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TRANSFERENCE ON CERTAIN MULTILINEAR MULTIPLIER OPERATORS

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Abstract

We study DeLeeuw type theorems for certain multilinear operators on the Lebesgue spaces and on the Hardy spaces. As applications, on the torus we obtain an analog of Lacey–Thiele's theorem on the bilinear Hilbert transform, as well as analogies of some recent theorems on multilinear singular integrals by Kenig–Stein and by Grafakos–Torres.

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1. Introduction

Let $\mathbb{R}^n$ be the n-dimensional Euclidean space and $\mathbb{R}^{nm} = \mathbb{R}^n \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n$ be the m-fold product space. Suppose that $\mathscr{S}(\mathbb{R}^n)$ is the space of all Schwartz test functions on $\mathbb{R}^n$ and $\lambda(u_1, \dots, u_m)$ is any function on $\mathbb{R}^{nm}$, where $u_j \in \mathbb{R}^n$ for $j = 1, 2, \dots, m$. The multilinear operators T_ε, $\varepsilon > 0$, associated with this λ are defined by

$$(1.1)\quad T_\varepsilon(f_1, f_2, \dots, f_m)(x) \\ = \int_{\mathbb{R}^{nm}} \prod_{j=1}^{m} \hat{f}_j(u_j)\lambda(\varepsilon u_1, \dots, \varepsilon u_m) \exp\left(2\pi i \sum_{j=1}^{m} \langle u_j, x\rangle\right) du_1 \cdots du_m,$$

for all $f_j \in \mathscr{S}(\mathbb{R}^n)$, $j = 1, 2, \dots, m$, where $\hat{f}_j$ is the Fourier transform of f_j, $x \in \mathbb{R}^n$ and $\langle u_j, x\rangle$ is the inner product of u_j and x. We denote $T = T_\varepsilon$ if $\varepsilon = 1$.

The significance of studying such kind of multilinear operators can be illustrated by following two simple model cases. First, in the case $m = 1$, T is the classical multiplier which plays very important roles in harmonic analysis and in partial differential equations (see [S]). Secondly, the study of the case $m > 1$ is much more involved. This topic can be dated back by the pioneering work of Coifman–Meyer started from 70's [CM1, CM2, CM3, CM4], as well as some recent works by Lacey–Thiele, Kenig–Stein and many others [KeS, CG, GK, GT, GW, LT]. Readers can see these references for more details about the background and significance in this topic. Here we list a simple example by letting $n = 1$, $m = 2$ and taking $\lambda(u_1, u_2) = \lambda(u_2 - u_1)$ with $\lambda(t) = i\operatorname{sgn}(t)$, where $\operatorname{sgn}(t)$ is the sign function on $\mathbb{R}^1$. Then it is easy to check that

$$T(f, g)(x) = \operatorname{pv} \pi^{-1} \int_{\mathbb{R}^1} f(x - t)g(x + t)t^{-1}\, dt$$

is the bilinear Hilbert transform, which is related to a famous conjecture by Calderón in studying certain problems of Cauchy integrals. Very recently, Lacey and Thiele [LT, La] solved this conjecture by proving that

$$\|T(f, g)\|_p < C\|f\|_q\|g\|_r$$

provided $1/p = 1/q + 1/r$, $1 < q, r \le \infty$ and $2/3 < p < \infty$.

Analogously, we can define multilinear operators on the torus. The n-torus $\mathbb{T}^n$ can be identified with $\mathbb{R}^n/\Lambda$, where Λ is the unit lattice which is an additive group of points in $\mathbb{R}^n$ having integer coordinates. Let $\mathbb{T}^{nm}$ be the m-fold product space $\mathbb{T}^n \times \mathbb{T}^n \times \cdots \times \mathbb{T}^n$. The multilinear operators $\tilde{T}_\varepsilon$, $\varepsilon > 0$, on $\mathbb{T}^{nm}$ associated with the function λ are defined by

$$(1.2) \quad \tilde{T}_\varepsilon\left(\tilde{f}_1, \dots, \tilde{f}_m\right)(x)$$
$$= \sum_{k_1\in\Lambda}\sum_{k_2\in\Lambda}\cdots\sum_{k_m\in\Lambda} \lambda(\varepsilon k_1, \dots, \varepsilon k_m) a_{k_1} a_{k_2} \cdots a_{k_m} \exp\left(2\pi i \sum_{j=1}^{m} \langle k_j, x\rangle\right)$$

for all $C^\infty(\mathbb{T}^n)$ functions

$$\tilde{f}_j(x) = \sum_{k_j\in\Lambda} a_{k_j} \exp\left(2\pi i \langle k_j, x\rangle\right), \quad j = 1, 2, \dots, m.$$

We denote $\tilde{T} = \tilde{T}_\varepsilon$ if $\varepsilon = 1$.

As we mentioned before, in the case $m = 1$, T becomes the ordinary multiplier operator. One of the well-known results in that case is a theorem by DeLeeuw [L] (see also [SW, page 260]) which says that if $\lambda(u)$ is a continuous function on $\mathbb{R}^n$ and if $p \ge 1$, then T is bounded on $L^p(\mathbb{R}^n)$ if and only if $\tilde{T}_\varepsilon$ is uniformly bounded on $L^p(\mathbb{T}^n)$ for $\varepsilon > 0$. This theorem was later extended to many different settings. Readers can see [K, KT, AC, F, T, KaS] for further details of these generalizations.

The main purpose of this paper is to extend DeLeeuw's theorem to the case $m \ge 2$. Letting $1/p = \sum_{j=1}^m 1/p_j$, we will establish the following theorems.

THEOREM 1. *Suppose that λ is an L^∞-function which is continuous on $\mathbb{R}^{mn}$ except on a countable set. Let T and $\tilde{T}_\varepsilon$ be the multilinear operators associated with λ. If*

$$\left\| \tilde{T}_\varepsilon(\tilde{f}_1, \ldots, \tilde{f}_m) \right\|_{L^p(\mathbb{T}^n)} \le \tilde{A} \prod_{j=1}^{m} \|\tilde{f}_j\|_{L^{p_j}(\mathbb{T}^n)} \tag{1.3}$$

uniformly for $\varepsilon > 0$, where $\tilde{A}$ is a constant independent of $\varepsilon > 0$ and $\tilde{f}_j$'s, then

$$\|T(f_1, \ldots, f_m)\|_{L^p(\mathbb{R}^n)} \le A \prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \tag{1.4}$$

where A is a constant independent of f_j's and $A \le \tilde{A}$.

For a set E, denote its Lebesgue measure by $\mu(E)$. We have the following weak type theorem.

THEOREM 2. *Let T, $\tilde{T}_\varepsilon$, λ be as in Theorem 1. If*

$$\mu\left\{x \in \mathbb{T}^n : \left|\tilde{T}_\varepsilon(\tilde{f}_1, \tilde{f}_2, \ldots, \tilde{f}_m)(x)\right| > \alpha\right\} \le \tilde{B}\left\{\prod_{j=1}^{m} \|\tilde{f}_j\|_{L^{p_j}(\mathbb{T}^n)} \alpha^{-1}\right\}^p, \tag{1.5}$$

where $\tilde{B}$ is independent of $\tilde{f}_j$'s, $\varepsilon > 0$ and $\alpha > 0$, then

$$\mu\left\{x \in \mathbb{R}^n : |T(f_1, f_2, \ldots, f_m)(x)| > \alpha\right\} \le B\left\{\prod_{j=1}^{m} \|f_j\|_{L^{p_j}(\mathbb{R}^n)} \alpha^{-1}\right\}^p, \tag{1.6}$$

where $B \le \tilde{B}$ is independent of f_j's, and $\alpha > 0$.

Conversely, we have the following two theorems.

THEOREM 3. *Suppose that λ is an L^∞-function on $\mathbb{R}^{nm}$. For a fixed $\varepsilon_0 > 0$ assume that all multi-integers $(k_1, k_2, \ldots, k_m) \in \Lambda \times \Lambda \times \cdots \times \Lambda$ are Lebesgue points of $\lambda(\varepsilon_0 \cdot)$. If* (1.4) *holds, then we have*

$$\left\| \tilde{T}_{\varepsilon_0}(\tilde{f}_1, \ldots, \tilde{f}_m) \right\|_{L^p(\mathbb{T}^n)} \le \tilde{A} \prod_{j=1}^{m} \|\tilde{f}_j\|_{L^{p_j}(\mathbb{T}^n)} \tag{1.3$'$}$$

with $\tilde{A} \le A$ being independent of $\varepsilon_0 > 0$ and $\tilde{f}_j$'s.

THEOREM 4. *Let ε_0 and λ be the same as in Theorem 3. If* (1.6) *holds, then we have*

$$\mu\left\{x \in \mathbb{T}^n : \left|\tilde{T}_{\varepsilon_0}(\tilde{f}_1, \ldots, \tilde{f}_m)(x)\right| > \alpha\right\} \le \tilde{B}\left\{\prod_{j=1}^{m} \|\tilde{f}_j\|_{L^{p_j}(\mathbb{R}^n)} \alpha^{-1}\right\}^p,$$

where $\tilde{B} \le B$ is independent of $\varepsilon_0 > 0$, $\alpha > 0$ and $\tilde{f}_j$'s.

As applications of Theorem 3, we will obtain an analog of Lacey–Thiele's theorem for bilinear conjugate Fourier series, as well as analogies of recent works on multilinear singular integrals by Kenig–Stein and Grafakos–Torres. It is worth remarking that recently Grafakos and Weiss studied an alternating definition of T in a more general amenable group and obtained some other transference results similar to Theorem 3 and Theorem 4 (see [GW]). But it seems that, by their theorems, one is not able to obtain Lacey–Thiele's theorem on the torus. On the other hand, their method does not work on the H^p-spaces, which we will work on later in this paper. We also want point out a few remarks.

REMARK 1. (1) If λ is L^∞ and continuous on $\mathbb{R}^{nm}$, clearly λ satisfies the condition in Theorem 3 and Theorem 4.

(2) Since the proofs for cases $m = 2$ and $m > 2$ are essentially the same, for the sake of simplicity, we will prove theorems for the case $m = 2$. We denote $f_1(x) = f(x), f_2(x) = g(x), \tilde{f}_1(x) = \tilde{f}(x), \tilde{f}_2(x) = \tilde{g}(x)$, and $p_1 = q, p_2 = r$ so that $1/p = 1/q + 1/r$ throughout this paper.

(3) The maximal operators are defined by

$$T^*(f_1, \dots, f_m)(x) = \sup_{\varepsilon>0} |T_\varepsilon(f_1, \dots, f_m)(x)|,$$
$$\tilde{T}^*(\tilde{f}_1, \dots, \tilde{f}_m)(x) = \sup_{\varepsilon>0} \left|\tilde{T}_\varepsilon(\tilde{f}_1, \dots, \tilde{f}_m)(x)\right|.$$

Noting

$$T^*(\cdot)(x) = \lim_{R\to\infty} \sup_{0<\varepsilon\le R} |T_\varepsilon(\cdot)(x)|, \quad \tilde{T}^*(\cdot)(x) = \lim_{R\to\infty} \sup_{0<\varepsilon\le R} \left|\tilde{T}_\varepsilon(\cdot)(x)\right|$$

for each x, without any changes in the proofs of Theorem 1–Theorem 4, we may use a limit argument to obtain Theorem 1–Theorem 4 for the maximal operators.

(4) In this paper, we do not intend to pursue the study of boundedness of T as those in the previous papers mentioned above. What we emphasize is to establish certain DeLeeuw type theorem, which says that, under some very mild condition, the boundedness of T on the Euclidean spaces is equivalent to the boundedness of its corresponding family $\{\tilde{T}_\varepsilon\}$ on the torus, so that one can easily obtain an analogous theorem on $\tilde{T}$ as soon as a new theorem of T is obtained.

(5) In the direction of generalization, one might expect to formulate a theorem that transfers not only the bilinear Hilbert transform, but also the multilinear fractional integrals in [KeS] and [GT]. It is known that, in general, DeLeeuw's theorem fails even in the one parameter case if $p \neq q$ (for example see [KaS]). So one might need some extra condition to establish such a theorem.

(6) Following the ideas in [AC] and [T], it is possible to establish transference theorem of multilinear operators between $\mathbb{R}^n$ and $\mathbb{Z}^n$. The proof for this case is in a different style, to avoid that this paper becomes too long, we will study this problem in our future papers.

The proofs of Theorem 1 and Theorem 2 use a standard argument involving the definition of Riemann integrals (see [SW]). For completeness, we present them in the second section. However, the duality argument used to prove Theorem 3 for the case $m = 1$ (see [SW, page 260]) is difficult to adopt. We use an alternating method to study Theorem 3 and Theorem 4 in Section 3. In Section 4, we study DeLeeuw's theorem on the Hardy spaces by using the atomic characterization of H^p.

Finally, in this paper, we use letter 'C' to denote (possibly different) constants that are independent of the essential variables in the argument.

2. Proofs of Theorem 1 and Theorem 2

Let $\mathscr{D}(\mathbb{R}^n) = \{f \in \mathscr{S}(\mathbb{R}^n) : f$ has compact support$\}$. The space $\mathscr{D}(\mathbb{R}^n)$ is dense in the space $L^p(\mathbb{R}^n)$, so it is enough to show the theorem for functions $f, g \in \mathscr{D}(\mathbb{R}^n)$. In order to do so, define $\tilde{f}_\varepsilon$ and $\tilde{g}_\varepsilon$, for $\varepsilon > 0$, to be the dilated and periodized versions of f and g, viz

$$\tilde{f}_\varepsilon = \varepsilon^{-n} \sum_{k \in \Lambda} f\left(\frac{x+k}{\varepsilon}\right), \quad \tilde{g}_\varepsilon = \varepsilon^{-n} \sum_{l \in \Lambda} g\left(\frac{x+l}{\varepsilon}\right) \tag{2.1}$$

Then by the Poisson summation formula we obtain

$$\tilde{f}_\varepsilon = \sum_{k \in \Lambda} \hat{f}(\varepsilon k) e^{2\pi i \langle k, x\rangle}, \quad \tilde{g}_\varepsilon = \sum_{l \in \Lambda} \hat{g}(\varepsilon l) e^{2\pi i \langle l, x\rangle}. \tag{2.2}$$

By the definition of the Riemann integral (see also [SW]), we know that

$$\lim_{\varepsilon \to 0} \varepsilon^{2n} \tilde{T}_\varepsilon(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)(\varepsilon x) = T(f, g)(x). \tag{2.3}$$

Let

$$Q = \left\{x \in \mathbb{R}^n : -1/2 \le x_j < 1/2, j = 1, 2 \dots, n\right\}$$

be the fundamental cube on which

$$\int_{\mathbb{T}_n} \tilde{f}(x)\, dx = \int_Q \tilde{f}(x)\, dx$$

for all function $\tilde{f}$ on $\mathbb{T}^n$. We choose $\{\varepsilon\}$ as a discrete sequence going to 0.

In order to prove Theorem 1, we choose $\eta(x) \ge 0$ to be a function in $\mathscr{D}(\mathbb{R}^n)$ satisfying $\eta(0) = 1$ and $\sum_{m \in \Lambda} \eta(x+m) = 1$. By Fatou's lemma, we have

$$\|T(f, g)\|^p_{L^p(\mathbb{R}^n)} \le \liminf_{\varepsilon \to 0} \int_{\mathbb{R}^n} \eta(\varepsilon x) \left|\varepsilon^{2n} \tilde{T}_\varepsilon(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)(\varepsilon x)\right|^p dx.$$

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By changing variables on x and using the fact $\sum \eta(x+k) = 1$, it is easy to see that

$$\begin{aligned}\|T(f,g)\|_{L^p(\mathbb{R}^n)} &\le \liminf_{\varepsilon\to 0}\left\{\varepsilon^{2np-n}\int_{\mathbb{R}^n}\eta(x)\left|\tilde{T}_\varepsilon\left(\tilde{f}_\varepsilon,\tilde{g}_\varepsilon\right)(x)\right|^p dx\right\}^{1/p}\\ &= \liminf_{\varepsilon\to 0}\varepsilon^{2n-n/p}\left\|\tilde{T}_\varepsilon\left(\tilde{f}_\varepsilon,\tilde{g}_\varepsilon\right)\right\|_{L^p(\mathbb{T}^n)}.\end{aligned}$$

By the assumption, we have that

$$\|T(f,g)\|_{L^p(\mathbb{R}^n)} \le \tilde{A}\liminf_{\varepsilon\to 0}\varepsilon^{2n-n/p}\|\tilde{f}_\varepsilon\|_{L^q(\mathbb{T}^n)}\|\tilde{g}_\varepsilon\|_{L^r(\mathbb{T}^n)}.$$

Finally, by [SW, page 266], we know that if ε is sufficiently small, then

$$\|\tilde{f}_\varepsilon\|_{L^q(\mathbb{T}^n)} = \varepsilon^{-n+n/q}\|f\|_{L^q(\mathbb{T}^n)}, \quad \|\tilde{g}_\varepsilon\|_{L^r(\mathbb{T}^n)} = \varepsilon^{-n+n/r}\|g\|_{L^r(\mathbb{T}^n)}.$$

Thus we obtain

$$\|T(f,g)\|_{L^p(\mathbb{R}^n)} \le \tilde{A}\|f\|_{L^q(\mathbb{T}^n)}\|g\|_{L^r(\mathbb{T}^n)}.$$

Theorem 1 is proved. □

We now turn to prove Theorem 2. Let $\chi_Q(x)$ be the characteristic function of Q. By Fatou's lemma, we have

$$\begin{aligned}(2.4)\qquad &\mu\{x\in\mathbb{R}^n : |Tf(x)| > \alpha\}\\ &\le \liminf_{\varepsilon\to 0}\mu\left\{x\in\mathbb{R}^n, \chi_Q(\varepsilon x)\left|\varepsilon^{2n}\tilde{T}_\varepsilon\left(\tilde{f}_\varepsilon,\tilde{g}_\varepsilon\right)(\varepsilon x)\right| > \alpha\right\}\\ &= \liminf_{\varepsilon\to 0}\varepsilon^{-n}\mu\left\{x\in\mathbb{R}^n, \chi_Q(x)\left|\varepsilon^{2n}\tilde{T}_\varepsilon\left(\tilde{f}_\varepsilon,\tilde{g}_\varepsilon\right)(x)\right| > \alpha\right\}\\ &= \liminf_{\varepsilon\to 0}\varepsilon^{-n}\mu\left\{x\in Q, \left|\tilde{T}_\varepsilon\left(\tilde{f}_\varepsilon,\tilde{g}_\varepsilon\right)(x)\right| > \alpha\varepsilon^{-2n}\right\}\\ &\le \liminf_{\varepsilon\to 0}\tilde{B}\varepsilon^{-n}\left\{\|\tilde{f}_\varepsilon\|_{L^q(\mathbb{T}^n)}\|\tilde{g}_\varepsilon\|_{L^r(\mathbb{T}^n)}\alpha^{-1}\varepsilon^{2n}\right\}^p\\ &= \tilde{B}\left\{\|f\|_{L^q(\mathbb{T}^n)}\|g\|_{L^r(\mathbb{T}^n)}\alpha^{-1}\right\}^p,\end{aligned}$$

which proves Theorem 2. □

3. Proofs of Theorem 3 and Theorem 4

Let $\|T_\varepsilon\| = \sup\{\|T_\varepsilon(f,g)\|_p : \|f\|_q = \|g\|_r = 1\}$. It is easy to see that $\|T_\varepsilon\| = \|T\|$ for all $\varepsilon > 0$. So to prove Theorem 3, without loss of generality, we may assume $\varepsilon_0 = 1$ (we may make the same assumption in proving Theorem 4, for the same reason).

Fix a positive integer K, define the set Ω_K by

$$\Omega_K = [-1/2 - 1/K, 1/2 + 1/K]^n.$$

Let Ψ be a function in $\mathscr{S}(\mathbb{R}^n)$ satisfying $\operatorname{supp}\Psi \subseteq \Omega_K$, $0 \le \Psi(x) \le 1$, and $\Psi(x) \equiv 1$ on Q. We denote $\Psi^{1/N}(x) = \Psi(x/N)$ for an integer N. For any C^∞ functions $\tilde{f}(x) = \sum_{k\in\Lambda} a_k e^{2\pi i\langle k,x\rangle}$ and $\tilde{g}(x) = \sum_{\nu\in\Lambda} b_\nu e^{2\pi i\langle \nu,x\rangle}$, we let

$$E_N(\tilde{f},\tilde{g})(x) = \Psi(x/N)^2 \tilde{T}(\tilde{f},\tilde{g})(x) - T\left(\Psi^{1/N}\tilde{f}, \Psi^{1/N}\tilde{g}\right)(x). \tag{3.1}$$

By checking the Fourier transform, it is easy to see that

$$\begin{aligned} -E_N(\tilde{f},\tilde{g})(x) = \sum_{k\in\Lambda}\sum_{\nu\in\Lambda} a_k b_\nu e^{2\pi i\langle k+\nu,x\rangle} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\Psi}(u)\hat{\Psi}(v)\{\lambda(k+u/N, \nu+v/N) \\ - \lambda(k,\nu)\} e^{2\pi i\langle u,x/N\rangle} e^{2\pi i\langle v,x/N\rangle}\, du\, dv. \end{aligned}$$

Since $\{a_k\}$, $\{b_\nu\}$ decay rapidly, λ is L^∞ and all (k,ν) are Lebesgue points of λ, clearly $E_N(\tilde{f},\tilde{g})(x) \to 0$ uniformly for $x \in \mathbb{R}^n$ as $N \to \infty$.

Noting $\tilde{T}(\tilde{f},\tilde{g})(x)$ is a periodic function, we have

$$\|\tilde{T}(\tilde{f},\tilde{g})\|_{L^p(\mathbb{T}^n)} = \left\{N^{-n}\int_{NQ} |\tilde{T}(\tilde{f},\tilde{g})|^p\,dx\right\}^{1/p}.$$

By the choice of Ψ, we further obtain

$$\|\tilde{T}(\tilde{f},\tilde{g})\|_{L^p(\mathbb{T}^n)} = \left\{N^{-n}\int_{NQ} |\Psi(x/N)^2\tilde{T}(\tilde{f},\tilde{g})|^p\,dx\right\}^{1/p}.$$

Thus by (3.1), we have that if $p \ge 1$, then

$$\begin{aligned} \|\tilde{T}(\tilde{f},\tilde{g})\|_{L^p(\mathbb{T}^n)} \le \left\{N^{-n}\int_{\mathbb{R}^n} \left|T\left(\Psi^{1/N}\tilde{f}, \Psi^{1/N}\tilde{g}\right)\right|^p dx\right\}^{1/p} \\ + \left\{N^{-n}\int_{NQ} |E_N(\tilde{f},\tilde{g})(x)|^p\,dx\right\}^{1/p}, \end{aligned}$$

and that the second integral on the right-hand side of the above inequality goes to zero as $N \to \infty$. On the other hand, the first integral on the right-hand side of the above inequality is equal to

$$N^{-n/p}\|T(\Psi^{1/N}\tilde{f}, \Psi^{1/N}\tilde{g})\|_{L^p(\mathbb{R}^n)}.$$

Thus by the assumption and the choice of Ψ, it is bounded by

$$\begin{aligned} &N^{-n/p}A\left\|\Psi^{1/N}\tilde{f}\right\|_{L^q(\mathbb{R}^n)}\left\|\Psi^{1/N}\tilde{g}\right\|_{L^r(\mathbb{R}^n)} \\ &\qquad \le AN^{-n/p}\left\{\int_{N\Omega_K}|\tilde{f}(x)|^q dx\right\}^{1/q}\left\{\int_{N\Omega_K}|\tilde{g}(x)|^r dx\right\}^{1/r}, \end{aligned}$$

where $N\Omega_K = [-N/2-N/K, N/2+N/K]^n$. Choose N such that N/K are integers. Then as $N \to \infty$ we have, since $\tilde{f}$ and $\tilde{g}$ are periodic functions, that

$$\begin{aligned} \|\tilde{T}(\tilde{f},\tilde{g})\|_{L^p(\mathbb{T}^n)} &\le AN^{-n/p}\left\{(N+2N/K)^n\int_Q|\tilde{f}(x)|^q dx\right\}^{1/q} \\ &\qquad \times\left\{(N+2N/K)^n\int_Q|\tilde{g}(x)|^r dx\right\}^{1/r} + o(1) \\ &= A(1+2/K)^{n/(1/q+1/r)}\|\tilde{f}\|_{L^p(\mathbb{T}^n)}\|\tilde{g}\|_{L^p(\mathbb{T}^n)} + o(1). \end{aligned}$$

Letting first $N \to \infty$, then $K \to \infty$, we prove Theorem 3 for $p \ge 1$.

For $0 < p < 1$, we have

$$\begin{aligned} \|\tilde{T}(\tilde{f},\tilde{g})\|^p_{L^p(\mathbb{T}^n)} &\le N^{-n}\int_{\mathbb{R}^n}\left|T\left(\Psi^{1/N}\tilde{f},\Psi^{1/N}\tilde{g}\right)(x)\right|^p dx \\ &\qquad + N^{-n}\int_{NQ}\left|E_N(\tilde{f},\tilde{g})(x)\right|^p dx. \end{aligned}$$

Thus the proof is the same as that for $p \ge 1$. □

To prove Theorem 4, fixing any $\alpha > 0$, we have

$$\begin{aligned} \mu\left\{x\in Q : \left|\tilde{T}(\tilde{f},\tilde{g})(x)\right| > \alpha\right\} &= N^{-n}\mu\left\{x \in NQ : \left|\tilde{T}(\tilde{f},\tilde{g})(x)\right| > \alpha\right\} \\ &= N^{-n}\mu\left\{x\in NQ : \left|\Psi(x/N)^2\tilde{T}(\tilde{f},\tilde{g})(x)\right| > \alpha\right\}. \end{aligned}$$

Thus by (3.1) and the fact that $E_N(\tilde{f},\tilde{g})(x) \to 0$ uniformly in x as $N \to \infty$, for any $\beta \in (0,\alpha)$ we can choose N sufficiently large such that

$$\mu\left\{x\in Q : \left|\tilde{T}(\tilde{f},\tilde{g})(x)\right| > \alpha\right\} \le N^{-n}\mu\left\{x\in\mathbb{R}^n : \left|T\left(\Psi^{1/N}\tilde{f},\Psi^{1/N}\tilde{g}\right)(x)\right| > \alpha-\beta\right\}.$$

Thus by the assumption we have

$$\begin{aligned} &\mu\left\{x\in Q : \left|\tilde{T}(\tilde{f},\tilde{g})(x)\right| > \alpha\right\} \\ &\qquad \le N^{-n}B\left\{\left\|\Psi^{1/N}\tilde{f}\right\|_{L^q(\mathbb{R}^n)}\left\|\Psi^{1/N}\tilde{g}\right\|_{L^r(\mathbb{R}^n)}(\alpha-\beta)^{-1}\right\}^p \\ &\qquad \le B\left\{\|\tilde{f}\|_{L^q(\mathbb{T}^n)}\|\tilde{g}\|_{L^r(\mathbb{T}^n)}(\alpha-\beta)^{-1}\right\}^p(1+2/K)^{n(1/q+1/r)p}. \end{aligned}$$

Since $\beta > 0$ is arbitrary, letting $K \to \infty$, the theorem is proved. $\square$

We now present some applications of Theorem 3. First we consider the bilinear Hilbert transform on the one-dimensional torus

$$\tilde{H}(\tilde{f}, \tilde{g})(x) = \text{p.v.} \int_{-1/2}^{1/2} \tilde{f}(x-t)\cot(\pi t)\tilde{g}(t+x)\,dt.$$

Then it is easy to check

$$\tilde{H}(\tilde{f}, \tilde{g})(x) = \sum_{-\infty}^{\infty}\sum_{-\infty}^{\infty} a_k b_\nu i \operatorname{sgn}(\nu - k) e^{2\pi i(k+\nu)x}.$$

By the known result of $H(f, g)$ (see [LT, La]) and the proof of Theorem 3 we have

COROLLARY 1. *$\tilde{H}$ maps $L^q(\mathbb{T}^1) \times L^r(\mathbb{T}^1)$ into $L^p(\mathbb{T}^1)$ for $1 < q, r \leq \infty$, $2/3 < p < \infty$ and $1/p = 1/q + 1/r$.*

PROOF. Let $\lambda(u, v) = i \operatorname{sgn}(v - u)$ and $E_N(\tilde{f}, \tilde{g})$ be as in the proof of Theorem 3. Note that $\lambda(u, v) = -\lambda(v, u)$. Therefore, by symmetry we have

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \hat{\Psi}(u)\hat{\Psi}(v)\lambda(k + u/N, k + v/N)e^{2\pi i\langle u, x/N\rangle}e^{2\pi i\langle v, x/N\rangle}\,du\,dv = 0$$

for all $k \in \Lambda$, $x \in \mathbb{R}$ and N. So, though the points $(k, k) \in \Lambda$ are not the Lebesgue points of $\lambda(u, v)$, we have $E_N(\tilde{f}, \tilde{g})(x) \to 0$ uniformly in $x \in \mathbb{R}$ as $N \to \infty$. Thus, by the proof of Theorem 3 and [LT, La], we get the corollary. $\square$

Secondly, we recall the multilinear singular integrals T_K on $\mathbb{R}^{nm}$:

$$T_K(f_1, \dots, f_m)(x) = \text{p.v.} \int_{\mathbb{R}^{nm}} f_1(x - y_1)\cdots f_m(x - y_m)K(y_1, \dots, y_m)\,dy_1\cdots dy_m,$$

where K is the Calderón-Zygmund kernel (see [GT, KeS]). We define

$$\text{(3.2)}\quad \tilde{T}_K(\tilde{f}_1, \dots, \tilde{f}_m)(x) = \sum_{k_1, \dots, k_m} \hat{K}(k_1, \dots, k_m)a_{k_1}a_{k_2}\cdots a_{k_m}\exp\left(2\pi i\sum_{j=1}^{m}\langle k_j, x\rangle\right),$$

where $(k_1, \dots, k_m)$ ranges over $\Lambda^m \backslash \{(0, \dots, 0)\}$.

COROLLARY 2. *Let K be a locally integrable function on $\mathbb{R}^{nm} - \{0\}$ which satisfies the size condition*

$$\text{(3.3)}\qquad |K(u_1, \dots, u_m)| \leq C|(u_1, \dots, u_m)|^{-nm},$$

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the cancellation condition

$$\left|\int_{R_1<|(u_1,\dots,u_m)|<R_2} K(u_1,\dots,u_m)\,du_1\cdots du_m\right| \le C < \infty, \tag{3.4}$$

for all $0 < R_1 < R_2 < \infty$, *and the smoothness condition*

$$\begin{aligned} &\left|K(u_1,\dots,u_j,\dots,u_m) - K(u_1,\dots,u_j',\dots,u_m)\right| \\ &\qquad \le C\frac{|u_j-u_j'|^\delta}{|(u_1,\dots,u_m)|^{nm+\delta}} \end{aligned} \tag{3.5}$$

for some $\delta > 0$ *whenever* $|u_j - u_j'| < |u_j|/2$. *Suppose that for some monotonically decreasing sequence* ε_j *convergent to zero, the limit*

$$\lim_{j\to\infty}\int_{\varepsilon_j<|(u_1,\dots,u_m)|\le 1} K(u_1,\dots,u_m)\,du_1\cdots du_m \tag{3.6}$$

exists. Then $\tilde{T}_K$ *maps* $L^{p_1}(\mathbb{T}^n)\times\cdots\times L^{p_m}(\mathbb{T}^n)$ *into* $L^p(\mathbb{T}^n)$ *with* $1 < p_j < \infty$, $1/m < p < \infty$ *and* $1/p = \sum_{j=1}^m 1/p_j$.

PROOF. We prove the corollary for the case $m = 2$. The proofs for the other cases are essentially the same as that for the case $m = 2$. From [GT], we know that $\|\hat{K}\|_\infty \le C < \infty$. Thus by (3.4)–(3.6) it is easy to see that $\hat{K}$ is a continuous function on $\mathbb{R}^{2n}\setminus\{0\}$. We write

$$\tilde{f}(x) = a_0 + \sum_{k\neq 0} a_k e^{2\pi i\langle k,x\rangle} = a_0 + \tilde{f}_1(x),$$

$$\tilde{g}(x) = b_0 + \sum_{k\neq 0} b_k e^{2\pi i\langle k,x\rangle} = b_0 + \tilde{g}_1(x),$$

Then,

$$\tilde{T}_K(\tilde{f},\tilde{g})(x) = \tilde{T}_K(a_0,\tilde{g}_1)(x) + \tilde{T}_K(\tilde{f}_1,b_0)(x) + \tilde{T}_K(\tilde{f}_1,\tilde{g}_1)(x).$$

Let $E_N(\tilde{f},\tilde{g})(x)$ be defined as in (3.1) with T_K and $\tilde{T}_K$ in place of T and $\tilde{T}$, respectively. Since $\hat{K}$ is continuous on $\mathbb{R}^{2n}\setminus\{0\}$, we see that $E_N(a_0,\tilde{g}_1)(x) \to 0$, $E_N(\tilde{f}_1,b_0)(x) \to 0$, $E_N(\tilde{f}_1,\tilde{g}_1)(x) \to 0$ uniformly in $x \in \mathbb{R}^n$ as $N \to \infty$. Thus by the proof of Theorem 3 and Theorem 5 in [GT], we have

$$\begin{aligned} \left\|\tilde{T}_K(\tilde{f},\tilde{g})\right\|_{L^p(\mathbb{T}^n)} &\le C\|a_0\|_{L^{p_1}(\mathbb{T}^n)}\left\|\tilde{g}_1\right\|_{L^{p_2}(\mathbb{T}^n)} + C\left\|\tilde{f}_1\right\|_{L^{p_1}(\mathbb{T}^n)}\|b_0\|_{L^{p_2}(\mathbb{T}^n)} \\ &\quad + C\left\|\tilde{f}_1\right\|_{L^{p_1}(\mathbb{T}^n)}\left\|\tilde{g}_1\right\|_{L^{p_2}(\mathbb{T}^n)}. \end{aligned}$$

This proves the corollary, since

$$\begin{aligned} &\|a_0\|_{L^{p_1}(\mathbb{T}^n)} \le \left\|\tilde{f}\right\|_{L^{p_1}(\mathbb{T}^n)}, &&\left\|\tilde{f}_1\right\|_{L^{p_1}(\mathbb{T}^n)} \le 2\left\|\tilde{f}\right\|_{L^{p_1}(\mathbb{T}^n)}, \\ &\|b_0\|_{L^{p_2}(\mathbb{T}^n)} \le \left\|\tilde{g}\right\|_{L^{p_2}(\mathbb{T}^n)}, &&\left\|\tilde{g}_1\right\|_{L^{p_2}(\mathbb{T}^n)} \le 2\left\|\tilde{g}\right\|_{L^{p_2}(\mathbb{T}^n)}. \end{aligned}$$

□

REMARK 2. Theorem 5 in [GT] was studied by Coifman–Meyer [CM1] if K is the kernel in the following Corollary 3.

COROLLARY 3. *Suppose that K is homogeneous of degree $-nm$, smooth away from the origin, and has mean value* 0 *on the unit sphere in* $\mathbb{R}^{nm}$. *Then $\tilde{T}_K$ maps* $L^{p_1}(\mathbb{T}^n) \times \cdots \times L^{p_m}(\mathbb{T}^n)$ *into* $L^p(\mathbb{T}^n)$, *with* $1/m < p < \infty$, $1 < p_j < \infty$ *and* $1/p = \sum_{j=1}^m 1/p_j$.

PROOF. Clearly, K satisfies (3.3)–(3.6) so that by Corollary 2, we obtain Corollary 3. Here we note that for the kernel considered in this corollary, the boundedness of T_K used in the proof of Corollary 2 also comes from [KeS, Theorem 8]. □

REMARK 3. It is possible to extend Theorem 1 and Theorem 3 to the Lorentz spaces so that we can obtain some weak-type estimates for $\tilde{T}_K$, which are analogous to those in [GT, Theorem 5] and [KeS, Theorem 8].

4. Bilinear operators in Hardy spaces

For $j = 1, 2, \ldots, \gamma$, let λ_j and μ_j be bounded functions on $\mathbb{R}^n$. Let U_j and $\tilde{U}_j$ be multipliers associate to λ_j on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively; V_j and $\tilde{V}_j$ be multipliers associate to μ_j on $\mathbb{R}^n$ and $\mathbb{T}^n$, respectively.

The bilinear operators $B_\gamma(f, g)(x)$ is defined by, for any $f, g \in \mathscr{S}(\mathbb{R}^n)$,

$$B_\gamma(f, g)(x) = \sum_{j=1}^{\gamma} U_j(f)(x)\, V_j(g)(x). \tag{4.1}$$

Similarly, the operator $\tilde{B}_\gamma$ is defined by

$$\tilde{B}_\gamma(\tilde{f}, \tilde{g})(x) = \sum_{j=1}^{\gamma} \tilde{U}_j(\tilde{f})(x)\, \tilde{V}_j(\tilde{g})(x) \tag{4.2}$$

for all

$$\tilde{f}(x) = \sum_{k \in \Lambda} a_k e^{2\pi i \langle k, x\rangle} \in C^\infty(\mathbb{T}^n), \qquad \tilde{g}(x) = \sum_{\nu \in \Lambda} b_\nu e^{2\pi i \langle \nu, x\rangle} \in C^\infty(\mathbb{T}^n),$$

where

$$\tilde{U}_j(\tilde{f})(x) = \sum_{k \in \Lambda} a_k \lambda_j(k) e^{2\pi i \langle k, x\rangle}, \qquad \tilde{V}_j(\tilde{g})(x) = \sum_{\nu \in \Lambda} b_\nu \mu_j(\nu) e^{2\pi i \langle \nu, x\rangle}.$$

The boundedness of bilinear operator $B_\gamma(f, g)$ on the Hardy spaces was studied by Coifman and Grafakos in [CG] (actually, in their study, U_j's and V_j's can be general Calderón-Zygmund operators of non-convolution type). Since there is no essential difference between $\gamma = 1$ and $\gamma > 1$, for simplicity, we study the case $\gamma = 1$. By the

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definition, it is easy to see that if $\gamma = 1$, then $B_1(f, g)$ is a special case of $T(f, g)$ and $\tilde{B}_1(\tilde{f}, \tilde{g})$ is a special case of $\tilde{T}(\tilde{f}, \tilde{g})$ with $\lambda(u, v) = \lambda_1(u)\mu_1(v)$. Therefore, naturally we will study DeLeeuw's theorem for $T(f, g)$ on the Hardy spaces. Below we first review the definition of the Hardy spaces.

Let $H^p(\mathbb{R}^n)$, $0 < p < \infty$, be the Hardy spaces defined by [FS]

$$H^p(\mathbb{R}^n) = \left\{ f \in \mathscr{S}'(\mathbb{R}^n),\ \|\Phi^+ f\|_{L^p(\mathbb{R}^n)} < \infty \right\},$$

where $\Phi^+ f(x) = \sup_{t>0} |\Phi_t * f(x)|$, $\Phi_t(x) = t^{-n}\Phi(x/t)$, and $\Phi \in \mathscr{S}(\mathbb{R}^n)$ is a radial function satisfying $\int \Phi = 1$. The corresponding periodic Hardy spaces are $H^p(\mathbb{T}^n) = \{\tilde{f} \in \mathscr{S}'(\mathbb{T}^n),\ \|\tilde{\Phi}^+ \tilde{f}\|_{L^p(\mathbb{R}^n)} < \infty\}$, where $\tilde{\Phi}^+ \tilde{f}(x) = \sup_{t>0} |\tilde{\Phi}_t * \tilde{f}(x)|$, $\tilde{\Phi}_t(x) = \sum_{k\in\Lambda} \hat{\Phi}(tk) e^{2\pi i (k,x)} + Ct^{-n} \sum_{k\in\Lambda} \Phi((x+k)/t)$.

In this section we will establish the following theorem.

THEOREM 5. *Let λ be a continuous and bounded functions on $\mathbb{R}^n$, and $T(f, g)$ and $\tilde{T}(\tilde{f}, \tilde{g})$ be the same as in Section 3. Suppose $1/p = 1/q + 1/r$. If there is a $C > 0$ such that $\|T(f, g)\|_{H^p(\mathbb{R}^n)} \le C\|f\|_{H^q(\mathbb{R}^n)}\|g\|_{H^r(\mathbb{R}^n)}$ for all $f \in H^q(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$ and $g \in H^r(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$, then $\|\tilde{T}(\tilde{f}, \tilde{g})\|_{H^p(\mathbb{T}^n)} \le C\|\tilde{f}\|_{H^q(\mathbb{T}^n)}\|\tilde{g}\|_{H^r(\mathbb{T}^n)}$ for all $f, g \in C^\infty(\mathbb{T}^n)$.*

To prove Theorem 5, we need to use the atomic characterization of the Hardy space. A regular $(p, 2, s)$ atom is a function $\alpha(x)$ supported in some ball $B(x_0, \rho)$ satisfying:

(i) $\|\alpha\|_2 \le \rho^{-n/p+n/2}$;
(ii) $\int_{\mathbb{R}^n} \alpha(x) P(x)\, dx = 0$

for all polynomials $P(x)$ of degree less than or equal to s.

The space $H_a^{p,s}(\mathbb{R}^n)$, $0 < p \le 1$, is the space of all distributions $f \in \mathscr{S}'(\mathbb{R}^n)$ having the form

$$f = \sum c_k \alpha_k \tag{4.3}$$

and satisfying

$$\sum |c_k|^p < \infty, \tag{4.4}$$

where each α_k is a $(p, 2, s)$ atom. The 'norm' $\|f\|_{H_a^{p,s}(\mathbb{R}^n)}$ is the infimum of all expressions $(\sum |c_k|^p)^{1/p}$ for which we have a representation (4.3) of f. A well-known fact (see [FS]) is that $\|f\|_{H_a^{p,s}(\mathbb{R}^n)} \cong \|f\|_{H^p(\mathbb{R}^n)}$ and in particular, $\|a\|_{H^p(\mathbb{R}^n)} \le C$, with a constant C independent of the $(p, 2, s)$ atom $a(x)$ if $s \ge [n(1/p - 1)]$.

We also have a similar decomposition theorem for any function $\tilde{f} \in H^p(\mathbb{T}^n)$. In particular, suppose $\tilde{f} \in C^\infty(\mathbb{T}^n)$ and its Fourier coefficient

$$a_0(\tilde{f}) = \int_Q \tilde{f}(x)\, dx = 0.$$

Then we have the following lemma.

LEMMA 1. *Suppose $\tilde{f} \in C^\infty(\mathbb{T}^n)$ with $a_0(\tilde{f}) = 0$. If we restrict x to Q, then for any fixed positive integer s*

$$\tilde{f}(x) = \sum c_k \alpha_k(x),$$

where each $\alpha_k(x)$ is a $(p, 2, s)$ atom satisfying $\alpha_k(x + l) = \alpha_k(x)$ for $l \in \Lambda$, and $\|\tilde{f}\|^p_{H^p(\mathbb{T}^n)} \cong \sum |c_k|^p$.

The proof can be found in [BF].

Now we are in a position to prove Theorem 5. For any $\tilde{f}, \tilde{g} \in C^\infty(\mathbb{T}^n)$, we have

$$\tilde{f}(x) = \sum_{k\in\Lambda} a_k e^{2\pi i\langle k,x\rangle} \quad \text{and} \quad \tilde{g}(x) = \sum_{\nu\in\Lambda} b_\nu e^{2\pi i\langle \nu,x\rangle}$$

with rapidly decaying coefficients. Recalling that $0 < p \le q$ and $0 < p \le r$ and a well-known fact $H^p = L^p$ if $p > 1$, we can use the same argument as in proving Theorem 3 to prove Theorem 5 in the case $p > 1$. It now suffices to show the case $0 < p \le 1$ and $0 < q, r \le 1$, the case $0 < p \le q \le 1 < r$ and the case $0 < p \le 1 < q \le r$. We prove these three cases separately.

CASE 1. $0 < p \le 1 < q \le r$. In this case $H^q = L^q$ and $H^r = L^r$. By definition and the Lebesgue dominated convergence theorem, we have

$$\|\tilde{T}(\tilde{f}, \tilde{g})\|_{H^p(\mathbb{T}^n)} = \lim_{R\to\infty} \left\{ \int_Q \sup_{0<t\le R} \left|\tilde{\Phi}_t * \tilde{T}(\tilde{f}, \tilde{g})(x)\right|^p dx \right\}^{1/p}.$$

Thus it suffices to show

$$\left\{ \int_Q \sup_{0<t\le R} \left|\tilde{\Phi}_t * \tilde{T}(\tilde{f}, \tilde{g})(x)\right|^p dx \right\}^{1/p} \le C\|\tilde{f}\|_{L^q(\mathbb{T}^n)}\|\tilde{g}\|_{L^r(\mathbb{T}^n)} \tag{4.5}$$

with C being independent of $R > 0$.

By definition, it is easy to check that, for each fixed $t > 0$,

$$\tilde{\Phi}_t * \tilde{T}(\tilde{f}, \tilde{g})(x) = \sum_k \sum_\nu a_k b_\nu \lambda(k, \nu)\hat{\Phi}(t(k + \nu))e^{2\pi i\langle k+\nu,x\rangle},$$

$$\Phi_t * T(f, g)(x) = \iint_{\mathbb{R}^n\times\mathbb{R}^n} \hat{f}(u)\hat{g}(v)\lambda(u, v)\hat{\Phi}(t(u + v))e^{2\pi i\langle u+v,x\rangle}\, du\, dv.$$

Let Ω_K and $\Psi^{1/N}(x)$ be the same as in Section 3. For each $t > 0$, using $\tilde{\Phi}_t * \tilde{T}$ and $\Phi_t * T$ instead of $\tilde{T}$ and T in (3.1), respectively, we obtain

$$
\begin{aligned}
(4.6)\qquad E_{N,t}(\tilde{f}, \tilde{g})(x) &= \Psi(x/N)^2 \tilde{\Phi}_t * \tilde{T}(\tilde{f}, \tilde{g})(x) - \Phi_t * T(\Psi^{1/N}\tilde{f}, \Psi^{1/N}\tilde{g})(x) \\
&= -\sum_{k\in\Lambda}\sum_{\nu\in\Lambda} a_k b_\nu e^{2\pi i\langle k+\nu, x\rangle} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\Psi}(u)\hat{\Psi}(v) \\
&\quad \times \Big\{\lambda(k+u/N, \nu+v/N)\hat{\Phi}(t(k+u/N+\nu+v/N)) \\
&\qquad -\hat{\Phi}(t(k+\nu))\lambda(k,\nu)\Big\} e^{2\pi i\langle u, x/N\rangle} e^{2\pi i\langle v, x/N\rangle}\, du\, dv.
\end{aligned}
$$

Since $\{a_k\}$, $\{b_\nu\}$ decay rapidly, λ and $\hat{\Phi}$ are L^∞ and continuous, it is clear that $E_{N,t}(\tilde{f}, \tilde{g})(x) \to 0$ uniformly for $x \in \mathbb{R}^n$ and $t \in [0, R]$ as $N \to \infty$. Thus we can obtain (4.5) by emulating the proof of Theorem 3.

CASE 2. $0 < p \le 1$ and $0 < q, r \le 1$.

We note that, for a C^∞ function $\tilde{f}(x) = \sum a_k(\tilde{f}) e^{2\pi i\langle k,x\rangle}$,

$$\|a_0(\tilde{f})\|_{H^p(\mathbb{T}^n)} = C|a_0(\tilde{f})| = \left|\int_Q \tilde{f}(x)\,dx\right|$$

and

$$\sup_{t>0}|\tilde{\Phi}_t * \tilde{f}(x)| \ge \lim_{t\to\infty}|\tilde{\Phi}_t * \tilde{f}(x)| = |a_0(\tilde{f})|.$$

Thus

$$\|a_0(\tilde{f})\|^p_{H^p(\mathbb{T}^n)} = \int_Q \|a_0(\tilde{f})\|^p_{H^p(\mathbb{T}^n)} dx \le C\int_Q \sup_{t>0}|\tilde{\Phi}_t * \tilde{f}(x)|^p\,dx = C\|\tilde{f}\|^p_{H^p(\mathbb{T}^n)}.$$

Because we can write

$$\tilde{f}(x) = a_0 + \sum_{k\ne 0} a_k e^{2\pi i\langle k,x\rangle} = a_0 + \tilde{f}_1(x)$$

and

$$\tilde{g}(x) = b_0 + \sum_{\nu\ne 0} b_\nu e^{2\pi i\langle \nu,x\rangle} = b_0 + \tilde{g}_1(x),$$

we treat $\tilde{T}(a_0, b_0)$, $\tilde{T}(a_0, \tilde{g}_1)$, $\tilde{T}(\tilde{f}_1, b_0)$ and $\tilde{T}(\tilde{f}_1, \tilde{g}_1)$ separately. The first one is easily estimated by the above observation. To estimate the last one, we write $\tilde{f}$ and $\tilde{g}$ for $\tilde{f}_1$ and $\tilde{g}_1$, respectively, for the sake of simplicity. Then we have $\int_Q \tilde{f}\,dx = \int_Q \tilde{g}\,dx = 0$ so that by Lemma 1, we can write $\tilde{f}$ and $\tilde{g}$ in the forms of their atomic decompositions

$$\tilde{f}(x) = \sum c_k \alpha_k(x), \qquad \tilde{g}(x) = \sum \beta_\nu O_\nu(x),$$

where each α_k is a $(q, 2, [n(1/q-1)] + 2n)$ atom and each O_ν is a $(r, 2, [n(1/r-1)] + 2n)$ atom, and

$$\sum |c_k|^q \cong \|\tilde{f}\|^q_{H^q(\mathbb{T}^n)}, \qquad \sum |\beta_\nu|^r \cong \|\tilde{g}\|^r_{H^r(\mathbb{T}^n)}.$$

We take

$$\Psi(x)=\prod_{j=1}^{n}\left(1-4x_j^2\right)_+, \qquad \text{where} \quad f_+(x)=\begin{cases} f(x) & \text{if } f(x)\ge 0,\\ 0 & \text{if } f(x)<0.\end{cases}$$

For positive integers M and N, we denote the cube $[-N/2M, N/2M)^n$ by NQ/M. For large N, by (4.6) and the assumption of the theorem, we have

$$\begin{aligned}&\left\|\sup_{0<t\le R}\left|\tilde{\Phi}_t * \tilde{T}(\tilde{f},\tilde{g})\right|\right\|_{L^p(\mathbb{T}^n)}\\ &\quad\cong\left\{N^{-n}\int_{NQ/2}\Psi^2(x/N)\sup_{0<t\le R}\left|\tilde{\Phi}_t*\tilde{T}(\tilde{f},\tilde{g})(x)\right|^p dx\right\}^{1/p}\\ &\quad\le C\left\{N^{-n}\int_{\mathbb{R}^n}\sup_{0<t\le R}\left|\Phi_t*T\left(\tilde{f}\Psi^{1/N},\tilde{g}\Psi^{1/N}\right)(x)\right|^p dx\right\}^{1/p}+o(1).\end{aligned}$$

This shows that

$$\begin{aligned}(4.7)\qquad &\left\|\sup_{0<t\le R}\left|\tilde{\Phi}_t * \tilde{T}(\tilde{f},\tilde{g})\right|\right\|_{L^p(\mathbb{T}^n)}\\ &\quad\le CN^{-n/p}\left\|\tilde{f}\Psi^{1/N}\right\|_{H^q(\mathbb{R}^n)}\left\|\tilde{g}\Psi^{1/N}\right\|_{H^r(\mathbb{R}^n)}+o(1), \quad \text{as } N\to\infty.\end{aligned}$$

Therefore, it suffices to show that

$$(4.8)\qquad \liminf_{N\to\infty} N^{-n/p}\left\|\tilde{f}\Psi^{1/N}\right\|_{H^q(\mathbb{R}^n)}\left\|\tilde{g}\Psi^{1/N}\right\|_{H^r(\mathbb{R}^n)}\le C\left\|\tilde{f}\right\|_{H^q(\mathbb{T}^n)}\left\|\tilde{g}\right\|_{H^r(\mathbb{T}^n)}.$$

We note that

$$\begin{aligned}\left\|\tilde{f}\Psi^{1/N}\right\|^q_{H^q(\mathbb{R}^n)}&\le C\sum_k |c_k|^q\left\|\alpha_k\Psi^{1/N}\right\|^q_{H^q(\mathbb{R}^n)},\\ \left\|\tilde{g}\Psi^{1/N}\right\|^r_{H^r(\mathbb{R}^n)}&\le C\sum_\nu |\beta_\nu|^r\left\|O_\nu\Psi^{1/N}\right\|^r_{H^r(\mathbb{R}^n)}.\end{aligned}$$

Thus we only need to prove that for any $(q,2,s)$ periodic atom $\alpha(x)$ with support in $B(x_0,\rho)\subset Q$,

$$(4.9)\qquad \left\|\alpha\Psi^{1/N}\right\|_{H^q(\mathbb{R}^n)}\le CN^{n/q},$$

where C is a constant independent of $\alpha(x)$ and N. By the definition, we have

$$\begin{aligned}\left\|\alpha\Psi^{1/N}\right\|^q_{H^q(\mathbb{R}^n)}&\cong\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{\mathbb{R}^n}\Psi(x/N)\alpha(x)\Phi_t(y-x)\,dx\right|^q dy\\ &=\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{\mathbb{R}^n}\prod_{j=1}^{n}\left(1-4x_j^2/N^2\right)_+\alpha(x)\Phi_t(y-x)\,dx\right|^q dy\\ &=\int_{\mathbb{R}^n}\sup_{0<t<\infty}\left|\int_{|x_j|<N/2}\left\{\prod_{j=1}^{n}\left(1-4x_j^2/N^2\right)\alpha(x)\right\}\Phi_t(y-x)\,dx\right|^q dy.\end{aligned}$$

Now if we let $N = 2m+1$, then, up to a set of measure 0, the set $\{x \in \mathbb{R}^n : |x_j| < m + 1/2,\ j = 1, 2, \dots, n\}$ is the union of the disjoint sets $\{Q + k : k = (k_1, \dots, k_n), -m \le k_j \le m,\ j = 1, 2, \dots, n\} = \{Q_k\}$, where the k_j's are integers. Now the last integral above is bounded by

$$I_m = C \sum_{-m \le k_j \le m} \int_{\mathbb{R}^n} \sup_{0<t<\infty} \left| \int_{Q_k} \left\{ \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) \alpha(x) \right\} \Phi_t(y-x)\, dx \right|^q dy.$$

Noting that $\alpha(x)$ is a periodic function, we easily see that $\chi_{Q_k}(x)\alpha(x)$ is an atom with support in Q_k. Also since on Q_k, $\prod_{j=1}^n (1 - 4x_j^2/N^2)$ is a polynomial of degree $2n$ which is bounded by 1, clearly

$$A(x) = \prod_{j=1}^n \left(1 - 4x_j^2/N^2\right) \chi_{Q_k}(x)\alpha(x)$$

is a $(q, 2, [n(1/q - 1)])$ atom on $\mathbb{R}^n$. So by a well-known estimate, the above integral I_m is bounded by

$$C \sum_{-m \le k_j \le m} \|A\|_{H^q(\mathbb{R}^n)}^q \le CN^n,$$

which shows $\|\alpha \Psi^{1/N}\|_{H^q(\mathbb{R}^n)} \le CN^{n/q}$.

Finally we treat $\tilde{T}(\tilde{f_1}, b_0)$. Let Γ be a C^∞ function supported in $\Omega_K = [-1/2 - 1/K, 1/2 + 1/k]$ for some fixed positive constant K. Suppose that $\Gamma(x) \equiv 1$ on Q and $\|\Gamma^{1/N}\|_{H^r(\mathbb{R}^n)} \le CN^{n/r}$. (We assume a suitable cancellation condition to get the last property of Γ.) Let Ψ be as above. Put

$$\begin{aligned} E_{N,t}(\tilde{f}, \tilde{g})(x) &= \Psi(x/N)\Gamma(x/N)\tilde{\Phi}_t * \tilde{T}(\tilde{f}, \tilde{g})(x) - \tilde{\Phi}_t * T(\Psi^{1/N}\tilde{f}, \Psi^{1/N}\tilde{g})(x) \\ &= -\sum_{k\in\Lambda}\sum_{\nu\in\Lambda} a_k b_\nu e^{2\pi i\langle k+\nu, x\rangle} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \hat{\Psi}(u)\hat{\Gamma}(v) \\ &\quad \times \Big\{ \lambda(k + u/N, \nu + v/N)\hat{\Phi}(t(k + u/N + \nu + v/N)) \\ &\qquad - \lambda(u, v)\hat{\Phi}(t(k+\nu)) \Big\} e^{2\pi i\langle u, x/N\rangle} e^{2\pi i\langle v, x/N\rangle}\, du\, dv. \end{aligned}$$

Then, for any fixed $R > 0$, $E_{N,t}(\tilde{f}, \tilde{g})(x) \to 0$ uniformly in $x \in \mathbb{R}^n$ and $t \in [0, R]$ as $N \to \infty$. Therefore, arguing as in (4.7), we have

$$\begin{aligned} \text{(4.10)} \qquad & \left\| \sup_{0<t\le R} \left| \tilde{\Phi}_t * \tilde{T}(\tilde{f_1}, b_0) \right| \right\|_{L^p(\mathbb{T}^n)} \\ & \le CN^{-n/p} \left\| \tilde{f}\Psi^{1/N} \right\|_{H^q(\mathbb{R}^n)} \left\| b_0 \Gamma^{1/N} \right\|_{H^r(\mathbb{R}^n)} + o(1), \quad \text{as } N \to \infty. \end{aligned}$$

By (4.10) and the estimates on (4.8), we see that the left-hand side is bounded by $C\|\tilde{f}\|_{H^q(\mathbb{T}^n)}\|\tilde{g}\|_{H^r(\mathbb{T}^n)}$. Clearly, we have the same estimate for $\tilde{T}(a_0, \tilde{g}_1)$.

CASE 3. $0 < p \leq 1, 0 < q \leq 1 < r$. The proof for this case is an easy combination of those for Cases 1 and 2, we leave the proof to the reader. $\square$

The following theorem is the converse of Theorem 5.

THEOREM 6. *Let λ and p, q, r be as in Theorem 5, and $T(f, g)$ and $\tilde{T}_\varepsilon(\tilde{f}, \tilde{g})$ be as in Section 3. If there is a $C > 0$ such that*

$$\left\|\tilde{T}_\varepsilon(\tilde{f}, \tilde{g})\right\|_{H^p(\mathbb{T}^n)} \leq C\left\|\tilde{f}\right\|_{H^q(\mathbb{T}^n)}\left\|\tilde{g}\right\|_{H^r(\mathbb{T}^n)} \quad \text{for all } f, g \in C^\infty(\mathbb{T}^n)$$

uniformly for $\varepsilon > 0$, then

$$\|T(f, g)\|_{H^p(\mathbb{R}^n)} \leq C\|f\|_{H^q(\mathbb{R}^n)}\|g\|_{H^r(\mathbb{R}^n)}$$

for all $f \in H^q(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$ and $g \in H^r(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$.

To prove Theorem 6, we need the following lemma.

LEMMA 2. *Let $f \in H^p(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$, and define $\tilde{f}_\varepsilon$ as in (2.1). Then*

$$\lim_{\varepsilon \to 0} \varepsilon^{n(1-1/p)}\left\|\tilde{f}_\varepsilon\right\|_{H^p(\mathbb{T}^n)} = \|f\|_{H^p(\mathbb{R}^n)}.$$

See [LL, Lemma 3] for a proof.

Now we return to prove Theorem 6. Let $f \in H^q(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$ and $g \in H^r(\mathbb{R}^n) \cap \mathscr{S}(\mathbb{R}^n)$. As in the proof of Theorem 1, we have

$$\lim_{\varepsilon \to 0} \varepsilon^{2n}\tilde{\Phi}_{t\varepsilon} * \tilde{T}_\varepsilon(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)(\varepsilon x) = \Phi_t * T(f, g)(x)$$

by the definition of the Riemann integral. Let η be as in the proof of Theorem 1. By Fatou's lemma we see that

$$\begin{aligned}
\|T(f, g)\|^p_{H^p(\mathbb{R}^n)} &= \int_{\mathbb{R}^n} \sup_{t>0} |\Phi_t * T(f, g)(x)|^p \, dx \\
&\leq \liminf_{\varepsilon \to 0} \varepsilon^{2np} \int_{\mathbb{R}^n} \eta(\varepsilon x) \sup_{t>0} \left|\tilde{\Phi}_{t\varepsilon} * \tilde{T}_\varepsilon(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)(\varepsilon x)\right|^p dx \\
&\leq \liminf_{\varepsilon \to 0} \varepsilon^{2np-n} \int_Q \sup_{t>0} \left|\tilde{\Phi}_t * \tilde{T}_\varepsilon(\tilde{f}_\varepsilon, \tilde{g}_\varepsilon)(x)\right|^p dx.
\end{aligned}$$

By the assumption and Lemma 2, we have

$$\begin{aligned}
\|T(f, g)\|^p_{H^p(\mathbb{R}^n)} &\leq C^p \liminf_{\varepsilon \to 0} \varepsilon^{2np-n}\left\|\tilde{f}_\varepsilon\right\|^p_{H^q(\mathbb{T}^n)}\left\|\tilde{g}_\varepsilon\right\|^p_{H^r(\mathbb{T}^n)} \\
&\leq C^p\left\|\tilde{f}\right\|^p_{H^q(\mathbb{R}^n)}\left\|\tilde{g}\right\|^p_{H^r(\mathbb{R}^n)}.
\end{aligned}$$

This completes the proof of Theorem 6. □

Finally, we point out an application of Theorem 5. Suppose that $B_1(f, g)$ and $\tilde{B}_1(\tilde{f}, \tilde{g})$ are defined as in (4.1) and (4.2), with U_1 and V_1 being standard Calderón-Zygmund operators. Also assume

$$(\widehat{U_1 f)(u}) = \hat{f}(u)\lambda_1(u), \qquad (\widehat{V_1 g)(v}) = \hat{g}(v)\mu_1(v),$$

where both λ_1 and μ_1 are continuous and bounded. By results in [CG] and [G] and Theorem 5, we have

THEOREM 7. *Let $0 < q, r \leq 1$ and $1/p = 1/r + 1/q$. Assume that for some non-negative integer κ, there is an s such that*

$$\int_{\mathbb{R}^n} x^\beta B_1(f, g)\, dx = 0$$

for all multi-indices β with $|\beta| \leq \kappa$, and all $(r, 2, s)$ atoms g and $(q, 2, s)$ atoms f. Then for $n/(n + \kappa + 1) < p \leq 1$, $\tilde{B}_1(\tilde{f}, \tilde{g})$ can extend to a bounded operator from $H^q(\mathbb{T}^n) \times H^r(\mathbb{T}^n)$ into $H^p(\mathbb{T}^n)$.

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