A Generalization of Circulants.

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1. In connection with Bertrand's algebraical exercise of 1850, Muir[1] remarks that it is not unlikely that the divisibility of $a^3+b^3+c^3-3abc$ by $a+b+c$ had been previously noted, although there is no record of the fact. Bertrand's exercise is to the effect that the circulant of the third order repeats under multiplication or, what is the same, admits of composition; the formulae of composition are stated in the exercise precisely as they would follow from Spottiswoode's theorem on the linear factors of a circulant. The whole of this is implied as an immediate special case, and indeed as one that any reader would construct at once, in an identity due to Lagrange, reproduced by Legendre.[2]

It is interesting also to see that Legendre was acquainted with the origin of Lagrange's forms as eliminants—a common method of obtaining that special case of these forms subsequently called circulants. Neither Lagrange nor Legendre, of course, pointed out any connection with determinants as they are now written. It seems not unlikely that Bertrand constructed his exercise directly from Legendre or from Euler's Algebra.

It has not been noticed that Lagrange's theorems on homogeneous, completely factorable forms that admit of composition lead to a generalization of circulant determinants. Nor has it been stated, apparently, that the cofunctions of Schapira[3] are the simplest instances of a class of functions originating in Lagrange's compositions. These will be considered elsewhere, when we shall also give a generalization of the theory of block circulants (see Muir, vol. 4, pp. 385-395) in line with that of the present note concerning simple circulants. The generalizations of block circulants are reached by

[1] *History of the Theory of Determinants*, vol. 1, p. 401.

[2] *Theorie des Nombres*, tome II, § XVI, pp. 137-138. Bertrand's exercise is obtained from Legendre's formulas by putting $a=b=0$, $c=1$. The substance of the discussion is given also in the Supplements to Euler's Algebra.

[3] See Muir, vol. 4, p. 360. A short account of Schapira's extensive theory will be found in his obituary notices. The original is in Russian.

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norming irrational transforms of the determinants in this note; when all the irrationalities involved are roots of unity, the normed determinants degenerate to block circulants. The theorems thus attained include those of Torelli (see Muir, vol. 4, pp. 366-370).

2. Let θ be any root of

$$(1)\quad \Theta(\theta) \equiv \theta^n + k_1 \theta^{n-1} + \ldots + k_n = 0,$$

and consider the associated linear difference equation

$$(2)\quad u_{r+n} + k_1 u_{r+n-1} + \ldots + k_n = 0 \quad (r = 0, 1, \ldots).$$

Any solution u_h $(h = 0, 1, \ldots)$ of (2) is of the form

$$(3)\quad u_h = u_0 c_{0,h} + u_1 c_{1,h} + \ldots + u_{n-1} c_{n-1,h},$$

where $c_{j,h}$ $(j = 0, 1, \ldots, n-1;\ h = 0, 1, \ldots)$ is the jth fundamental sequence satisfying (2); namely, the $c_{j,h}$ are defined by

$$(4)\quad \begin{aligned} &c_{j,r+n} + k_1 c_{j,r+n-1} + \ldots + k_n c_{j,r} = 0, \\ &c_{j,j} = 1, \quad c_{j,h} = 0 \quad (h \leqslant n-1,\ h \neq j), \\ &(j = 0, 1, \ldots, n-1;\ h = 0, 1, \ldots). \end{aligned}$$

Let $T(\theta) \neq 0$ be any polynomial in θ, including the case $T(\theta) =$ a constant different from zero. Then, by the usual division transformation, we can write $T(\theta)$ uniquely in the form

$$(5)\quad T(\theta) = Q(\theta)\,\Theta(\theta) + R(\theta),$$

where $Q(\theta)$, $R(\theta)$ are polynomials in θ, the degree of $R(\theta)$ does not exceed $n-1$, and either or both of $Q(\theta)$, $R(\theta)$ may be constants, excluding only the case $Q(\theta) = R(\theta) = 0$. Hence, by (1), we have

$$(6)\quad T(\theta) = R(\theta).$$

Applying (6) to θ^r, where r is any integer $\geqslant 0$, we get readily

$$(7)\quad \theta^r = c_{0,r} + c_{1,r}\theta + \ldots + c_{n-1,r}\theta^{n-1}.$$

By n applications of (7) we next have the reduction

$$(8)\quad \begin{aligned} &b_0 \theta^r + b_1 \theta^{r+1} + \ldots + b_{n-1}\theta^{r+n-1} \\ &= B_{0,r} + B_{1,r}\theta + \ldots + B_{n-1,r}\theta^{n-1}, \end{aligned}$$

where $b_0, b_1, \ldots, b_{n-1}$ are independent variables, in which the B's do not contain θ. Hence, by what precedes, we get, for $j = 0, 1, \ldots, n-1$,

$$(9)\quad B_{j,r} = b_0 c_{j,r} + b_1 c_{j,r+1} + \ldots + b_{n-1} c_{j,r+n-1},$$

$$(10)\quad B_{j,r} = B_{j,0}\, c_{0,r} + B_{j,1}\, c_{1,r} + \ldots + B_{j,n-1}\, c_{n-1,r},$$

$$(11)\quad B_{j,r+n} + k_1 B_{j,r+n-1} + \ldots + k_n B_{j,r} = 0,$$

so that the sequences $B_{j,r}$ $(j = 0, 1, \ldots, n-1;\ r = 0, 1, \ldots)$ are a second set of n linearly independent solutions of (2).

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The *generalized circulant* $B(b) \equiv B(b_0, b_1, \ldots, b_{n-1})$, *of order* n *generated by the equation* (1) *is defined to be the determinant* (12)

$$B(b) \equiv \begin{vmatrix} B_{0,0} & B_{0,1} & \cdots\cdots & B_{0,n-1} \\ B_{1,0} & B_{1,1} & \cdots\cdots & B_{1,n-1} \\ \cdots & \cdots & \cdots & \cdots \\ B_{n-1,0} & B_{n-1,1} & \cdots\cdots & B_{n-1,n-1} \end{vmatrix}.$$

To justify this definition we shall prove that the determinant (12) is the product of n linear factors; that the product of two such determinants is a third of the same kind; and that, when

$$(13)\quad k_1 = k_2 = \ldots = k_{n-1} = 0, \quad k_n = -1$$

in (1), the first of the theorems just stated becomes Spottiswoode's for the factorization of an ordinary circulant, and hence that the second theorem degenerates, in the case (13), to the theorem for the product of two ordinary circulants. Moreover the arguments $z_0, \ldots, z_{n-1}$ in the product theorem

$$(14)\quad B(x_0, \ldots, x_{n-1})\; B(y_0, \ldots, y_{n-1}) = B(z_0, \ldots, z_{n-1})$$

will be explicitly determined as bilinear forms in the

$$x_h, y_h \; (h = 0, \ldots, n-1)$$

with coefficients given by (4).

Before proceeding to the very simple proofs, we give an example. Take $n = 3$ in (1), and write

$$\phi_{j,r}(x) \equiv x_0 c_{j,r} + x_1 c_{j,r+1} + x_2 c_{j,r+2},$$

$$B(x) \equiv \begin{vmatrix} \phi_{0,0}(x) & \phi_{0,1}(x) & \phi_{0,2}(x) \\ \phi_{1,0}(x) & \phi_{1,1}(x) & \phi_{1,2}(x) \\ \phi_{2,0}(x) & \phi_{2,1}(x) & \phi_{2,2}(x) \end{vmatrix},$$

so that $B(x) \equiv B(x_0, x_1, x_2)$ is the determinant to be discussed. From (4) with $n = 3$, we find, on calculating the $\phi_{j,r}(x)$,

$$B(x) = \begin{vmatrix} x_0 & -k_3 x_2 & -k_3 x_1 + k_1 k_3 x_2 \\ x_1 & x_0 - k_2 x_2 & -k_2 x_1 + (k_1 k_2 - k_3) x_2 \\ x_2 & x_1 - k_1 x_2 & x_0 - k_1 x_1 + (k^2{}_1 - k_2) x_2 \end{vmatrix},$$

and this is equal to the product

$$(x_0 + \theta_0 x_1 + \theta^2{}_0 x_2)(x_0 + \theta_1 x_1 + \theta^2{}_1 x_2)(x_0 + \theta_2 x_1 + \theta^2{}_2 x_2),$$

where θ_0, θ_1, θ_2 are the roots of

$$\theta^3 + k_1 \theta^2 + k_2 \theta + k_3 = 0.$$

When $k_1 = k_2 = 0$, $k_3 = -1$, the determinantbecomes

$$\begin{vmatrix} x_0 & x_2 & x_1 \\ x_1 & x_0 & x_2 \\ x_2 & x_1 & x_0 \end{vmatrix} = \begin{vmatrix} x_0 & x_1 & x_2 \\ x_2 & x_0 & x_1 \\ x_1 & x_2 & x_0 \end{vmatrix},$$

and we may take $\theta_0 = 1$, $\theta_1 = \omega$, $\theta_2 = \omega^2$, where ω is a complex root of $\theta^3 - 1 = 0$.

3. To obtain the linear factors of the determinant (12) we write $\beta \equiv b_0 + b_1 \theta + \ldots + b_{n-1} \theta^{n-1}$,
or, by (9),

$$(14) \quad \beta = B_{0,0} + B_{1,0}\,\theta + \ldots + B_{n-1,0}\,\theta^{n-1}.$$

Multiply (14) throughout by $1, \theta, \theta^2, \ldots, \theta^{n-1}$, reduce the right hand members by (8), change all signs in the resultant set of n equations, and eliminate $1, \theta, \theta^2, \ldots, \theta^{n-1}$, and thus obtain the characteristic equation for β in the form

$$(15) \quad \begin{vmatrix} \beta - B_{0,0} & -B_{1,0} & \cdots\cdots & -B_{n-1,0} \\ -B_{0,1} & \beta - B_{1,1} & \cdots\cdots & -B_{n-1,1} \\ \cdots & \cdots & \cdots & \cdots \\ -B_{0,n-1} & -B_{1,n-1} & \cdots & \beta - B_{n-1,n-1} \end{vmatrix} = 0.$$

Say this is

$$\beta^n + D_1 \beta^{n-1} + \ldots + D_n = 0,$$

so that

$$D_n = (-1)^n \begin{vmatrix} B_{0,0} & B_{1,0} & \cdots\cdots & B_{n-1,0} \\ B_{0,1} & B_{1,1} & \cdots\cdots & B_{n-1,1} \\ \cdots & \cdots & \cdots & \cdots \\ B_{0,n-1} & B_{1,n-1} & \cdots\cdots & B_{n-1,n-1} \end{vmatrix}.$$

But $D_n = (-1)^n$ times the product of all the conjugates (including β) of β. Hence the determinant (12), or $B(b_0, b_1, \ldots, b_{n-1})$, is the product

$$(16) \quad \prod_{i=0}^{n-1} [b_0 + b_1 \theta_i + \ldots + b_{n-1} \theta_i^{\,n-1}],$$

where θ_i $(i = 0, \ldots, n-1)$ are the roots of (1).

4. To see that the factorization in §3 becomes Spottiswoode's for a circulant in the case indicated in (13), we have in that case

$$c_{j,\,sn+h} = 0 \text{ if } h \neq j,$$
$$= 1 \text{ if } h = j,$$

where

$$0 \leqslant h \leqslant n-1, \quad 0 \leqslant j \leqslant n-1.$$

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Hence the $B_{i,j}$ now become

$$
\begin{array}{llll}
B_{0,0} = b_0, & B_{0,1} = b_{n-1}, & B_{0,2} = b_{n-2}, \ldots, & B_{0,n-1} = b_1, \\
B_{1,0} = b_1, & B_{1,1} = b_0\;, & B_{1,2} = b_{n-1}, \ldots, & B_{1,n-1} = b_2, \\
\ldots & \ldots & \ldots\ldots & \ldots \\
B_{n-1,0} = b_{n-1}, & B_{n-1,1} = b_{n-2}, & \ldots\ldots\ldots\ldots\ldots\ldots, & B_{n-1,n-1} = b_0;
\end{array}
$$

and (12) degenerates to the circulant

$$
\begin{vmatrix}
b_0 & b_{n-1} & b_{n-2} & \ldots & b_1 \\
b_1 & b_0 & b_{n-1} & \ldots & b_2 \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
b_{n-1} & b_{n-2} & b_{n-3} & \ldots & b_0
\end{vmatrix},
$$

which, by interchanges of rows, followed by the like for columns, is

$$
\begin{vmatrix}
b_0 & b_1 & b_2 & \ldots & b_{n-1} \\
b_{n-1} & b_0 & b_1 & \ldots & b_{n-2} \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
b_1 & b_2 & b_3 & \ldots & b_0
\end{vmatrix},
$$

the usual form. Since the equation (1) is now $\theta^n - 1 = 0$, the factorization (16) degenerates to Spottiswoode's.

5. It will be of interest to give the explicit forms of the $z_j\ (j = 0, \ldots, n-1)$ in the composition (14). These follow at once from applying to the product

$$(x_0 + x_1\theta + \ldots + x_{n-1}\theta^{n-1})\,(y_0 + y_1\theta + \ldots + y_{n-1}\theta^{n-1})$$

the reduction formula (8). The result is as follows. Define the functions p, P, by

$$p_j(x, y) \equiv \sum_{s=0}^{n-1} x_s\, y_{j-s}\ (j = 0, 1, \ldots, n \quad 1);$$

$$p_{n+r}(x, y) \equiv \sum_{s=1}^{n-r-1} x_{r+s}\, y_{n-s}\ (r = 0, 1, \ldots, n-2);$$

$$P_h(x, y) \equiv \sum_{t=h}^{2n-z} c_{h,t}\, p_t(x, y)\ (h = 0, 1, \ldots, n-1).$$

Then the required values are

$$z_h = P_h(x, y) \qquad (h = 0, 1, \ldots, n-1).$$

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