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On a conjecture of M. R. Murty and V. K. Murty

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Abstract. Let $\omega^*(n)$ be the number of primes p such that $p-1$ divides n. Recently, M. R. Murty and V. K. Murty proved that

$$x(\log\log x)^3 \ll \sum_{n\le x}\omega^*(n)^2 \ll x\log x.$$

They further conjectured that there is some positive constant C such that

$$\sum_{n\le x}\omega^*(n)^2 \sim Cx\log x,$$

as $x\to\infty$. In this short note, we give the correct order of the sum by showing that

$$\sum_{n\le x}\omega^*(n)^2 \asymp x\log x.$$

Let $\omega(n)$ be the number of distinct prime divisors of n. In about 100 years ago, Hardy and Ramanujan [3] found out that $\omega(n)$ has normal order $\log\log n$, which means that for almost all integers n we have $\omega(n)\sim\log\log n$. Later, Turán [6] provided a quite elegantly simplified proof by establishing

$$\sum_{n\le x}(\omega(n)-\log\log n)^2 \ll x\log\log x.$$

In 1955, Prachar [5] considered a variant arithmetic function of ω. Let $\omega^*(n)$ be the number of primes p such that $p-1$ divides n. Prachar proved that

$$\sum_{n\le x}\omega^*(n) = x\log\log x + Bx + O(x/\log x)$$

and

$$\sum_{n\le x}\omega^*(n)^2 = O\left(x(\log x)^2\right),$$

where B is a constant. Motivated by Prachar's work, Erdős and Prachar [2] proved that the number of pairs of primes p and q so that the least common multiple

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$[p-1,q-1]\le x$ is bounded by $O(x\log\log x)$. Following a remark of Erdős and Prachar, M. R. Murty and V. K. Murty [4] improved this to $O(x)$. By this improvement, they reached the nice bounds

$$x(\log\log x)^3 \ll \sum_{n\le x}\omega^*(n)^2 \ll x\log x.$$

With these in hands, M. R. Murty and V. K. Murty conjectured that there is some positive constant C such that

$$\sum_{n\le x}\omega^*(n)^2 \sim Cx\log x,$$

as $x\to\infty$. In this note, the author shall give a slight improvement of the result due to M. R. Murty and V. K. Murty toward the correct direction of their conjecture.

Theorem 1 *There are two absolute constants a_1 and a_2 such that*

$$a_1 x\log x \le \sum_{n\le x}\omega^*(n)^2 \le a_2 x\log x.$$

Proof We only need to prove the lower bound as the upper bound is displayed by M. R. Murty and V. K. Murty. Throughout our proof, the number x is sufficiently large. From the paper of M. R. Murty and V. K. Murty [4, equation (4.10)], we have

$$\sum_{n\le x}\omega^*(n)^2 = x\sum_{d\le x}\varphi(d)\left(\sum_{\substack{p\le x\\ p\equiv 1 \pmod d}}\frac{1}{p-1}\right)^2 + O(x). \tag{1}$$

Integrating by parts gives

$$\sum_{\substack{p\le x\\ p\equiv 1 \pmod d}}\frac{1}{p} = \frac{\pi(x;d,1)}{x} + \int_2^x \frac{\pi(t;d,1)}{t^2}dt \ge \int_{x^{3/4}}^x \frac{\pi(t;d,1)}{t^2}dt, \tag{2}$$

where $\pi(t;d,1)$ is the number of primes $p\equiv 1 \pmod d$ up to t. Thus, from equations (1) and (2), we obtain

$$\sum_{n\le x}\omega^*(n)^2 \ge x\sum_{d\le x^{1/3}}\varphi(d)\left(\int_{x^{3/4}}^x \frac{\pi(t;d,1)}{t^2}dt\right)^2 + O(x). \tag{3}$$

For any integer $0\le j\le \left\lfloor \frac{\log x}{13\log 2}\right\rfloor$, let $Q_j = 2^j x^{1/4}$. Then $Q_j < x^{1/3}$ for all integers j. From a weak form of the Bombieri–Vinogradov theorem (see, for example, [1]), we have

$$\sum_{Q_j<d\le 2Q_j}\max_{y\le z}\left|\pi(y;d,1)-\frac{\mathrm{li}\, y}{\varphi(d)}\right| \ll \frac{z}{(\log z)^5},$$

for any $0\le j\le \left\lfloor \frac{\log x}{13\log 2}\right\rfloor$ and $x^{3/4}\le z\le x$, where the implied constant is absolute. It follows immediately that

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$$\max_{y\le z}\left|\pi(y;d,1)-\frac{\operatorname{li} y}{\varphi(d)}\right|<\frac{\operatorname{li} z}{\varphi(d)\log z} \tag{4}$$

hold for all $Q_j < d \le 2Q_j$ but at most $O\left(Q_j/(\log x)^2\right)$ exceptions. From equation (4), we have

$$\pi(y;d,1)>\frac{\operatorname{li} y}{2\varphi(d)}\quad (z/2<y\le z)$$

for all $Q_j < d \le 2Q_j$ with at most $O\left(Q_j/(\log x)^2\right)$ exceptions. A little thought with the dichotomy of z between the interval $[x^{3/4}, x]$ leads to the fact

$$\pi(y;d,1)>\frac{\operatorname{li} y}{2\varphi(d)}>\frac{y}{3\varphi(d)\log y}\quad \left(\forall\, x^{3/4}\le y\le x\right) \tag{5}$$

for all $Q_j < d \le 2Q_j$ except for $O\left(Q_j/\log x\right)$ exceptions. For any integer $0 \le j \le \left\lfloor \frac{\log x}{13\log 2}\right\rfloor$, let S_j be the set of all integers $Q_j < d \le 2Q_j$ such that equation (5) holds. Thus, from the analysis above and equations (3) and (5), we conclude that

$$\begin{aligned}\sum_{n\le x}\omega^*(n)^2 &\ge \frac{x}{9}\sum_{0\le j\le\left\lfloor\frac{\log x}{13\log 2}\right\rfloor}\sum_{d\in S_j}\varphi(d)\left(\int_{x^{3/4}}^{x}\frac{1}{\varphi(d)t\log t}dt\right)^2+O(x)\\ &\gg x\sum_{0\le j\le\left\lfloor\frac{\log x}{13\log 2}\right\rfloor}\sum_{d\in S_j}\frac{1}{\varphi(d)}+x\gg x\log x.\end{aligned}$$

■

It is worth here mentioning that we have the following corollary:

$$\sum_{p,q\le x}\frac{1}{[p-1,q-1]}\asymp\log x$$

due to (see [4, p. 6, last line])

$$\sum_{n\le x}\omega^*(n)^2=\sum_{p,q\le x}\frac{x}{[p-1,q-1]}+O(x).$$

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