# THE POST - NEWTONIAN ROTATION OF EARTH: A FIRST APPROACH *

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ABSTRACT. The problems of dynamics of extended bodies in metric theories of gravity are reviewed. In a first approach towards the relativistic description of the Earth's rotational motion the post - Newtonian treatment of the free precession of a pseudo - rigid and axially symmetric model Earth is presented. Definitions of angular momentum, pseudo - rigidity, the corotating frame, tensor of inertia and axial symmetry of the rotating body are based upon the choice of the standard post - Newtonian (PN) coordinates and the full PN energy momentum complex. In this framework, the relation between angular momentum and angular (coordinate) velocity is obtained. Since the PN Euler equations for the angular velocity here formally take their usual Newtonian form it is concluded that apart from PN modifications (renormalizations) of the inertia tensor, the rotational motion of our pseudo - rigid and axially symmetric model Earth essentially is "Newtonian".

## 1. INTRODUCTION

Seventy years now have passed since Einstein's paper on the foundations of General Relativity appeared in the literature but there is no satisfactory and practicable description of the motion of extended bodies in Einstein's theory of gravity yet.

The full problem of dynamics of extended bodies in Einstein's theory of gravity is summarized in Fig.1, which arose out of a discussion with Prof. J. Ehlers. The starting points of theoretical considerations are: (I) Einstein's field equations for the metric tensor **g**, (II) equations of state specifying a model of matter and (III) a certain set of boundary conditions such as e.g. asymptotic flatness for **g** or no incoming gravitational radiation. If the model of matter is compatible with the field equations all information about the dynamics of matter is embodied in the local equations of motion as given by the vanishing of the divergence of the energy momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0 \qquad (1)$$

The metric tensor **g** appears explicitly in (1); therefore it is called an equation of motion of the first kind (Ehlers). Expressing the metric tensor in terms of matter variables is still an unsolved problem of Einstein's theory of gravity.

In the Newtonian framework one successfully proceeds with the introduction of collective variables such as centres of mass, mass multipole moments, angular and translational momenta etc. and derives global equations of motion for the momentum and



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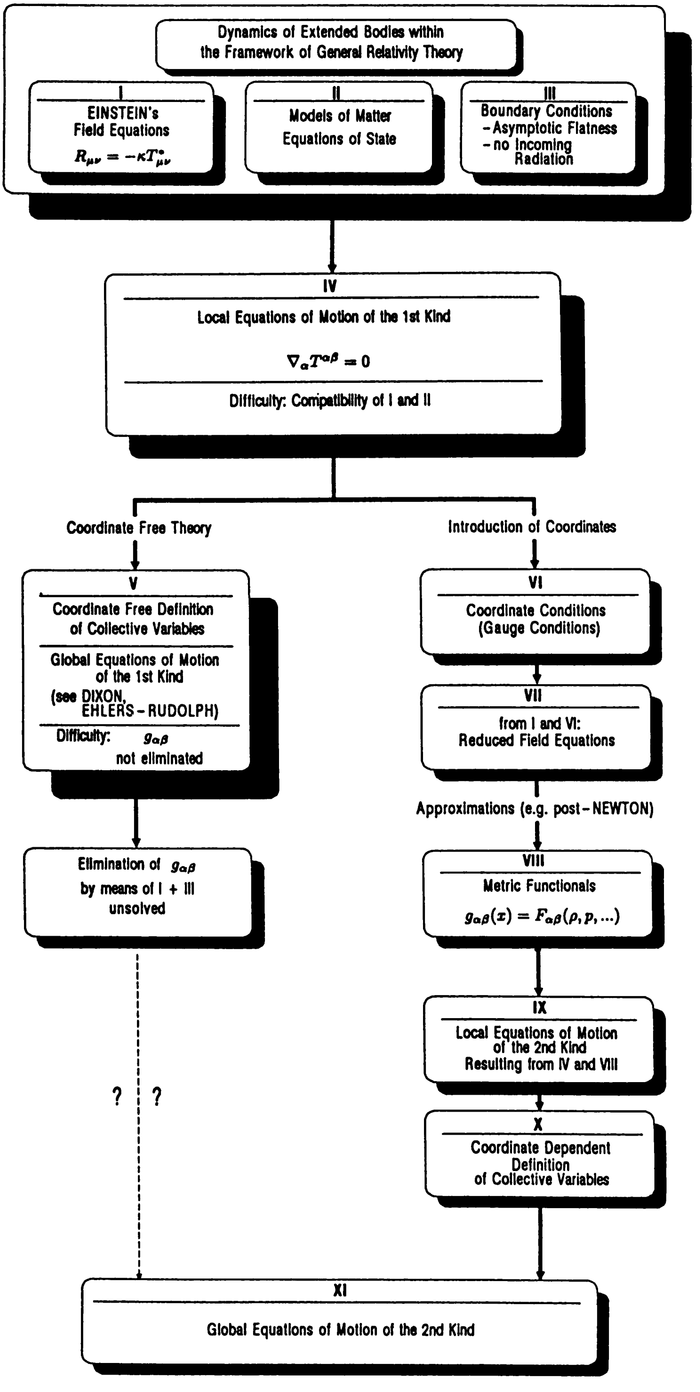


Fig. 1

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angular momentum vector where the Newtonian potential can be expressed in terms of the multipole moments of the sources. Such an equation of motion is called to be of the second kind. Dixon (1979) has characterized the steps that are necessary to generalize this Newtonian route to the full Einstein theory:

i) suitable choice of a representative point (mass centre) within each body
ii) derivation of a momentum - velocity relation
iii) evaluation of the total force and torque exerted on an extended body in a gravitational field
iv) characterization of the self - field
v) evaluation of the self - force and self - torque
vi) determination of the external field in terms of matter variables (multipole moments) of the field - generating bodies

Whereas most of the steps (especially ii) are conceptually trivial in Newtonian space - time they are highly problematic in a relativistically curved space - time; the separation of the total field into an external and a self - part (iv) and especially the determination of the gravitational field in terms of matter variables of the sources seems to be an almost hopeless task at present. This situation is characterized by question marks in Fig.1. Dixon (1979) (see also Ehlers and Rudolph 1977) has solved the first three parts i)-iii) of the whole problem in an exact way (i.e. without resorting to approximation schemes) and in terms of geometrical (i.e. coordinate independent) quantities. However, the complexity in the construction of dynamical quantities such as momentum vector, angular momentum tensor and reduced mass multipole moments is so enormous that so far no application of Dixon's theory exists in the literature even for simple external fields and it is not very likely that much progress will be achieved here in the near future.

The other route, which is usually chosen in the literature, selects a certain set of coordinates from the beginning by requiring a certain gauge condition. If the coordinates are chosen it is not difficult to formulate some approximation scheme e.g. for slow motion and weak fields and to express the components of the metric tensor $g_{\mu\nu}$ as functionals of matter variables such as density of mass, pressure etc. Inserting these functionals in equation (1) leads to the local equations of motion of the second kind. For the (parametrized) post - Newtonian treatment the equations of motion are given by eqs. (39.46) and (39.47) of Misner et al. (1973). Then one can introduce a set of collective variables like e.g. momentum vector and angular momentum tensor with components $p^\mu$ and $J^{\mu\nu}$, according to the set of coordinates chosen. In the post - Newtonian framework one usually defines these quantities by:

$$p^\mu = \int_\Sigma \Theta^{\mu\nu}\, d^3\Sigma_\nu \tag{2}$$

and

$$J^{\mu\nu} = 2\int_\Sigma x^{[\mu}\Theta^{\nu]\lambda}\, d^3\Sigma_\lambda \tag{3}$$

where the integrals are taken over a spacelike hypersurface as given by the post - Newtonian $3+1$ split of space - time into space and time and $\Theta^{\mu\nu}$ is the energy momentum complex. From the form of eqs. (2) and (3) it is obvious that these definitions depend upon the the chosen (PN) gauge and orientation of spacelike hypersurfaces though these quantities define vectors and tensors w.r.t. a restricted set of coordinate transformations. Besides this, the quantities $p^\mu$ and $J^{\mu\nu}$ defined in such a way can be interpreted as momentum and angular momentum only in the asymptotic regime ($r \to \infty$) where space - time is sufficiently flat. These expressions for momentum and angular momentum clearly have the great advantage that their evaluation is simple in the PN framework, whereas the corresponding geometric quantities as given by Dixon (1979) involve e.g. knowledge of the world function and derivatives thereof (see e.g Synge 1966) which for concrete problems will not be easy to obtain. The non - geometric character of collective variables generally presents no difficulty in principle, since they are not directly observable. On the other hand it is clear that such a theory of "chrono - geometrical corrections" (to the Newtonian theory) only makes sense if the observables are described by scalars in the full geometrical sense e.g. in the frame of tetrad formalism of reference frames. Such an introduction of collective variables finally allows to deduce the desired global equations of the second kind where the last step requires a tedious piece of work.

For the translational motion in the (parametrized) post - Newtonian scheme this has been done e.g by Will (1981) who neglects higher multipole moments of the bodies and their intrinsic angular momenta and assumes the whole system to be secularly stationary, a condition that has been further analyzed in detail by Spyrou (1978). For the full post - Newtonian formalism the global equations of the second kind (including e.g. tidal effects) for translational motion have not been written down.

The situation is even worse for rotational motion where PN - results are known only for very special cases (e.g. Barker and O'Connell 1975, Börner et al. 1975, Caporali 1979). Clearly, much work will have to be done in the near future to obtain the desired global PN - equations of translational and rotational motion in their complete form and to assess the magnitudes of the various "PN - corrections" to Newtonian dynamics.

In the following we will concentrate upon the free precession of some (almost) axially symmetric body (Earth) in the post - Newtonian framework. Our considerations were motivated by a paper presented by Fukushima (1986) which we found to contain some subtle flaws that are corrected in this article. For example he neglected contributions of the gravitational field to momentum and angular momentum which is correct only for the linearized theory that will fail for the treatment of the Earth's rotation (see Misner et al. 1973, Chapter 19.1).

## 2. POST - NEWTONIAN FREE PRECESSION OF A PSEUDO - RIGID AND AXIALLY SYMMETRIC MODEL EARTH

Let us consider the free precession of a single body in the Einstein post - Newtonian framework and choose the origin of our PN - frame at the centre of mass $Z$ (Will 1981) of the body. The angular momentum tensor w.r.t. an asymptotic observer is then given by expression (3) which is is a conserved quantity provided the energy momentum complex

$$\Theta^{\mu\nu} = (-g)(T^{\mu\nu} + t^{\mu\nu}) \tag{4}$$

is symmetrical, which is the case if $t^{\mu\nu}$ denotes the Landau - Lifshitz complex; $g = \det g_{\mu\nu} = -1 - 4U/c^2$, where $U$ is the Newtonian potential. If constant - time hypersurfaces are chosen, eq. (3) can be written as

$$J^{\mu\nu} = 2\int x^{[\mu}\Theta^{\nu]0}\, d^3x \tag{5}$$

which is time independent if the matter distribution is confined to a compact support. The angular momentum tensor defines the corresponding vector by

$$J^{\sigma} = \frac{1}{2c^2}\eta^{\sigma\mu\nu\kappa}u_{\kappa}J_{\mu\nu} \tag{6}$$

where $\eta^{\sigma\mu\nu\kappa}$ is the completely antisymmetric tensor with

$$\eta^{0123} = \frac{-1}{\sqrt{-g}} \qquad \eta_{0123} = -\sqrt{-g}$$

and $u_{\kappa}$ denotes the four - velocity of an observer resting in the asymptotic flat regime. This implies that $u_{\kappa} = (c, 0)$ and since $u_{\sigma}J^{\sigma} = 0$ ($J^{\sigma}$ is purely spacelike in our frame) we see that $J^0 = 0$ and

$$J^i = \epsilon_{ijk}\int x^j \Theta^{k0}\, d^3x \tag{7}$$

with $\epsilon_{ijk} = 1$. (Since all further calculations are done w.r.t. the asymptotic observer, we will no longer distinguish between lower and upper spacelike indices.)

The relevant space - time components of the energy momentum tensor and complex are given by Will (1981):

$$T^{i0} = \rho(c^2 + \Pi + v^2 + 2U)\frac{v^i}{c} + p^{ik}\frac{v^k}{c} \tag{8a}$$

$$t^{i0} = \frac{1}{4\pi c}(3U_{,i}U_{,0} + 8U_{,j}V_{[j,i]}) \tag{8b}$$

where $\rho$ is the matter density, $\Pi$ the internal energy, $p^{ik}$ the components of the stress tensor $\hat{\mathbf{p}}$ (pressure $p = 1/3\, Tr(p^{ik})$) and

$$V_i = \int \frac{\rho' v_i'}{|\mathbf{x} - \mathbf{x}'|}\, d^3x'$$

Inserting eqs. (8) and $g = -1 - 4U/c^2$ into expression (7) yields:

$$\mathbf{J} = \int \mathbf{x} \times \left[\rho(1 + \frac{v^2}{c^2} + \frac{6U}{c^2} + \frac{\Pi}{c^2})\mathbf{v} + \frac{1}{c^2}\hat{\mathbf{p}}\mathbf{v} - \frac{7}{2c^2}\rho\mathbf{V} - \frac{1}{2c^2}\rho\mathbf{W}\right] d^3x \tag{9}$$

with

$$W_i = \int \frac{\rho'\mathbf{v}' \cdot (\mathbf{x} - \mathbf{x}')(x - x')_i}{|\mathbf{x} - \mathbf{x}'|^3}\, d^3x'$$

We now introduce a system of coordinates rotating w.r.t. our PN - rest - frame with angular velocity $\mathbf{\Omega}$ in which the coordinate velocity written in Euclidean vector space notation takes the form

$$\tilde{\mathbf{v}} = \mathbf{v} - \mathbf{\Omega} \times \mathbf{x}$$

and require $\tilde{\mathbf{v}}' = 0$ for all mass elements of the rotating body. This requirement generalizes in a coordinate dependent way the notion of rigidity in the Newtonian framework and leads to the Newtonian equilibrium condition

$$\frac{1}{\rho}\nabla\hat{\mathbf{p}} = \nabla[U + \frac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2] - \dot{\mathbf{\Omega}} \times \mathbf{x} \tag{10}$$

One possible solution of this relation reads (Fukushima 1986):

$$\rho = \text{const} \qquad p^{ik} = \begin{cases} \rho[U + \frac{1}{2}(\mathbf{\Omega} \times \mathbf{x})^2] & i = k \\ -\frac{1}{2}\rho[(x^i)^2 - (x^k)^2]\epsilon_{ikl}\dot{\Omega}_l & i \neq k \end{cases}$$

(no summation over $i$ and $k$).

If we furthermore assume the body to be axially symmetric in the sense that all integrals which are odd in the spatial coordinates $x$ or $y$ vanish in the corotating frame, we are left with:

$$J^x = A\Omega_x + \frac{1}{c^2}(A_1\Omega_x^2 + A_2\Omega_y^2 + A_3\Omega_z^2)\Omega_x - \frac{D}{c^2}\dot{\Omega}_y\Omega_z \tag{11a}$$

$$J^y = A\Omega_y + \frac{1}{c^2}(A_2\Omega_x^2 + A_1\Omega_y^2 + A_3\Omega_z^2)\Omega_y + \frac{D}{c^2}\dot{\Omega}_x\Omega_z \tag{11b}$$

$$J^z = C\Omega_z + \frac{1}{c^2}[A_3(\Omega_x^2 + \Omega_y^2) + C_3\Omega_z^2]\Omega_z - \frac{D}{c^2}(\dot{\Omega}_y\Omega_x - \dot{\Omega}_x\Omega_y) \tag{11c}$$

where

$$A = I_1 + I_3 + K_1 + K_3 + K_{13} \qquad C = 2I_1 + 2K_1 + K_{12} \qquad I_i = \int \rho(1 + \frac{\Pi}{c^2} + \frac{7U}{c^2})x^i x^i \, d^3x$$

$$K_i = -\frac{7}{2c^2}\int \frac{\rho\rho'}{|\mathbf{x} - \mathbf{x}'|}x^i x'^i \, d^3x' \, d^3x \qquad K_{ij} = -\frac{1}{2c^2}\int \frac{\rho\rho'}{|\mathbf{x} - \mathbf{x}'|^3}(x^i x'^j - x^j x'^i)^2 \, d^3x' \, d^3x$$

$$A_1 = \frac{3}{2}(I_{11} + 2I_{13} + I_{33}) \qquad A_2 = \frac{3}{2}(3I_{12} + 2I_{13} + I_{33}) = A_1 \qquad A_3 = \frac{3}{2}(I_{11} + I_{12} + 4I_{13})$$

$$C_3 = 3(I_{11} + I_{12}) \qquad D = \frac{1}{2}(I_{12} - I_{13}) \qquad I_{ij} = \int \rho x^i x^i x^j x^j \, d^3x$$

(no summation over equal indices). †

We find that the Euler equations of torque free rotation in the corotating frame $d\mathbf{J}/dt + \mathbf{\Omega} \times \mathbf{J} = 0$ (as in the Newtonian case) are satisfied up to post - Newtonian order by the Newtonian solution for the Poinsot motion of a "rigid and axially symmetric" body with modified moments of inertia $\tilde{A}$ and $\tilde{C}$

$$\tilde{A} = A + \frac{1}{c^2}(A_1\Omega_0^2 + A_3\Omega_z^2) - \frac{D}{c^2}\frac{C - A}{A}\Omega_z^2$$

$$\tilde{C} = C + \frac{1}{c^2}(A_3\Omega_0^2 + C_3\Omega_z^2) - \frac{D}{c^2}\frac{C - A}{A}\Omega_0^2$$

We therefore see that the free precession of such a body is essentially "Newtonian" (for a similar result for strongly gravitating and slowly rotating bodies see Thorne and Gürsel 1983).

† The main discrepancy between our results and those obtained by Fukushima (1986) arises from computational errors preventing him to derive $A_1 = A_2$ for the axially symmetric case with the consequence that the solution for $\mathbf{\Omega}$ is completely altered.

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