Can. J. Math., Vol. XXXI, No. 4, 1979, pp. 881–889

REMARKS ON k-LEVIFLAT COMPLEX MANIFOLDS

B. GILLIGAN AND A. HUCKLEBERRY

0. Introduction. In the theory of functions of several complex variables one is naturally led to study non-compact complex manifolds which have certain types of exhaustions. For example, on a Stein manifold X there is a strictly plurisubharmonic function $\varphi: X \rightarrow \mathbf{R}^+$ so that the pseudoballs $B_c = \{\varphi < c\}$ exhaust X. Conversely, a manifold which has such an exhaustion is Stein. The purpose of this note is to study a class of manifolds which have exhaustions along the lines of those on holomorphically convex manifolds, namely the k-Leviflat complex manifolds. Unlike the Stein case, the Levi form may have positive dimensional 0-eigenspaces. In the holomorphically convex case these are tangent to the generic fiber of the Remmert reduction.

Definition. A complex manifold X is said to be *k-Leviflat* if it possesses a proper exhaustion function $\varphi: X \rightarrow \mathbf{R}^+$ so that for $c \gg 0$ and for every $p \in \{\varphi = c\}$ the Levi form $L(\varphi)(p)$ (i.e. the full complex Hessian of φ restricted to the complex tangent space of the surface $\{\varphi = c\}$ at p) is positive semi-definite with rank $k - 1$.

If $k = n$, then the hypersurfaces $\{\varphi = c\}$ above can be defined by strictly plurisubharmonic functions. In this setting it is a well-known (although deep) fact that X is a proper modification of a Stein space [**4**]. For $k < n$ it is clear that X is not a Stein space. In fact, there exist at most k analytically independent holomorphic functions on a k-Leviflat manifold [**9**].

Furthermore, it is quite possible for a given k-Leviflat manifold to possess no non-constant holomorphic functions. This is precisely the case for "reduced groups" $\mathbf{C}^n/\Gamma_{n+m}$, where Γ_{n+m} is a lattice of rank $n + m$ and $0 < m \leqq n$ (see [**13**]). So, contrary to the case when $k = n$, if $k < n$ then X may or may not be holomorphically convex.

The first part of this paper is devoted to giving necessary and sufficient conditions for a k-Leviflat manifold to be holomorphically convex. For a statement of this theorem we need the notion of the rank of a function algebra: Let A be a set of holomorphic functions on a complex manifold X and let $x \in X$. Then the *level set* of A at x is

$$L(A)_x: = \{y \in X | f(y) = f(x) \text{ for all } f \in A\}.$$

It turns out that $L(A)_x$ is an analytic subvariety of X and the *rank of A at x*,

Received February 23, 1978. This research was partially supported by NRC Operating Grant A-8739 and by NSF Grant MCS 75-07086A01.

$\text{rank}_x A$, is defined to be the codimension of $L(A)_x$ at x. Moreover rank A: $= \max_{x \in X}(\text{rank}_x A)$.

THEOREM 1. *Let X be a k-Leviflat complex manifold. Then X is holomorphically convex if and only if* $\text{rank}_x \mathcal{O}(X) = k$ *for all x in some neighborhood of infinity.*

This theorem was proved in [**7**] under rather strong additional assumptions on the exhaustion (e.g. plurisubharmonic!).

The theory of k-Leviflat manifolds is certainly not void of examples (e.g. see [**7**], [**9**], [**10**]). However the interesting such seem to be either holomorphically convex or every holomorphic function is constant. Of course it is possible to make trivial combinations of these. But there are numerous non-trivial examples (in fact big classes of complex homogeneous spaces) which are k-Leviflat, but where the holomorphic separation map (see [**3**]) is not proper and $0 < \text{rank}\,\mathcal{O}(X) < k$. The second point of this note is to give a detailed description of one such homogeneous space.

The third point of this paper is to make some remarks which pertain to mapping properties of bounded domains in k-Leviflat spaces. It is known that if Ω is a bounded domain in a Stein manifold X and $\partial\Omega$ is smooth, then Ω is not equivalent to the total space of a holomorphic fiber bundle whose base and fiber are positive dimensional [**8**]. Of course an example of this is the fact that the ball and the polydisk in $\mathbf{C}^n$ are inequivalent. Now many k-Leviflat manifolds arise as bundle spaces and in fact contain bounded domains which are themselves disk bundles. Hence the above inequivalency theorem can not be carried over verbatim. However we are able to prove the following:

THEOREM 2. *Let X be a k-Leviflat manifold and* $\Omega \subset\subset X$ *be a bounded domain with* $\partial\Omega$ *smooth. Assume that* Ω *is big enough to have non-empty intersection with the region of X where the hypersurfaces* $\{\varphi = c\}$ *are k-Leviflat. Then* Ω *is not holomorphically equivalent to the total space E of a holomorphic fiber bundle* $F \to E \to B$ *where* $\text{codim}_{\mathbf{C}} F$ *and* $\text{codim}_{\mathbf{C}} B$ *are both less than k.*

We note that the trivial product of a k-dimensional ball and any compact complex manifold of dimension $n - k$, where $n \geqq 2k$, shows that the inequalities on codimension are both needed. We also remark that the Stein case (i.e. n-Leviflat) is contained in this theorem, because the codimension conditions reduce to the base and fiber being positive dimensional.

A key ingredient for all of the above is that the k-Leviflat hypersurfaces $\{\varphi = c\}$ are compact CR-manifolds which are foliated by $(n - k)$-dimensional complex manifolds. Such foliations have not been studied in great detail, but the standard examples have vector spaces or parabolic manifolds (e.g. $\mathbf{C}^*$) as leaves. We end this paper by noting a rather general procedure by which one can generate k-Leviflat manifolds where the typical leaf of the CR-foliation can be chosen from a very large class; in particular it could be hyperbolic. This says that the classical argument for the non-existence of analytic functions (e.g. "Grauert's example", [**5**]) which uses the parabolic nature of the leaves does not work in general. It in fact seems necessary to use something like the function algebra techniques of [**9**].

1. The proof of theorem 1. Up to a certain point, the proof is the same as in [**7**]. Thus we try to be as brief as possible. We begin by collecting some function algebra facts. If K is a subset of a complex space, then the *analytic dimension* of K is the smallest integer k such that K is contained in a countable union of k-dimensional local analytic sets. If K is a compact subset of $\mathbf{C}^m$ with analytic dimension k and S is the Shilov boundary of $\mathcal{O}(K)$, then S also has analytic dimension k. This is a consequence of a much more general theorem in [**7**].

Let X be an n-dimensional, k-Leviflat manifold with exhaustion function $\varphi: X \to \mathbf{R}^+$. Let $B_c: = \{\varphi < c\}$ and $S_c: = \{\varphi = c\}$ be the pseudoball and pseudosphere of radius c respectively. For c large enough, S_c is a k-Leviflat hypersurface and is therefore foliated by $(n - k)$-dimensional complex manifolds (see [**2**] for basic properties of such foliations). Our first goal is to prove that if $\text{rank}_x \mathcal{O}(X) = k$ for all x near infinity, then X is holomorphically convex. For this we first note that there is a *holomorphic separation map*

$$F: X \to \mathbf{C}^{2n+1}, n = \dim_{\mathbf{C}} X,$$

such that $F(x) = F(y)$ exactly when $f(x) = f(y)$ for every $f \in \mathcal{O}(X)$. From now on we assume that for $x \in X \backslash B_{c_0}$ the rank of F at x is k, and that for $c \geqq c_0$ we also have S_c a k-Leviflat hypersurface.

The main idea of the proof of this direction of the theorem is to show that the leaves of the foliation of S_c are compact and agree with the connected components of the fibers of F. For this let $K: = F(S_c)$. Since rank $F = k$, the analytic dimension of K is also k. Let $\mathcal{S}: = \{x \in X | \text{rank}(dF) < k\}$. Then the analytic dimension of $F(\mathcal{S})$ is at most $k - 1$. In particular, since the Shilov boundary of $\mathcal{O}(K)$ has analytic dimension k, there is a peak point $q \in K \backslash F(\mathcal{S})$. Let $p \in S_c$ be such that $F(p) = q$. Denote by $\mathcal{M}_p$ the leaf of the foliation which contains p. If $F|\mathcal{M}_p$ were non-constant, then K would contain a local analytic set through q. This can't happen, because q is a peak point. So $F(\mathcal{M}_p) = \{q\}$.

Now let $\mathcal{C}$ be the union of the leaves of the foliation of S_c which do not intersect $\mathcal{S}$ and which are compact. Obviously the above $\mathcal{M}_p$ is contained in $\mathcal{C}$. If $\mathcal{M}_x$ is any leaf in $\mathcal{C}$, then the connected component $\mathcal{F}_x$ of the fiber of F at x must coincide with $\mathcal{M}_x$. Thus there is a neighborhood U of such an $\mathcal{M}_x$ and a neighborhood U' of $F(x)$ in $\mathbf{C}^{2n+1}$ so that $F|U: U \to U'$ is a proper map [**14**]. We may take U small enough so that $U \cap \mathcal{S} = \emptyset$.

We want to show that the fibers of $F|U$ which have non-empty intersection with S_c in fact coincide with leaves of the foliation of S_c. Suppose there is such a fiber $\mathcal{F}_y$, $y \in S_c$, which has non-empty intersection with $S_{c'}$ for some $c' > c$. By taking c' larger if necessary, we may assume that $\max\varphi|\mathcal{F}_y = c'$. Further we may take a curve x_t, $0 < t < 1$, in S_c so that $x_0 = x$, $x_1 = y$ and such that

$\max\varphi|\mathscr{F}_{x_t} < c'$ for all $t < 1$. Let $y' \in \mathscr{F}_y$ be such that $\varphi(y') = c'$. Since $\mathscr{F}_y \cap \mathscr{S} = \emptyset$, we may choose coordinates $z_1, \ldots, z_n$ in a polydisk neighborhood Δ of y' so that

$$\mathscr{F}_{x_t} \cap \Delta = \{(z_1(t), \ldots, z_k(t))\} \times \Delta',$$

where Δ' is the $(n - k)$-dimensional polydisk

$$\{(z_{k+1}, \ldots, z_n) | \, |z_j| < 1 \text{ for all } j\}, \; t > t_0.$$

Now we can redefine the hypersurface $\{\varphi = c'\}$ by a Diederich-Fornaess exhaustion ψ [**1**], where ψ is continuous on $\Delta \cap \{\varphi \leqq c'\} = \{\psi \leqq 0\}$ and ψ is strictly plurisubharmonic and negative on $\Delta \cap \{\varphi < c'\} = \{\psi < 0\}$.

Now let ψ_t be the function defined on Δ' by restricting ψ to $\mathscr{F}_{x_t} \cap \Delta$ in the obvious way, $t_0 < t < 1$, and let $r_t := \sup_{s \leqq t}(\psi_s)$. Since $\{r_t\}$ is a bounded, non-decreasing sequence of subharmonic functions, the limit $r := \lim_{t \to 1} r_t$ exists and is subharmonic. But $r(0) = 0 = \sup_{\Delta'} r$. So $r \equiv 0$. Thus $\mathscr{F}_{y'} \cap \Delta \subset S_{c'}$, and consequently $\mathscr{F}_y = \mathscr{F}_{y'} = \mathscr{M}_{y'}$. Since $\mathscr{F}_y$ has non-trivial intersection with S_c, this is contrary to $c' > c$. Thus for all $y \in U \cap S_c$ we have $\mathscr{F}_y \subset \bar{B}_c$. An argument exactly the same as the above leads to a contradiction under the assumption that $\mathscr{F}_y \not\subset S_c$.

We have now shown that if $\mathscr{M}_x$ is a leaf in $\mathscr{C}$, then for every $y \in U \cap S_c$ it follows that $\mathscr{F}_y$ is contained in $\mathscr{C}$ and thus $\mathscr{M}_y = \mathscr{F}_y$ is compact. Hence $\mathscr{C}$ is open. Now if $\mathscr{N} := S_c \backslash \mathscr{C}$ had interior, then we could apply the same argument as above (e.g. $K' := F(\mathscr{N})$ has analytic dimension k, etc.). Thus we would find a compact leaf in $\mathscr{N}$ which has empty intersection with $\mathscr{S}$. But such a leaf must be in $\mathscr{C}$. Hence $\mathscr{C}$ is also dense.

Let $\mathscr{M}_x$ be an arbitrary leaf in S_c. Since $\mathscr{C}$ is dense, $\mathscr{M}_x$ is locally the limit of compact leaves on which F is constant. Thus F is constant on $\mathscr{M}_x$ and, since $\text{rank}_x F = k$, it follows that $\mathscr{M}_x$ is compact. Hence every leaf of the foliation of S_c is compact.

If a branch $\mathscr{B}$ of a fiber of F were transversal to some S_c, $c > c_0$, then $\mathscr{B}$ would have non-empty intersection with infinitely many of the compact leaves of the foliation of S_c. This can't happen, because $\text{rank}_x F = k$ for all $x \in S_c$. Therefore, for $x \in X \backslash \bar{B}_{c_0}$, the component $\mathscr{F}_x$ of the fiber of F at x and the leaf $\mathscr{M}_x$ must coincide. Furthermore, for all $x \in X$, the component $\mathscr{F}_x$ must be compact. In particular we have the Stein factorization of F, $\sigma: X \to X'$, where X' is a normal complex space and σ is a proper, surjective holomorphic map whose fibers are just the connected components of the fibers of F. Note that $X' \backslash \sigma(\bar{B}_c)$ must be a manifold, because on $X \backslash \bar{B}_c$ the fibers of σ are just the leaves of the foliations. Moreover there is an exhaustion $\varphi': X' \to \mathbf{R}^+$ of X' so that on $X \backslash B_c$ we have $\varphi = \varphi' \circ \sigma$. Thus the Levi form of φ' is positive definite on $X' \backslash \varphi(\bar{B}_c)$ and X' must be Stein. Consequently X itself is holomorphically convex, and $\sigma: X \to X'$ is just the Remmert reduction.

We have shown that if X satisfies the rank condition, then it is holomorphically convex. Conversely if X is holomorphically convex, then the fibers of

https://doi.org/10.4153/CJM-1979-083-1 Published online by Cambridge University Press

the separation map must be compact. Let $\mathscr{B}$ be a branch of such a fiber so that $\max \varphi|\mathscr{B} = c > c_0$. Then $\mathscr{B} \subset \bar{B}_c$ and $\mathscr{B} \cap S_c \supset \{p\}$. The existence of a $(k-1)$-dimensional positive eigenspace of $L(\varphi)(p)$ rules out $\mathscr{B}$ being less than k-codimensional. Thus if $\varphi(x) > c_0$ then $\operatorname{rank}_x F \geqq k$. But the general results on analytic dependency show that rank $F \leqq k$.

2. A homogeneous example. In this section we give an example of a 5-dimensional manifold X which is 3-Leviflat and which possesses only 2 independent holomorphic functions. The separation map realizes X as a principal abelian group bundle over $\mathbf{C}^* \times \mathbf{C}^*$. This bundle is not even topologically trivial. The generic fiber of the separation map is a reduced group having no non-constant analytic functions. The manifold X is homogeneous, being realizable as $\mathbf{C}^5$ modulo a certain elementary discrete group of affine transformations. Although we only give this one example, we note that there is obviously a large class of such Leviflat affine manifolds.

Let

$$G := \left\{ \begin{pmatrix} 1 & z_1 & z_3 & z_5 \\ 0 & 1 & z_2 & z_4 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; z_i \in \mathbf{C} \right\}.$$

It is obvious that G is a nilpotent complex Lie group with underlying manifold $\mathbf{C}^5$. Our example X will be G/Γ, where Γ where is a certain discrete subgroup of G. At this point we should note that a structure theorem for nilpotent homogeneous spaces (i.e. G/H, where G is a connected nilpotent complex Lie group and H is a closed subgroup) has been proved in [**3**]: Let $X = G/H$, where G is nilpotent. Then there exists a closed subgroup J in G so that $J \supset H$, $G/H \to G/J$ is the holomorphic separation map, J/H is a principal abelian group tower with $\mathscr{O}(J/H) = \mathbf{C}$, and G/J is a Stein principal abelian group tower.

Let e_i, $1 \leqq i \leqq 5$, be the element of G which has δ_{ij} in the spot for the z_j-variable. Let

$$\gamma := \begin{pmatrix} 1 & 0 & a & ia \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad \delta := \begin{pmatrix} 1 & b & i & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad a, b \in \mathbf{R}\backslash\mathbf{Q},$$

and let Γ be the discrete subgroup of G generated by the e_i's, γ and δ. If $f \in \mathscr{O}(G/\Gamma)$, then its lift to G (which we also denote by f) must be periodic under the action of Γ. The fact that f must be periodic under the action of the subgroup generated by e_3, e_5, and γ implies that f is a constant function of z_3 and z_5. Hence the periodicity with respect to e_1 and δ further shows that f is independent of z_1.

We have shown that f being periodic under the action of Γ reduces to saying that f depends only on z_2 and z_4, and that $f(z_2 + n, z_4 + m) = f(z_2, z_4)$ for all integral n and m. In particular if $X: = G/\Gamma$, then rank $\mathcal{O}(X) = 2$. The holomorphic separation map is $G/\Gamma \to G/J$, where $J = L \cdot \Gamma$ and

$$L = \left\{ \begin{pmatrix} 1 & z_1 & z_3 & z_5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \; z_j \in \mathbf{C} \right\}.$$

This realizes X as a holomorphic fiber bundle over $\mathbf{C}^* \times \mathbf{C}^*$ whose fiber is the reduced group $L/L \cap \Gamma$ which is equivalent to $\mathbf{C}^3$ modulo the lattice

$$\Gamma' = \langle (1, 0, 0), (0, 1, 0), (0, 0, 1), (0, a, ia), (b, i, 0) \rangle_{\mathbf{Z}}.$$

An easy calculation using the natural exhaustion by tube neighborhoods about the maximal compact subgroup shows that the fiber $\mathbf{C}^3/\Gamma'$ is itself 1-Leviflat and possesses no non-constant holomorphic functions.

Before constructing a 3-Leviflat exhaustion of X, we would like to point out that X and its flatness arose in a non-trivial way. In particular the bundle

$$X = G/\Gamma \to G/J = \mathbf{C}^* \times \mathbf{C}^*$$

is not even topologically trivial. To see this just note that if it were trivial, then X would be homotopic to a real torus, and thus $\pi_1(X)$ would be abelian. However Γ is a non-abelian group.

We will now construct the exhaustion φ. Let $z = (z_1, \ldots, z_5) \in G = \mathbf{C}^5$, where the group operation is given by the matrix multiplication above. The group Γ "spans" a connected real subgroup $\hat{G}$ of G so that $\Gamma \subset \hat{G}$ and $\hat{G}/\Gamma$ is compact [**12**]. If $z_j = x_j + iy_j$, then

$$z = \sum_{j=1}^{5} x_j e_j + iy_3 e_3 + iy_5 e_5 + iy_1 e_1 + iy_2 e_2 + iy_4 e_4$$

with the first three terms in $\hat{G}$, and the last three terms normal to $\hat{G}$. Define $\varphi(z_1, \ldots, z_5): = (y_1)^2 + (y_2)^2 + (y_4)^2$. It is easy to check that φ is $\hat{G}$-invariant. Hence φ pushes down to an exhaustion of the quotient $\varphi: G/\Gamma \to \mathbf{R}^+$. One should think of the pseudoballs as tubular neighborhoods of $\hat{G}/\Gamma$ in X.

The full complex Hessian of φ computed in the coordinates $(z_1, \ldots, z_5)$ of the universal cover G is always the constant diagonal matrix $d(1/2, 1/2, 0, 1/2, 0)$. Since the pseudospheres are all foliated by leaves corresponding to the space $\langle e_3, e_5 \rangle_{\mathbf{C}}$ in G, and since the rank of the full Hessian is always 3, it is clear that the rank of the Levi form is always 2. Hence the exhaustion $\varphi: X \to \mathbf{R}^+$ realizes X as a 3-Leviflat manifold.

3. Proof of theorem 2. Let $F \to E \to B$ be a holomorphic fiber bundle with entire space E, base B, and fiber F. Let X be a complex manifold and $\Omega \subset X$ a bounded domain in X with $\partial\Omega$ at least twice differentiable. Suppose that $p \in \partial\Omega$ is such that the Levi form of a defining function φ for $\partial\Omega$ in a neighborhood U of p has rank $k - 1$ at p. The main remark of this section is the following.

PROPOSITION. *Let $F \to E \to B$, X, and Ω be as above and suppose that Ω is holomorphically equivalent to E. Then either* $\operatorname{codim}_{\mathbf{C}} F \geqq k$ *or* $\operatorname{codim}_{\mathbf{C}} B \geqq k$.

Proof. We take $(z_1, \ldots, z_n)$ to be coordinates on a polydisk neighborhood U of p, where p corresponds to 0 and the function $\varphi: U \to \mathbf{R}$ defines $\partial\Omega \cap U$ (i.e. $\partial\Omega \cap U = \{\varphi = 0\}$, $\Omega \cap U = \{\varphi < 0\}$, and $d\varphi \neq 0$ on U). Since $\operatorname{rank} L(\varphi)(0) = k - 1$, we may choose the coordinates so that if $H = \{z_{k+1} = 0, \ldots, z_n = 0\}$, then $\partial\Omega \cap (H \cap U)$ is a strongly pseudoconvex hypersurface in $H \cap U$. Suppose $\operatorname{codim}_{\mathbf{C}} F < k$. For $q \in \Omega$ we may use the assumed holomorphic equivalence to speak of the *fiber* F_q *of* E *through* q. If $q \in H$, then $\dim_{\mathbf{C}}(F_q \cap H) \geqq 1$.

To simplify notation, let $U' = H \cap U$, $\Omega' = \Omega \cap U'$, φ' be the restricted defining function, and for $q \in U'$ let $F_q' = F_q \cap U'$. Let $\bar{F}_q'$ be the closure of F_q' in $\bar{\Omega}'$. Suppose that for all $q \in U'$ no point of $\{\varphi' = 0\}$ is in $\bar{F}_q'$. This will lead to a contradiction: Let $f \in \mathcal{O}(\bar{U}')$ be a peaking function at 0 satisfying $f(0) = 1$, $|f|_{\Omega'} = 1$, $|f|_{\partial U \cap \Omega'} \leqq 1/2$ ($|f|_S$ denotes the sup-norm of f on the set S.). Hence, for $q \in U'$ sufficiently near 0, $f|F_q'$ must take its maximum at an interior point of F_q'. This is the desired contradiction. So some point in $\partial\Omega$, which we might as well assume is 0, is in the closure of some F_q'.

Now let $\pi: E \to B$ be the bundle projection, $F_q = \pi^{-1}(b)$, and W be a neighborhood of b in B so that there is a polydisk Δ of the appropriate dimension and a holomorphic trivialization $\tau: F \times \Delta \to \pi^{-1}(W)$ of the bundle over W with $\tau(F \times \{0\}) = F_q$. Let $\{x_m\} \subset F$ be a sequence so that $\tau(x_m, 0) = : p_m \to p$. Let $\tau_m: \Delta \to \Omega$ be defined by $z \mapsto \tau(x_m, z)$. By Montel's Theorem, a subsequence τ_{m_k} converges compactly to a map $T: \Delta \to \partial\Omega$ with $T(0) = p$.

We will now suppose that $\operatorname{codim}_{\mathbf{C}} B < k$ and derive a contradiction. Under this assumption, the fact that $L(\varphi)(p)$ has rank $k - 1$ at p implies that $\operatorname{rank}_\xi dT < \dim_{\mathbf{C}} \Delta$ for all $\xi \in \Delta$. By taking Δ smaller if necessary we may assume that $T(\Delta) \subset U$. Letting $(\xi_1, \ldots, \xi_d)$ be the coordinates of Δ and $T = (t_1, \ldots, t_n)$, we have the jacobian matrix $(\partial t_i/\partial \xi_j)$. Let $D(T)$ be the determinant of any one of the $(d \times d)$-minors of this jacobian. It is clear that $D(T) \equiv 0$. Furthermore if x_m is any sequence in F_q' with $x_m \to x \in \partial\Omega'$, then the rank of any limiting map can not be maximal and thus any $(d \times d)$-minor must vanish identically.

Let $p' \in F_q'$. Then we can write $\tau(p', \xi) = (\tau^1(\xi), \ldots, \tau^n(\xi))$ and consider any associated function $D(p') := D(\tau)(p', 0)$ as a holomorphic function of p' in F_q'. Clearly $D(x_m) \to 0$ as $x_m \to x \in \partial\Omega'$. Now let $f \in \mathcal{O}(\bar{U}')$ be a peaking function at 0: $f(0) = 1$, $|f|_{\Omega'} = 1$, $|f|_{\partial U' \cap \Omega'} \leqq 1/2$. Let K be the closure of F_q' in $\bar{\Omega}'$ and suppose that D is not identically zero on K. Define

$$h_n := \frac{D}{|D|_K} \left(\frac{4f}{3}\right)^n.$$

Let $K' = K \cap \{f > 4/5\}$. Note that $h_n \equiv 0$ on $\partial\Omega' \cap K$. Hence

$$|h_n|_K = |h_n|_{\partial U' \cap K} \leqq (2/3)^n.$$

Then $h_n \to 0$ as $n \to \infty$. But if $x \in K'$, then $|h_n(x)| > (16/15)^n \to \infty$. Since $K' \neq \emptyset$, this is the desired contradiction. That is, we have shown that if $\operatorname{codim}_{\mathbf{C}} F < k$ then $\operatorname{codim}_{\mathbf{C}} B \geqq k$.

The proof of Theorem 2 is now quite simple: Let Ω be a relatively compact bounded domain in a k-Leviflat manifold X so that $\partial\Omega$ is at least twice differentiable. Further assume that $\varphi: X \to \mathbf{R}^+$ is the k-Leviflat exhaustion of X and that $\max\varphi|\Omega = c_1 > c_0$, where S_c is k-Leviflat for all $c > c_0$. Then in particular there exists $p \in \partial\Omega$ so that $\varphi(p) = c_1$. Thus $L(\psi)(p)$ has at least a $(k-1)$-dimensional positive eigenspace, where ψ is a defining function for $\partial\Omega$ near p. The proof is now immediate from the above proposition.

Remark 1. In the known examples the assumption that Ω is big enough to get into the k-Leviflat range of φ seems to always be satisfied, because S_c is k-Leviflat for all $c > 0$ and $\{\varphi = 0\}$ has no interior.

Remark 2. In the next section we note how to construct a number of k-Leviflat manifolds which contain holomorphically fibered bounded domains. But in fact both inequalities of Theorem 2 are violated.

4. Some examples. In this section we want to point out that there are many possibilities for leaves in the k-Leviflat hypersurfaces. In particular it is quite possible to have hyperbolic manifolds (see [**11**] for generalities) as leaves. Thus any argument using Liouville's Theorem on the leaves will not work in general. Our examples will arise as foliated bundle spaces. This type of example is well-known in the differentiable setting.

Let X be a compact complex manifold, S^1 be the circle group, and let $\rho: \pi_1(X) \to S^1$ be a representation. If $\tilde{X}$ is the universal cover of X, then one can construct a line bundle on X as follows:

$$\begin{array}{ccc} \tilde{X} \times \mathbf{C} & \rightarrow & L := \tilde{X} \times \mathbf{C}/\sim \\ \tilde{\pi}\downarrow & & \downarrow \pi \\ \tilde{X} & \longrightarrow & X \end{array}$$

where $\tilde{\pi}$ is the standard projection, $(\tilde{x}, z) \sim (\tilde{y}, \xi)$ whenever $\tilde{y} = \gamma(\tilde{x})$ and $\xi = \rho(\gamma^{-1})z$ for some $\gamma \in \pi_1(X)$, and π is the induced map. It is easy to check that $\pi: L \to X$ is a holomorphic line bundle which is trivial over a cover $\{U_i\}$ so that the transition functions $\{f_{ij}\}$ are constant with modulus 1. Thus the trivial functions $h_i: \equiv 1 \in C^\infty(U_i)$ satisfy $h_i = |f_{ij}|^{-2}h_j$ on $U_i \cap U_j$, and consequently they yield a hermitian metric on L.

Let $\varphi: L \to \mathbf{R}^+$ be the exhaustion of L given by this metric (i.e. $\varphi(p) = \log\|p\|^2$). Since up to appropriate constants the Levi form of the exhaustion and the Chern form $c(L)$ of the bundle relative to this metric are just $\partial\bar{\partial}\log(h_i)$, it is clear that $c(L) = 0$ and that the exhaustion realizes L as a 1-Leviflat complex manifold.

Let $S := \{\varphi = 1\}$ be the unit pseudosphere in L. Then S is foliated by complex manifolds which have real codimension 1 in S. Let $\mathscr{M}$ be a leaf of this foliation. Then $\pi|\mathscr{M}: \mathscr{M} \to X$ realizes $\mathscr{M}$ as a covering space of X. Let λ be a closed loop in $\mathscr{M}$ (with base point x_0). Then $\gamma := \pi(\lambda)$ is a closed loop in X such that $\rho(\gamma) = 1 \in S^1$ (This follows directly from the definition of the bundle, because λ begins and ends at x_0.). So in particular if $\pi_1(X)/\ker(\rho)$ is non-trivial, then $\mathscr{M}$ is a non-trivial cover of X. For example, if $b_1(X) \neq 0$, then one can always represent $H_1(X, \mathbf{Z})$ faithfully (i.e. $\ker(\rho) = [\pi_1 : \pi_1]$) and thus the cover $\mathscr{M} \to X$ is infinite, implying the non-compactness of $\mathscr{M}$.

If $\pi_1(X)$ is abelian, then the above observation shows that one can construct L so that the leaf $\mathscr{M}$ is the universal cover $\tilde{X}$ of X. Now if A is an abelian variety of dimension at least 3 in some projective space and X is a complete intersection in A given by hyperplanes with $\dim_{\mathbf{C}} X \geqq 2$, then an application of the Lefschetz Theorem shows that $\pi_1(X)$ is a free abelian group. Hence $\mathscr{M} = \tilde{X}$ is non-compact. Furthermore, if A is simple (so that X can't be a torus), then recent work of Green [**6**] proves that X is hyperbolic.

As a closing remark we note that if $\pi_1(X)$ is nilpotent, then one can construct a non-trivial tower of line bundle spaces (corresponding to the central series of $\pi_1(X)$) which is k-Leviflat.

References

1. K. Diederich and J. E. Fornaess, *Pseudoconvex domains: Bounded strictly plurisubharmonic exhaustion functions*, Inv. Math. *39* (2) (1977), 129–141.

2. M. Freeman, *Local complex foliations of real submanifolds*, Math. Ann. *209* (1974), 1–30.

3. B. Gilligan and A. Huckleberry, *On non-compact complex nil-manifolds*, (to appear).

4. H. Grauert, *On Levi's problem and the imbedding of real-analytic manifolds*, Ann. Math. *68* (1958), 460–472.

5. ——— *Die Bedeutung des Levischen Problems für die analytische und algebraische Geometrie*, Pro. Int. Cong. Math. Stockholm (1962), 86–101.

6. M. Green, *Holomorphic maps into complex tori*, (to appear in Amer. J. of Math.)

7. A. Huckleberry, *The Levi problem on pseudoconvex manifolds which are not strongly pseudoconvex*, Math. Ann. *219* (1976), 127–137.

8. ——— *Holomorphic fibrations of bounded domains*, Math. Ann. *227* (1977), 61–66.

9. A. Huckleberry and R. Nirenberg, *On k-pseudoflat complex spaces*, Math. Ann. *200* (1973), 1–10.

10. H. Kazama, *On pseudoconvexity of complex abelian Lie groups*, J. Math. Soc. Japan, *25* (2) (1973), 329–333.

11. S. Kobayashi, *Hyperbolic manifolds and holomorphic mappings* (Marcel Dekker, Inc., New York, 1970).

12. Y. Matsushima, *On the discrete subgroups and homogeneous spaces of nilpotent Lie groups*, Nagoya Math. J. *2* (1951), 95–110.

13. A. Morimoto, *Non-compact complex Lie groups without non-constant holomorphic functions*, Proceedings of the Conference on Complex Analysis, Minneapolis (1964), 256–272.

14. K. Stein, *Analytische Zerlegungen komplexer Räume*, Math. Ann. *132* (1956), 63–93.

University of Regina,
Regina, Saskatchewan;
University of Notre Dame,
Notre Dame, Indiana