\mu p.\varphi}, w \vDash \Box^+(\mathsf{tr}(\varphi) \to p)$. By assumption, it follows that $\mathfrak{M}^{\mu p.\varphi}, w \vDash p$. Therefore, $\mathfrak{M}, w \vDash \mu p.\varphi$. □

We can now conclude with the desired result.

Theorem 20. *For all formulas $\varphi \in \mathscr{L}_\mu$, we have $\mu\mathbf{wK4} + \varphi = \mu\mathbf{wK4} + \mathsf{tr}(\varphi)$.*

Proof. Let $\varphi \in \mathscr{L}_\mu$. From Lemma 17, we know that $\mu\mathbf{wK4} \vdash \varphi \to \mathsf{tr}(\varphi)$ and therefore $\mu\mathbf{wK4} + \varphi \vdash \mathsf{tr}(\varphi)$. Conversely, let $\overline{\psi} = (\mu p_1.\psi_1, \ldots, \mu p_n.\psi_n)$ be the tuple of all formulas of the form $\mu p.\psi$ occurring in φ, and $\overline{p} := (p_1, \ldots, p_n)$. Given a pointed $\mathbf{wK4}$ model $(\mathfrak{M}, w)$, we prove that $\mathfrak{M}, w \vDash \mathsf{tr}(\varphi)[\overline{\psi}/\overline{p}] \to \varphi$. For suppose $\mathfrak{M}, w \vDash \mathsf{tr}(\varphi)[\overline{\psi}/\overline{p}]$. This yields $\mathfrak{M}^{\varphi}, w \vDash \mathsf{tr}(\varphi)$, and then $\mathfrak{M}, w \vDash \varphi$ from Lemma 19. By Theorem 15, it follows that $\mu\mathbf{wK4} \vdash \mathsf{tr}(\varphi)[\overline{\psi}/\overline{p}] \to \varphi$. By uniform substitution, we also have $\mu\mathbf{wK4} + \mathsf{tr}(\varphi) \vdash \mathsf{tr}(\varphi)[\overline{\psi}/\overline{p}]$, and therefore $\mu\mathbf{wK4} + \mathsf{tr}(\varphi) \vdash \varphi$. □

As an immediate consequence, we obtain $\mathscr{C}(\varphi) = \mathscr{C}(\mathsf{tr}(\varphi))$ for all $\varphi \in \mathscr{L}_\mu$, and this yields the following result.

Corollary 21. *For all classes $\mathscr{C}_0$ of derivative spaces, the language $\mathscr{L}_\mu$ is as expressive as $\mathscr{L}^0_\mu$ over $\mathscr{C}_0$.*

4. Classes Defined by Greatest Fixed Points

The goal of this section is to exhibit μ-definable classes that are not modally definable. Thanks to the previous section, we know that we can restrict our attention to formulas without least fixed points. It turns out that a large family of axioms of the form $\theta \vee \nu p.\Diamond p$ will yield the desired result. We easily see that given a pointed Kripke model $(\mathfrak{M}, w)$, we have $\mathfrak{M}, x \vDash \nu p.\Diamond p$ if and only if there exists an infinite path beginning on w. Topologically, $\nu p.\Diamond p$ holds in the *perfect core* of X, the largest dense-in-itself subset of X. While the existence of an infinite path is not in general modally definable, it is not hard to check that $\mathscr{C}(\nu p.\Diamond p) = \mathscr{C}(\Diamond\top)$, as this is just the class of dense-in-themselves spaces. However, the story becomes more complicated if we only require certain points in the space to satisfy $\nu p.\Diamond p$. In this case, the following can be applied to exhibit many modally undefinable classes of spaces.

Theorem 22. *Let $\theta \in \mathscr{L}_\mu$ and suppose that for all $n \in \mathbb{N}$, there exists a* **wK4** *frame $\mathfrak{F}_n = (W_n, R_n)$ and $r_n \in W_n$ such that:*

(1) $\mathfrak{F}_n$ is rooted in r_n and $\mathfrak{F}_n, r_n \nvDash \theta \vee \nu p.\Diamond p$;
(2) $\mathfrak{F}_n$ contains a path of size n;
(3) for all $w \in W_n \setminus \{r_n\}$ we have $\mathfrak{F}_n, w \vDash \theta$.

Then, $\mathscr{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathbf{K4}}$. If in addition every $\mathfrak{F}_n$ is finite, then $\mathscr{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathsf{fin}}$ and $\mathscr{C}_{\mathsf{Kripke}} \cap \mathscr{C}_{\mathsf{fin}} \cap \mathscr{C}_{\mathbf{K4}}$.

Remark 23. We recall that both Kripke frames and topological spaces are identified with their respective derivative spaces, so $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathbf{K4}}$ can equivalently be regarded as the class of all T_d Aleksandroff spaces, and $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathsf{fin}}$ as the class of finite topological spaces. Thus, Theorem 22 applies to classes of topological spaces, as well as Kripke frames.

Remark 24. It is easily observed that if $\mathscr{C}$ is not modally definable within $\mathscr{C}_0$ and $\mathscr{C}_0 \subseteq \mathscr{C}_1$, then $\mathscr{C}$ is not modally definable within $\mathscr{C}_1$ as well. This allows us to draw interesting consequences

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

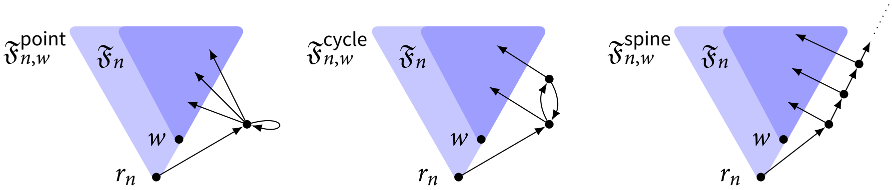

Figure 1. The frames $\mathfrak{F}_{n,w}^{\text{point}}$, $\mathfrak{F}_{n,w,}^{\text{cycle}}$ and $\mathfrak{F}_{n,w}^{\text{spine}}$.

from Theorem 22, as $\mathscr{C}_{\text{irrefl}} \cap \mathscr{C}_{\mathbf{K4}}$ is a subclass of $\mathscr{C}_{\text{all}}$, $\mathscr{C}_{\text{Kripke}}$, $\mathscr{C}_{\text{topo}}$, $\mathscr{C}_{\text{topo}} \cap \mathscr{C}_{\mathbf{K4}}$, and many other relevant classes.

From now on, we fix a formula θ and a family of frames $(\mathfrak{F}_n)_{n\in\mathbb{N}}$ satisfying the assumptions of Theorem 22. For all $n \in \mathbb{N}$, we assume that $W_n \cap \omega = \varnothing$. We start with an elementary observation.

Claim 25. For all $n \in \mathbb{N}$, the frame $\mathfrak{F}_n$ is irreflexive and transitive.

Proof. First assume that $\mathfrak{F}_n$ is not irreflexive, so that there is w with wR_nw. Then, $(r_n, w, w\ldots)$ is an infinite path beginning on r_n, contradicting $\mathfrak{F}_n, r_n \not\models \nu p.\Diamond p$. If instead $\mathfrak{F}_n$ is not transitive, then since $\mathfrak{F}_n$ is weakly transitive, this can only occur if there exist $w, u \in W_n$ such that wR_nu, uR_nw and not wR_nw. Then, $(r_n, w, u, w, u, \ldots)$ is an infinite path beginning on r_n – or else $(w, u, w, u, \ldots)$ in case $w = r_n$. $\square$

Given a world $w \in W_n$, we define the **wK4** frames $\mathfrak{F}_{n,w}^{\text{point}} = (W^0, R^0)$, $\mathfrak{F}_{n,w}^{\text{cycle}} = (W^1, R^1)$ and $\mathfrak{F}_{n,w}^{\text{spine}} = (W^2, R^2)$ by:

$$\begin{aligned}
W^0 &:= W_n \cup \{0\}\\
R^0 &:= R_n \cup \{(r_n, 0), (0, 0)\} \cup \{(0, u) \mid wR_n^+u\}\\
W^1 &:= W_n \cup \{0, 1\}\\
R^1 &:= R_n \cup \{(r_n, 0), (r_n, 1), (0, 1), (1, 0)\} \cup \{(k, u) \mid k \in \{0, 1\} \text{ and } wR_n^+u\}\\
W^2 &:= W_n \cup \omega\\
R^2 &:= R_n \cup \{(r_n, k) \mid k \in \omega\} \cup \{(m, k) \mid m < k < \omega\} \cup \{(k, u) \mid k \in \omega, wR_n^+u\}
\end{aligned}$$

In words, $\mathfrak{F}_{n,w}^{\text{point}}$ is the frame $\mathfrak{F}_n$ endowed with a reflexive point reachable from the root, and which sees all the successors of w (as well as w itself). The frames $\mathfrak{F}_{n,w}^{\text{cycle}}$ and $\mathfrak{F}_{n,w}^{\text{spine}}$ are constructed similarly but with respectively a two-element loop and an infinite branch, instead of a reflexive point. The three frames are depicted in Fig. 1.

If some modal formula ψ defines the same class of spaces as $\theta \vee \nu p.\Diamond p$, then by construction ψ should be refuted at $(\mathfrak{F}_n, r_n)$ for all n but not at $(\mathfrak{F}_{n,w}^{\text{spine}}, r_n)$ or $(\mathfrak{F}_{n,w}^{\text{cycle}}, r_n)$ or $(\mathfrak{F}_{n,w}^{\text{point}}, r_n)$, since in all three of them there is an infinite path beginning on the root. Yet we will prove that if n is big enough and $\neg\psi$ is satisfiable on $(\mathfrak{F}_n, r_n)$, then it is also satisfiable on $(\mathfrak{F}_{n,w}^{\text{point}}, r_n)$ for some w, leading to a contradiction.[3] The proof is rather technical, but we can sketch the main lines of our strategy. First, it is clear that transferring the satisfiability of a diamond formula (i.e., of the form $\Diamond\varphi$) or a Boolean formula from $(\mathfrak{F}_n, r_n)$ to $(\mathfrak{F}_{n,w}^{\text{point}}, r_n)$ is immediate, so the challenge really comes from box formulas (of the form $\Box\varphi$). The central argument is that since n may be arbitrarily large, we can select some $\mathfrak{F}_n$ with an arbitrarily long path. By means of a pigeonhole argument, we will

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

then manage to show that on some point w of this path, if $\Box\varphi$ is satisfied, then so is $\Box^+\varphi$ (when $\Box\varphi$ is any subformula of $\neg\psi$). Then, transferring the truth of $\Box\varphi$ to the reflexive point of $\mathfrak{F}_{n,w}^{\mathsf{point}}$ will be straightforward. First, we will need a notion of *type* of a possible world.

Definition 26. *Let φ be a modal formula. We write $\psi \trianglelefteq \varphi$ whenever ψ is a subformula of φ. We also call the* box size $|\varphi|_\Box$ *of φ the number of subformulas of φ of the form $\Box\psi$. If $\mathfrak{M}$ is a derivative model and w a world in $\mathfrak{M}$, we define the* box type *of w relative to φ as the set $t^{\varphi}_{\mathfrak{M}}(w) := \{\Box\psi \mid \Box\psi \trianglelefteq \varphi \text{ and } \mathfrak{M}, w \vDash \Box\psi\}$.*

As explained above, the following result allows to transfer the satisfiability of box formulas, as soon as the parameter n is large enough.

Claim 27. Let φ be a modal formula in NNF and $n > 2^{|\varphi|_\Box}$. Suppose that there exists a valuation V over $\mathfrak{F}_n$ such that $\mathfrak{F}_n, V, r_n \vDash \Box\varphi$. Then there exists a world $w \in W_n$ and a valuation V' over $\mathfrak{F}_{n,w}^{\mathsf{point}}$ such that $\mathfrak{F}_{n,w}^{\mathsf{point}}, V', r_n \vDash \Box\varphi$, and V and V' coincide over $\mathfrak{F}_n$.

Proof. First, we know that $\mathfrak{F}_n$ contains a path $(w_i)_{i\in[1,n]}$ of size n. By construction, there are $2^{|\varphi|_\Box}$ different box types relative to φ. Thus, by the pigeonhole principle, there exists $i, j \in \mathbb{N}$ such that $1 \le i < j \le n$ and $t^{\varphi}_{\mathfrak{M}}(w_i) = t^{\varphi}_{\mathfrak{M}}(w_j)$. We then define a valuation V' over $\mathfrak{F}_{n,w_j}^{\mathsf{point}}$ by setting, for all $p \in \mathsf{Prop}$:

$$V'(p) := \begin{cases} V(p) \cup \{0\} & \text{if } w_j \in V(p) \\ V(p) & \text{otherwise} \end{cases}.$$

So V and V' coincide over $\mathfrak{F}_n$, and V' is defined over 0 so that this point satisfies the same atomic propositions as w_j. We then prove by induction on $\psi \trianglelefteq \varphi$ that $\mathfrak{F}_n, V, w_j \vDash \psi$ implies $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', 0 \vDash \psi$:

- If ψ is of the form $\psi = p$ or $\psi = \neg p$ with $p \in \mathsf{Prop}$ this is just true by construction.
- If ψ is of the form $\psi = \psi_1 \wedge \psi_2$, then $\mathfrak{F}_n, V, w_j \vDash \psi_1 \wedge \psi_2$ implies $\mathfrak{F}_n, V, w_j \vDash \psi_1$ and $\mathfrak{F}_n, V, w_j \vDash \psi_2$ and it suffices to apply the induction hypothesis. If ψ is of the form $\psi = \psi_1 \vee \psi_2$, then $\mathfrak{F}_n, V, w_j \vDash \psi_1 \vee \psi_2$ implies $\mathfrak{F}_n, V, w_j \vDash \psi_1$ or $\mathfrak{F}_n, V, w_j \vDash \psi_2$ and the result follows in the same way.
- Suppose that ψ is of the form $\psi = \Diamond\psi_0$ and $\mathfrak{F}_n, V, w_j \vDash \psi$. Then since V and V' coincide over $\mathfrak{F}_n$, we have $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', w_j \vDash \psi$ as well. By transitivity, it follows that $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', 0 \vDash \psi$.
- Suppose that ψ is of the form $\psi = \Box\psi_0$ and that $\mathfrak{F}_n, V, w_j \vDash \psi$. Then since $t^{\varphi}_{\mathfrak{M}}(w_i) = t^{\varphi}_{\mathfrak{M}}(w_j)$, we have $\mathfrak{F}_n, V, w_i \vDash \psi$ as well. Since $w_i R_n w_j$ it follows $\mathfrak{F}_n, V, w_j \vDash \psi_0$, and then $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', 0 \vDash \psi_0$ by the induction hypothesis. Since V and V' coincide over $\mathfrak{F}_n$, we also have $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', w_j \vDash \Box^+\psi_0$. All in all, we obtain $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', 0 \vDash \Box\psi_0$ as desired.

Now observe that since $w_i R_n w_j$ we must have $w_j \neq r_n$; otherwise, we would obtain $r_n R_n r_n$ by transitivity, contradicting Claim 25. Thus, $r_n R_n w_j$, and from $\mathfrak{F}_n, V, r_n \vDash \Box\varphi$ we obtain $\mathfrak{F}_n, V, w_j \vDash \varphi$, whence $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', 0 \vDash \varphi$. Since V and V' coincide over $\mathfrak{F}_n$, we conclude that $\mathfrak{F}_{n,w_j}^{\mathsf{point}}, V', r_n \vDash \Box\varphi$. □

We can then extend the result to any modal formula.

Claim 28. Let φ be a modal formula. There exists $n \in \mathbb{N}$ such that if φ is satisfiable on $(\mathfrak{F}_n, r_n)$, then there exists a world $w \in W_n$ such that φ is satisfiable on $\mathfrak{F}_{n,w}^{\mathsf{spine}}$ and $\mathfrak{F}_{n,w}^{\mathsf{cycle}}$ and $\mathfrak{F}_{n,w}^{\mathsf{point}}$.

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

Proof. Applying the theorem of disjunctive normal form for propositional logic, and using the fact that $\Box$ and $\wedge$ commute, we can assume that φ is of the form $\varphi = \bigvee_{i=1}^{m} \sigma_i$ with

$$\sigma_i = \rho_i \wedge \Box\psi_i \wedge \bigwedge_{j=1}^{m_i} \Diamond\theta_{i,j}$$

for all $i \in [1, m]$, where ρ_i is a propositional formula. Note that since $\Box\top$ is a tautology, we can always assume the presence of $\Box\psi_i$. We also suppose that ψ_i is presented in NNF. We then define

$$n := 1 + \max\{2^{|\psi_i|_\Box} \mid 1 \leq i \leq m\}$$

and assume that there exists a valuation V such that $\mathfrak{F}_n, V, r_n \models \varphi$. Then there exists $i \in [1, m]$ such that $\mathfrak{F}_n, V, r_n \models \sigma_i$. It follows that $\mathfrak{F}_n, V, r_n \models \Box\psi_i$ with $n > 2^{|\psi_i|_\Box}$, so by Claim 27 there exists $w \in W_n$ and a valuation V' over $\mathfrak{F}_{n,w}^{\mathsf{point}}$ such that $\mathfrak{F}_{n,w}^{\mathsf{point}}, V', r_n \models \Box\psi_i$, and V and V' coincide over $\mathfrak{F}_n$. It is then clear that $\mathfrak{F}_{n,w}^{\mathsf{point}}, V', r_n \models \sigma_i$, and thus $\mathfrak{F}_{n,w}^{\mathsf{point}}, V', r_n \models \varphi$.

This proves that φ is satisfiable on $\mathfrak{F}_{n,w}^{\mathsf{point}}$. Now consider the function $f : W_1 \to W_0$ defined by $f(0) := f(1) := 0$ and $f(w) := w$ for all $w \in W_n$. Likewise, we define a function $g : W_2 \to W_0$ by $g(n) := 0$ for all $n \in \omega$, and $g(w) := w$ for all $w \in W_n$. Then, f defines a bounded morphism from $\mathfrak{F}_{n,w}^{\mathsf{cycle}}$ to $\mathfrak{F}_{n,w}^{\mathsf{point}}$, and g defines a bounded morphism from $\mathfrak{F}_{n,w}^{\mathsf{spine}}$ to $\mathfrak{F}_{n,w}^{\mathsf{point}}$. It follows that φ is satisfiable on $\mathfrak{F}_{n,w}^{\mathsf{spine}}$ and $\mathfrak{F}_{n,w}^{\mathsf{cycle}}$. $\Box$

We are now ready to prove Theorem 22:

Proof. Suppose toward a contradiction that there is a formula $\psi \in \mathscr{L}_\Diamond$ defining the same class as $\theta \vee \nu p.\Diamond p$ within $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathbf{K4}}$. Let n be the integer obtained by applying Claim 28 to $\neg\psi$. By Claim 25, the frame $\mathfrak{F}_n$ is irreflexive and transitive, and we also have $\mathfrak{F}_n \not\models \theta \vee \nu p.\Diamond p$ by assumption, so $\mathfrak{F}_n \not\models \psi$ as well.

Thus, $\neg\psi$ is satisfiable on $(\mathfrak{F}_n, v)$ for some $v \in W_n$. If $v \neq r_n$, we denote by $\mathfrak{F}$ the subframe of $\mathfrak{F}_n$ generated by v. Then, $\mathfrak{F}$ does not contain r_n; otherwise, we would have $vR_n r_n R_n v$ and thus $vR_n v$, a contradiction. The assumption on $\mathfrak{F}_n$ yields $\mathfrak{F} \models \theta$, so $\mathfrak{F} \models \theta \vee \nu p.\Diamond p$ and thus $\mathfrak{F} \models \psi$. Therefore, $\mathfrak{F}_n, v \models \psi$, a contradiction. Hence, we have $r_n = v$. Then, by Claim 28, there exists $w \in W_n$ such that $\neg\psi$ is satisfiable on $\mathfrak{F}_{n,w}^{\mathsf{spine}}$. Yet $\mathfrak{F}_{n,w}^{\mathsf{spine}} \in \mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathbf{K4}}$ and $\mathfrak{F}_{n,w}^{\mathsf{spine}} \models \theta \vee \nu p.\Diamond p$, so $\mathfrak{F}_{n,w}^{\mathsf{spine}} \models \psi$, a contradiction.

Now suppose that every $\mathfrak{F}_n$ is finite. By the same reasoning, we can show that $\mathscr{C}(\theta \vee \nu p.\Diamond p)$ is not modally definable within $\mathscr{C}_{\mathsf{irrefl}} \cap \mathscr{C}_{\mathsf{fin}}$ and $\mathscr{C}_{\mathsf{Kripke}} \cap \mathscr{C}_{\mathsf{fin}} \cap \mathscr{C}_{\mathbf{K4}}$. To that end, it suffices to replace $\mathfrak{F}_{n,w}^{\mathsf{spine}}$ by, respectively, $\mathfrak{F}_{n,w}^{\mathsf{cycle}}$, which is irreflexive and finite, and $\mathfrak{F}_{n,w}^{\mathsf{point}}$, which is transitive and finite. $\Box$

Theorem 22 remains a very general statement, and it is worth instantiating it with examples. The following result shows the existence of infinitely many non-modally definable classes of spaces.

Proposition 29. *Given $m \in \mathbb{N}$ we define $.2_m^+ := (\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q) \vee \Box^m\bot$ and $\mathsf{IP}.2_m^+ := .2_m^+ \vee \nu p.\Diamond p$. Then the class of topological spaces X such that $X \models \mathsf{IP}.2_m^+$ is not modally definable. In addition, whenever $m, k \geq 1$ and $m \neq k$ we have $\mu\mathbf{wK4} + \mathsf{IP}.2_m^+ \neq \mu\mathbf{wK4} + \mathsf{IP}.2_k^+$.*

Proof. It suffices to prove that the assumptions of Theorem 22 are satisfied for $\theta := .2_m^+$. In $\Diamond^+\Box^+ q \to \Box^+\Diamond^+ q$, we recognize a variant of the axiom .2 – see Chagrov and Zakharyaschev (1997), but relative to the reflexive closure R^+; we call it $.2^+$, and this also explains the name $.2_m^+$. Thus, given a frame $\mathfrak{F} = (W, R)$ we have $\mathfrak{F} \models \mathsf{IP}.2_m^+$ iff for all $w \in W$ one of the following holds:

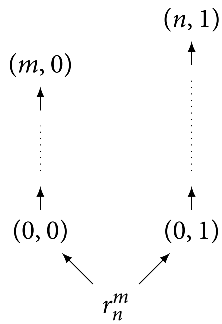

Figure 2. The fork-like frame $\mathfrak{F}_n^m$.

- for all $u, v \in W$ such that wR^+u and wR^+v, there exists $t \in W$ such that uR^+t and vR^+t;
- there exists no path of size $m+1$ beginning on w;
- there exists an infinite path beginning on w.

Consider, for all $n \in \mathbb{N}$, the frame $\mathfrak{F}_n^m := (W_n^m, R_n^m)$ depicted in Fig. 2. We can see that the $\mathfrak{F}_n^m$'s fulfill all the conditions of Theorem 22, so we are done (see Remark 24 for why the result applies to topological spaces). Finally, if $1 \leq m < k$ we can see that $\mathfrak{F}_1^{m-1} \vDash \mathsf{IP}.2_m^+$ whereas $\mathfrak{F}_1^{m-1} \nvDash \mathsf{IP}.2_k^+$, and this proves that $\mu\mathbf{wK4} + \mathsf{IP}.2_m^+ \neq \mu\mathbf{wK4} + \mathsf{IP}.2_k^+$. □

In Section 6, we will analyze these axioms further to see that they are well-behaved, but we find it appropriate to end this section by presenting an intuitive topological interpretation of the axiom $\mathsf{IP}.2_0^+$, which reduces to $.2^+ \vee \nu p.\Diamond p$. Given a formula θ and a space X, we say that X is *θ-imperfect* if there exist two disjoint subspaces Y and Z of X such that $X = Y \cup Z$, $Y \vDash \theta$ and Z is dense-in-itself.

Proposition 30. *Let $\theta \in \mathscr{L}_\mu$, and let X be a topological space. Then, $X \vDash \theta \vee \nu p.\Diamond p$ if and only if X is θ-imperfect.*

Proof. From left to right, assume that $X \vDash \theta \vee \nu p.\Diamond p$. We set $Z := \{x \in X \mid X, x \vDash \nu p.\Diamond p\}$ (the perfect core of X) and $Y := X \setminus Z$. The fixed point equation immediately gives $Z = \mathrm{d}(Z)$, so Z is dense-in-itself. From $\mathrm{d}(Z) \subseteq Z$, we also obtain that Z is closed and Y is open. Now, let $x \in Y$ and V be a valuation over Y. We have $X, V, x \vDash \theta \vee \nu p.\Diamond p$ and by construction $X, V, x \nvDash \nu p.\Diamond p$, so $X, V, x \vDash \theta$. Since Y is open, we obtain $Y, V, x \vDash \theta$. Therefore, $Y \vDash \theta$.

From right to left, suppose that such a decomposition $X = Y \cup Z$ exists. Let $x \in X$ and V be a valuation over X. Suppose that $x \in Z$. Since Z is dense-in-itself, we have $Z \subseteq \mathrm{d}(Z) = [\![\Diamond p]\!]_{X, V[p:=Z]}$ so $Z \subseteq [\![\nu p.\Diamond p]\!]_{X,V}$. Therefore, $X, V, x \vDash \nu p.\Diamond p$. Otherwise, we have $x \in Y$. If $x \notin \mathrm{Int}(Y)$, then $x \in \mathrm{Cl}(Z)$ and since $x \notin Z$ it follows that $x \in \mathrm{d}(Z)$. We have seen that $X, V, z \vDash \nu p.\Diamond p$ for all $z \in Z$, so $X, V, x \vDash \Diamond \nu p.\Diamond p$, and then the fixed point equation gives $X, V, x \vDash \nu p.\Diamond p$. Otherwise, we have $x \in \mathrm{Int}(Y)$. Since $Y \vDash \theta$ and $\mathrm{Int}(Y)$ is open in Y, we have $\mathrm{Int}(Y) \vDash \theta$. Then, $\mathrm{Int}(Y), V, x \vDash \theta$ and since $\mathrm{Int}(Y)$ is open, we finally get $X, V, x \vDash \theta$. In all cases, we obtain $X, V, x \vDash \theta \vee \nu p.\Diamond p$ as desired. □

Remark 31. By inspection of the proof for the left-to-right implication, we can also assume that Y is scattered and Z is perfect (i.e., closed and dense-in-itself). This explains and justifies the name "θ-imperfect."

In our example, the axiom $.2^+$ is known to define the class of extremally disconnected spaces – see Definition 11 and also van Benthem and Bezhanishvili (2007). We thus obtain the following result:

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

Corollary 32. *The class of spaces that can be written as the disjoint union of an extremally disconnected subspace and a perfect subspace is not modally definable.*

5. Completeness for Imperfect Spaces

We have shown that there are μ-definable classes that are not modally definable, including infinitely many classes of imperfect spaces. We can make these examples even stronger by showing that the logics we have exhibited are complete for these classes. To this end, we construct the canonical model and use the technique of the final model applied by Fine and Zakharyashev to modal logic (see Bezhanishvili et al. 2011; Chagrov and Zakharyaschev 1997) and by Baltag et al. (2021) to the μ-calculus. Central will be the notion of cofinal subframe logic.

Definition 33. *Let $\mathfrak{F} = (W, R)$ be a Kripke frame. A subframe $\mathfrak{F}' = (W', R')$ of $\mathfrak{F}$ is called a* cofinal subframe *of $\mathfrak{F}$ if $w' \in W'$ and $w'Rw$ implies the existence of $u' \in W'$ such that wR^+u'. Given $\mathfrak{M}$ based on $\mathfrak{F}$ and $\mathfrak{M}'$ a submodel of $\mathfrak{M}$, we call $\mathfrak{M}'$ a* cofinal submodel *of $\mathfrak{M}$ if it is based on a cofinal subframe $\mathfrak{F}'$ of $\mathfrak{F}$.*

Definition 34. *Let $\mathbf{L}$ be an extension of $\mathbf{K}$. The logic $\mathbf{L}$ is called* cofinal subframe *if whenever $\mathfrak{F} \vDash \mathbf{L}$ and $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}$, we have $\mathfrak{F}' \vDash \mathbf{L}$.*

Definition 35. *Let $\mathbf{L}$ be an extension of $\mathbf{K}$. Let $P \subseteq \mathsf{Prop}$. The* canonical model *of $\mathbf{L}$ over P is the model $\mathfrak{M} := (\Omega, R, V)$ with:*

- *Ω the set of maximal $\mathbf{L}$-consistent subsets of $\mathscr{L}_\Diamond$;*
- *$R := \{(\Gamma, \Delta) \mid \Box\varphi \in \Gamma \implies \varphi \in \Delta\}$;*
- *$V(p) := \{\Gamma \in \Omega \mid p \in \Gamma\}$ for all $p \in \mathsf{Prop}$.*

The so-called *Truth Lemma* then establishes an equivalence between truth and membership at the worlds of $\mathfrak{M}$, that is, $\mathfrak{M}, \Gamma \vDash \varphi$ if and only if $\varphi \in \Gamma$. Combined with the Lindenbaum's lemma, this yields completeness of $\mathbf{L}$ with respect to its canonical model – see Blackburn et al. (2001, Section 4.2). If $\mathbf{L}$ is an extension of $\mu\mathbf{wK4}$, the canonical model is defined in the same way, but the Truth Lemma then fails to hold. The technique designed by Baltag et al. (2021) consists in restricting oneself to an appropriate cofinal submodel of $\mathfrak{M}$. First, given a $\mathbf{L}$-consistent formula ψ, one can construct a finite set of formulas Σ containing ψ, closed under subformulas, and closed (up to logical equivalence in $\mathbf{L}$) under negation and $\Diamond^+$. We then define the so-called Σ-final model as follows.

Definition 36. *A world $\Gamma \in \Omega$ is called* Σ-final *if there exists $\varphi \in \Sigma \cap \Gamma$ such that whenever $\Gamma R \Delta$ and $\varphi \in \Delta$, we have $\Delta R \Gamma$. The* Σ-final model *is then the submodel $\mathfrak{M}_\Sigma$ of $\mathfrak{M}$ induced by $\Omega_\Sigma := \{\Gamma \in \Omega \mid \Gamma$ is Σ-final$\}$.*

Under the right assumptions, it can be proven that (1) $\mathfrak{M}_\Sigma$ is a cofinal submodel of $\mathfrak{M}$, (2) ψ belongs to some Σ-final world, and (3) the Truth Lemma holds in $\mathfrak{M}_\Sigma$ for the formulas in Σ. This yields Kripke completeness of $\mu\mathbf{wK4}$ and, in fact, of any logic of the form $\mu\mathbf{wK4} + \theta$ where $\theta \in \mathscr{L}_\Diamond$ and $\mathbf{wK4} + \theta$ is a canonical and cofinal subframe logic. Note that this result is limited to extensions of $\mu\mathbf{wK4}$ with *basic* modal axioms. By contrast, the present work is novel in that it offers completeness results for axioms with fixed points. First, we need a technical lemma.

Lemma 37. *If $\mu\mathbf{wK4} + \theta \vdash \varphi$, then $\mu\mathbf{wK4} + (\theta \vee \nu p.\Diamond p) \vdash \varphi \vee \nu p.\Diamond p$.*

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

Proof. We write $\mathbf{L}_0 := \mu\mathbf{wK4} + \theta$ and $\mathbf{L} := \mu\mathbf{wK4} + (\theta \vee \nu p.\Diamond p)$. We proceed by induction on the length of a proof.

- If φ is an axiom of $\mu\mathbf{wK4}$ or θ itself, then this is clear.
- Suppose that this holds for φ, and that $\mathbf{L}_0 \vdash \varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n]$ is obtained from $\mathbf{L}_0 \vdash \varphi$. By the induction hypothesis, we have $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$ and by substitution it follows that $\mathbf{L} \vdash \varphi[\psi_1, \ldots, \psi_n/p_1, \ldots, p_n] \vee \nu p.\Diamond p$.
- Suppose that this holds for φ and $\varphi \to \psi$, and that $\mathbf{L}_0 \vdash \psi$ is obtained from $\mathbf{L}_0 \vdash \varphi$ and $\mathbf{L}_0 \vdash \varphi \to \psi$. By the induction hypothesis, we have $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$ and $\mathbf{L} \vdash (\varphi \to \psi) \vee \nu p.\Diamond p$, and we deduce $\mathbf{L} \vdash \psi \vee \nu p.\Diamond p$.
- Suppose that this holds for $\varphi \to \psi$, and that $\mathbf{L}_0 \vdash \Diamond\varphi \to \Diamond\psi$ is obtained from $\mathbf{L}_0 \vdash \varphi \to \psi$. By the induction hypothesis, we have $\mathbf{L} \vdash (\varphi \to \psi) \vee \nu p.\Diamond p$, whence $\mathbf{L} \vdash \varphi \to (\psi \vee \nu p.\Diamond p)$. By monotonicity, it follows that $\mathbf{L} \vdash \Diamond\varphi \to \Diamond(\psi \vee \nu p.\Diamond p)$. Since $\Diamond$ and $\vee$ commute, we get $\mathbf{L} \vdash (\Diamond\varphi \to \Diamond\psi) \vee \Diamond\nu p.\Diamond p$. Further, by applying the induction rule to $\mathbf{L} \vdash \Diamond\nu p.\Diamond p \to \Diamond\nu p.\Diamond p$, we obtain $\mathbf{L} \vdash \Diamond\nu p.\Diamond p \to \nu p.\Diamond p$. Therefore, $\mathbf{L} \vdash (\Diamond\varphi \to \Diamond\psi) \vee \nu p.\Diamond p$.
- Suppose that this holds for $\varphi \to \psi[\varphi/p]$ and that $\mathbf{L}_0 \vdash \varphi \to \nu p.\psi$ is obtained from $\mathbf{L}_0 \vdash \varphi \to \psi[\varphi/p]$. By the induction hypothesis, we have
$$\mathbf{L} \vdash \nu p.\Diamond p \vee (\varphi \to \psi[\varphi/p])$$
and we prove that
$$\mu\mathbf{wK4} \vdash \psi[\varphi/p] \wedge \neg\nu p.\Diamond p \to \psi[\varphi \wedge \neg\nu p.\Diamond p/p].$$
Indeed, consider a **wK4** frame $\mathfrak{M}$ rooted in w and assume that $\mathfrak{M}, w \vDash \psi[\varphi/p] \wedge \neg\nu p.\Diamond p$. From $\vDash \neg\nu p.\Diamond p \to \Box\neg\nu p.\Diamond p$, we obtain $\mathfrak{M} \vDash \neg\nu p.\Diamond p$, so $\mathfrak{M} \vDash \varphi \leftrightarrow (\varphi \wedge \neg\nu p.\Diamond p)$ and thus $\mathfrak{M} \vDash \psi[\varphi/p] \leftrightarrow \psi[\varphi \wedge \neg\nu p.\Diamond p/p]$. Therefore, $\mathfrak{M}, w \vDash \psi[\varphi \wedge \neg\nu p.\Diamond p/p]$, and the result follows by Theorem 15. We then obtain
$$\mathbf{L} \vdash \varphi \wedge \neg\nu p.\Diamond p \to \psi[\varphi \wedge \neg\nu p.\Diamond p/p]$$
and by the induction rule, we derive $\mathbf{L} \vdash \varphi \wedge \neg\nu p.\Diamond p \to \nu p.\psi$, or equivalently $\mathbf{L} \vdash \nu p.\Diamond p \vee (\varphi \to \nu p.\psi)$.

□

Theorem 38. *Let θ be a modal formula such that* $\mathbf{wK4} + \theta$ *is cofinal subframe and canonical. Then,* $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ *is Kripke complete and has the finite model property.*

Proof. We write $\mathbf{L} := \mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ and $\mathbf{L}_0 := \mu\mathbf{wK4} + \theta$. Suppose that $\mathbf{L} \nvdash \neg\psi$, and let Σ be a finite set of formulas containing ψ and $\theta \vee \nu p.\Diamond p$, and with the closure properties enumerated above. We introduce

- $\mathfrak{M} = (\Omega, R, V)$ the canonical model of $\mathbf{L}$, based on $\mathfrak{F} = (\Omega, R)$;
- $\mathfrak{M}_\Sigma = (\Omega_\Sigma, R_\Sigma, V_\Sigma)$ the Σ-final submodel of $\mathfrak{M}$, based on $\mathfrak{F}_\Sigma = (\Omega_\Sigma, R_\Sigma)$;
- $\mathfrak{M}_0 = (\Omega_0, R_0, V_0)$ the canonical model of $\mathbf{L}_0$, based on $\mathfrak{F}_0 = (\Omega_0, R_0)$.

See Fig. 3 for a visual depiction of these frames. We know that $\mathfrak{F}_\Sigma$ is a cofinal subframe of $\mathfrak{F}$. In addition, we have $\mathbf{L} \subseteq \mathbf{L}_0$, so for all maximal consistent sets Γ such that $\mathbf{L}_0 \subseteq \Gamma$ we also have $\mathbf{L} \subseteq \Gamma$; it is also clear that R and R_0 coincide over Ω_0. Thus, $\mathfrak{F}_0$ is a subframe of $\mathfrak{F}$. We then introduce
$$\Omega' := \{\Gamma \in \Omega_\Sigma \mid \mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p\}$$
which induces a generated subframe $\mathfrak{F}' = (\Omega', R')$ of $\mathfrak{F}$. Indeed, if $\Gamma \in \Omega'$ and $\Gamma R_\Sigma \Delta$, then since $\mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p$ we have $\mathfrak{M}_\Sigma, \Delta \vDash \neg\nu p.\Diamond p$ too and thus $\Delta \in \Omega'$. Further, given $\Gamma \in \Omega'$ we have $\mathfrak{M}_\Sigma, \Gamma \vDash \neg\nu p.\Diamond p$, and we obtain $\neg\nu p.\Diamond p \in \Gamma$ by the Truth Lemma. If $\mathbf{L}_0 \vdash \varphi$, then $\mathbf{L} \vdash \varphi \vee \nu p.\Diamond p$

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

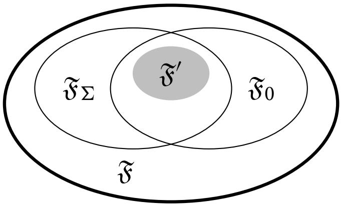

Figure 3. The canonical frame of **L** and its subframes.

by Lemma 37, and from $\varphi \vee \nu p.\Diamond p \in \Gamma$ and $\neg \nu p.\Diamond p \in \Gamma$ we deduce $\varphi \in \Gamma$. Therefore, $\mathbf{L}_0 \subseteq \Gamma$, and we obtain $\Gamma \in \Omega_0$. This proves that $\mathfrak{F}'$ is a subframe of $\mathfrak{F}_0$.

Now, suppose that $\Gamma \in \Omega'$, $\Delta \in \Omega_0$, and $\Gamma R \Delta$. Since $\mathfrak{F}_\Sigma$ is cofinal in $\mathfrak{F}$, there exists $\Lambda \in \Omega_\Sigma$ such that $\Delta R^+ \Lambda$. By weak transitivity, it follows that $\Gamma R^+ \Lambda$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$ it follows that $\Lambda \in \Omega'$. Therefore, $\mathfrak{F}'$ is a cofinal subframe of $\mathfrak{F}_0$. As observed by Baltag et al. (2021), that $\mathbf{wK4} + \theta$ is canonical implies that $\mu\mathbf{wK4} + \theta$ is canonical too, so $\mathfrak{F}_0 \vDash \theta$. Since $\mathbf{L}_0$ is cofinal subframe, it follows that $\mathfrak{F}' \vDash \theta$ as well.

We now show that $\mathfrak{F}_\Sigma \vDash \theta \vee \nu p.\Diamond p$. Let $V_\bullet$ be a valuation over Ω_Σ and $\Gamma \in \Omega_\Sigma$. If $\Gamma \in \Omega'$, let $(\Omega', R', V'_\bullet)$ be the submodel of $(\Omega_\Sigma, R_\Sigma, V_\bullet)$ induced by Ω'. We know that $(\Omega', R', V'_\bullet), \Gamma \vDash \theta$, and since $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, it follows that $(\Omega_\Sigma, R_\Sigma, V_\bullet), \Gamma \vDash \theta$. Otherwise, we have $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$, but obviously the truth value of $\nu p.\Diamond p$ does not depend on the valuation V_Σ, and thus $(\Omega_\Sigma, R_\Sigma, V_\bullet), \Gamma \vDash \nu p.\Diamond p$. This proves our claim. Finally, as mentioned earlier, ψ is satisfiable on $\mathfrak{M}_\Sigma$, and this concludes the proof of Kripke completeness.

There remains to prove that ψ is satisfiable on a finite Kripke model. In the work of Baltag et al. (2021), we find the construction of a finite model $\mathfrak{M}^*_\Sigma = (\Omega^*_\Sigma, R^*_\Sigma, V^*_\Sigma)$ – obtained as a quotient of $\mathfrak{M}_\Sigma$ by so-called Σ-*bisimilarity* – together with a surjection $\rho : \Omega_\Sigma \to \Omega^*_\Sigma$ such that for all $\Gamma \in \Omega_\Sigma$, the pointed models $(\mathfrak{M}_\Sigma, \Gamma)$ and $(\mathfrak{M}^*_\Sigma, \rho(\Gamma))$ satisfy the same formulas among those of Σ. In particular, this entails that $\mathfrak{M}^*_\Sigma \vDash \mathbf{L}$ and that ψ is satisfiable on $\mathfrak{M}^*_\Sigma$, as desired. □

In order to prove topological completeness, we apply the technique used by Baltag et al. (2021) to turn a **wK4** frame into an appropriate topological space. The construction essentially consists of replacing every reflexive point w of a frame by countably many copies of w and to arrange them all into a dense-in-itself subspace, so as to mimic the reflexivity of w in a topological manner.

Definition 39. *Let $\mathfrak{F} = (W, R)$ be a* **wK4** *frame. We denote by W^r the set of reflexive worlds of $\mathfrak{F}$ and by W^i the set of irreflexive worlds of $\mathfrak{F}$. We then introduce the* unfolding *of $\mathfrak{F}$ as the space $X_\mathfrak{F} := (W^r \times \omega) \cup (W^i \times \{\omega\})$ endowed with the topology $\tau_\mathfrak{F}$ of all sets U such that for all $(w, \alpha) \in U$:*

(1) there is $n^U_{w,\alpha} < \omega$ such that for all $(u, \beta) \in X_\mathfrak{F}$, if wRu, uRw, and $\beta \geq n^U_{w,\alpha}$ then $(u, \beta) \in U$;
(2) if $(u, \beta) \in X_\mathfrak{F}$, wRu, and not *uRw, then $(u, \beta) \in U$.*

Proposition 40. (Baltag et al. 2021). *The pair $(X_\mathfrak{F}, \tau_\mathfrak{F})$ is a topological space and the map $\pi : X_\mathfrak{F} \to W$ defined by $\pi(w, \alpha) := w$ is a surjective d-morphism.*

Theorem 41. *Let θ be a modal formula such that* **wK4** $+ \theta$ *is cofinal subframe and canonical. Then, $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$ is topologically complete.*

Proof. Suppose that ψ is consistent in $\mu\mathbf{wK4} + \theta \vee \nu p.\Diamond p$. We keep the notations of the proof of Theorem 38. We introduce the spaces $X := X_{\mathfrak{F}_\Sigma}$, $Y := \pi^{-1}[\Omega']$ and $Z := X \setminus Y$. We prove that Y and Z satisfy the conditions of Proposition 30. First, we know that $\mathfrak{F}'$ is a generated subframe of $\mathfrak{F}_\Sigma$, so Ω' is open, and thus so is $\pi^{-1}[\Omega'] = Y$. In addition, since $\mathfrak{F}' \vDash \neg\nu p.\Diamond p$, the frame $\mathfrak{F}'$ is

https://doi.org/10.1017/S0960129523000385 Published online by Cambridge University Press

irreflexive, so $Y = \Omega' \times \{\omega\}$ and $\pi_{|Y}$ is injective. Since π is a d-morphism, the maps π and π^{-1} are continuous, and since Y is open, so are $\pi_{|Y}$ and $\pi_{|Y}^{-1}$. Therefore, $\pi_{|Y}$ is a homeomorphism between Y and $\mathfrak{F}'$. From $\mathfrak{F}' \vDash \theta$ and Proposition 13, we conclude that $Y \vDash \theta$.

We then prove that Z is dense-in-itself. Let $(\Gamma, \alpha) \in Z$ and U be an open neighborhood of (Γ, α). From $(\Gamma, \alpha) \in Z$, we know that $\Gamma \notin \Omega'$, that is, $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$. If $\alpha \neq \omega$, then Γ is reflexive. We select some $\beta \geq n^U_{w,\alpha}$ such that $\beta \neq \alpha$, and by definition of $n^U_{w,\alpha}$ we obtain $(\Gamma, \beta) \in U$. We also have $(\Gamma, \beta) \in Z$. Otherwise, we have $\alpha = \omega$, and then Γ is irreflexive. From this and $\mathfrak{M}_\Sigma, \Gamma \vDash \nu p.\Diamond p$, we obtain the existence of $\Delta \neq \Gamma$ such that $\Gamma R \Delta$ and $\mathfrak{M}_\Sigma, \Delta \vDash \nu p.\Diamond p$. We set $\beta := n^U_{w,\alpha}$ if Δ is reflexive, and $\beta := \omega$ otherwise; we then have $(\Delta, \beta) \in Z$ by definition. Depending on whether $\Delta R \Gamma$ or not, we apply either item 1 or item 2 of Definition 39, and in both cases we obtain $(\Delta, \beta) \in U$. Both cases bring the existence of some element in $U \cap Z$ different from (Γ, α), and we are done.

It follows that $X \vDash \theta \vee \nu p.\Diamond p$. We know that ψ is satisfiable on $\mathfrak{F}_\Sigma$, and since π is a d-morphism it follows by Proposition 6 that ψ is satisfiable on X as well. This concludes the proof. □

In the following corollary, we finally apply these results to our examples.

Corollary 42. *For all $m \in \mathbb{N}$, the logic μ**wK4** $+$ IP.2^+_m is Kripke and topologically complete.*

Proof. Since $(\Diamond\Diamond p \to p \vee \Diamond p) \wedge .2^+_m$ is a Sahlqvist formula, the logic $\mathbf{L}_0 := \mathbf{wK4} + .2^+_m$ is canonical (Blackburn et al., 2001, Section 4.3). In order to apply Theorems 38 and 41, we prove that $\mathbf{L}_0$ is cofinal subframe. Let $\mathfrak{F} = (W, R)$ be a **wK4** frame such that $\mathfrak{F} \vDash \mathbf{L}_0$, and let $\mathfrak{F}' = (W', R')$ be a cofinal subframe of $\mathfrak{F}'$.

Let $w \in W'$. First, suppose that $\mathfrak{F}, w \vDash .2^+$. Then if wR^+u and wR^+v with $u, v \in W'$, we have by assumption uR^+t and vR^+t for some $t \in W$. Then since $\mathfrak{F}'$ is cofinal in $\mathfrak{F}$ we have tR^+t' for some $t' \in W'$, and thus uR^+t' and vR^+t'. This proves that $\mathfrak{F}', w \vDash .2^+$. Otherwise, there exists a valuation V such that $\mathfrak{F}, V, w \nvDash .2^+$, and since $\mathfrak{F}, V, w \vDash .2^+_m$ it follows that $\mathfrak{F}, V, w \vDash \Box^m \bot$. From this, we deduce $\mathfrak{F}', w \vDash \Box^m \bot$. In both cases, we obtain $\mathfrak{F}', w \vDash .2^+_m$. Therefore, $\mathfrak{F}' \vDash \mathbf{L}_0$, and this proves the claim. □

6. Conclusion

We have established some fundamental results regarding the expressivity of the topological μ-calculus as opposed to basic modal logic. We have shown that the latter is indeed more expressive axiomatically than the former, a fact that was surprisingly difficult to prove. Accordingly, the examples we have exhibited are optimal in the sense that they involve topologically complete logics, which we have argued correspond to natural classes of spaces. In particular, they are related to the perfect core of a space, equivalent to the unary version of the tangled derivative, perhaps the most fundamental topological fixed point. This suggests that we are only scratching the surface of the jungle of spatial μ-logics, and their classification could be a bold new direction in the study of topological modal logics.

Acknowledgements. We are grateful to Nick Bezhanishvili for his involvement in this project as a co-supervisor. We are also indebted to a number of anonymous referees who provided us with helpful feedback on an earlier version of this paper. DFD was partially supported by the FWO-FWF Lead Agency Grant G030620N.

Competing interests. The authors declare none.

Notes

1 Recall that the derivative $\mathrm{d}(A)$ of a set A consists of all limit points of A.

2 Note that a stronger notion of expressivity is also considered in the literature: namely, $\mathscr{L}'$ is at least as expressive as $\mathscr{L}$ if for every $\varphi \in \mathscr{L}$ there is an equivalent $\varphi' \in \mathscr{L}'$ (with respect to a fixed semantics). To avoid confusion, we may call the latter

local expressivity, and the notion we are concerned with *axiomatic expressivity.* Thus, local expressivity refers to the capacity of a language to define sets of points locally in a given model, while axiomatic expressivity refers to its capacity to axiomatize classes of models. With this terminology in mind, while it was known that μ-calculus is locally more expressive than the basic modal language over topological spaces – see, for example, Fernández-Duque (2011*a*) - here we will show that it is also axiomatically more expressive.

3 Later we will see that the same result with $(\mathfrak{F}^{\mathsf{spine}}_{n,w}, r_n)$ and $(\mathfrak{F}^{\mathsf{cycle}}_{n,w}, r_n)$ follows for free.

References

Afshari, B. and Leigh, G. (2017). Cut-free completeness for modal mu-calculus. In: *32nd Annual ACM/IEEE Symposium on Logic in Computer Science LICS*, IEEE Press, 1–12.

Aleksandroff, P. (1937). Diskrete Räume. *Matematicheskii Sbornik* **2** 501–518.

Baltag, A., Bezhanishvili, N. and Fernández-Duque, D. (2021). The topological mu-calculus: completeness and decidability. In: *36th Annual ACM/IEEE Symposium on Logic in Computer Science, LICS 2021, Rome, Italy, June 29–July 2, 2021*, IEEE, 1–13.

Bezhanishvili, G., Ghilardi, S. and Jibladze, M. (2011). An algebraic approach to subframe logics. Modal case. *Notre Dame Journal of Formal Logic* **52** (2) 187–202.

Blackburn, P., de Rijke, M. and Venema, Y. (2001). *Modal Logic*, Cambridge Tracts in Theoretical Computer Science, Cambridge, UK, Cambridge University Press.

Chagrov, A. and Zakharyaschev, M. (1997). *Modal Logic*, Oxford Logic Guides, vol. 35, Oxford, Clarendon Press.

Esakia, L. (2001). Weak transitivity–a restitution. *Logical Investigations* **8** 244–245.

Esakia, L. (2004). Intuitionistic logic and modality via topology. *Annals of Pure and Applied Logic* **127** (1–3) 155–170. Provinces of logic determined.

Fernández-Duque, D. (2011a). On the modal definability of simulability by finite transitive models. *Studia Logica* **98** (3) 347–373.

Fernández-Duque, D. (2011b). Tangled modal logic for spatial reasoning. In: *Twenty-Second International Joint Conference on Artificial Intelligence*.

Fernández-Duque, D. and Iliev, P. (2018). Succinctness in subsystems of the spatial μ-calculus. *FLAP* **5** (4) 827–874.

Goldblatt, R. and Hodkinson, I. (2017). Spatial logic of tangled closure operators and modal mu-calculus. *Annals of Pure and Applied Logic* **168** (5) 1032–1090.

Goldblatt, R. and Hodkinson, I. (2018). The finite model property for logics with the tangle modality. *Studia Logica* **106** (1) 131–166.

Kozen, D. (1983). Results on the propositional μ-calculus. *Theoretical Computer Science* **27** (3) 333–354.

Kudinov, A. and Shehtman, V. (2014). Derivational modal logics with the difference modality. In: *Leo Esakia on Duality in Modal and Intuitionistic Logics*, Springer, 291–334.

McKinsey, J. and Tarski, A. (1944). The algebra of topology. *Annals of Mathematics* **45** 141–191.

Santocanale, L. (2008). Completions of μ-algebras. *Annals of Pure and Applied Logic* **154** (1) 27–50.

Santocanale, L. and Venema, Y. (2010). Completeness for flat modal fixpoint logics. *Annals of Pure and Applied Logic* **162** (1) 55–82.

Shehtman, V. (1999). "Everywhere" and "here". *Journal of Applied Non-Classical Logics* **9** (2–3), 369–379.

van Benthem, J. (2006). Modal frame correspondences and fixed-points. *Studia Logica* **83** (1) 133–155.

van Benthem, J. and Bezhanishvili, G. (2007). Modal logics of space. In: *Handbook of Spatial Logics*, Springer, Dordrecht, 217–298.

Walukiewicz, I. (2000). Completeness of Kozen's axiomatisation of the propositional μ-calculus. *Information and Computation* **157** (1–2) 142–182.

Cite this article: Fernández-Duque D and Gougeon Q (2024). Fixed point logics and definable topological properties. *Mathematical Structures in Computer Science* **34**, 81–97. https://doi.org/10.1017/S0960129523000385