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Polystable Parabolic Principal G-Bundles and Hermitian–Einstein Connections

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Abstract. We show that there is a bijective correspondence between the polystable parabolic principal G-bundles and solutions of the Hermitian–Einstein equation.

1 Introduction

Parabolic vector bundles on curves were introduced by C. S. Seshadri [14]. Parabolic vector bundles on higher dimensional varieties were introduced by M. Maruyama and K. Yokogawa in [11]. The principal bundle analog of parabolic bundles was defined in [2]. Ramified principal bundles were defined in [3], where it was shown that the ramified principal G-bundles on a curve are in bijective correspondence with the parabolic principal G-bundles. The case of higher dimensions was treated in [8]; the details of this correspondence are recalled in Section 2. In [8, 9], connections on ramified principal bundles were investigated.

J. Li established a Hitchin–Kobayashi correspondence between polystable parabolic vector bundles on Kähler manifolds and parabolic vector bundles satisfying the Hermitian–Einstein equation (for parabolic vector bundles over Kähler surfaces see [10]; this was done earlier by O. Biquard [5]). Our aim here is to extend this Hitchin–Kobayashi correspondence to parabolic principal bundles (as mentioned above, they are same as ramified principal bundles).

Let X be a connected complex projective manifold and $D \subset X$ a simple normal crossing divisor. The smooth locus of D will be denoted by D^{sm}. Let $D_1, \dots, D_\ell$ be the irreducible components of D (so each D_i is a smooth divisor). Fix a Hermitian structure of the line bundle $\mathcal{O}_X(D_i)$ such that the pointwise norm of the section of $\mathcal{O}_X(D_i)$ given by the constant function 1 is strictly bounded by 1 (this is possible because X is compact). Let f_i be the continuous function on X given by the norm of this section; it is smooth outside D_i.

Fix a Kähler form ω on X. For any real number $\alpha \in (0, 2)$, let

$$\omega_\alpha := \frac{2\sqrt{-1}}{2-\alpha}\sum_{i=1}^{\ell} \partial\overline{\partial} f_i^{2-\alpha} + C_\alpha \cdot \omega$$

be the Kähler form on X, where the f_i are constructed above and C_α is a sufficiently large positive real number such that ω_α is positive. The significance of this Kähler form is explained in [10, Proposition 4.1].

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Let G be a connected reductive linear algebraic group defined over $\mathbb{C}$. Let $\psi\colon E_G \to X$ be a ramified principal G-bundle. For each point $x \in D^{\mathrm{sm}}$, let n_x be the order of the isotropy group of any point $z \in \psi^{-1}(x) \subset E_G$ for the natural action of G on E_G (it is independent of the choice of z in the fiber over x). Define $\delta := \mathrm{l.c.m.}\{n_x\}_{x\in D^{\mathrm{sm}}}$.

Let $E_K \subset E_G$ be a reduction of structure group to a maximal compact subgroup $K \subset G$. There is a unique complex connection on E_G that preserves E_K (Lemma 4.1); this connection will be denoted by ∇. Let ∇' be the connection on the principal G-bundle $E'_G := E_G|_{X\setminus D}$. The curvature of ∇' will be denoted by $K(\nabla')$. We have

$$\Lambda_{\omega_\alpha} K(\nabla') \in C^\infty(X \setminus D, \mathrm{ad}(E'_G)),$$

where Λ_{ω_α} is the adjoint of multiplication by the Kähler form ω_α, and $\mathrm{ad}(E'_G)$ is the adjoint vector bundle. The reduction $E_K \subset E_G$ is called Hermitian–Einstein if the above section $\Lambda_{\omega_\alpha} K(\nabla')$ corresponds to some element in the center of the Lie algebra of G.

We prove the following theorem.

Theorem 1.1 *Any polystable ramified principal G-bundle E_G admits a Hermitian structure satisfying the Hermitian–Einstein equation for all $\alpha \in (2(1-\delta), 2)$ (the number δ is defined above).*

If a ramified principal G-bundle E_G over X admits a Hermitian structure satisfying the Hermitian–Einstein equation for some $\alpha \in (2(1-\delta), 2)$, then E_G is polystable.

If all the parabolic Chern classes of E_G vanish, then Theorem 1.1 follows from [8, Theorem 5.2]. For $G = \mathrm{GL}_n(\mathbb{C})$, Theorem 1.1 was proved in [10].

2 Preliminaries

Let X be a connected complex projective manifold. Fix a simple normal crossing divisor $D \subset X$. So D is reduced and effective, each irreducible component of D is smooth, and the irreducible components of D intersect transversely. Let G be a linear algebraic group defined over $\mathbb{C}$.

Let $\psi\colon E_G \to X$ be a ramified principal G-bundle with ramification over D (see [3, 8, 9] for the definition). We briefly recall the defining properties. The total space E_G is a smooth complex variety equipped with an algebraic right action of G

$$f\colon E_G \times G \to E_G, \tag{2.1}$$

and the following conditions hold:

- $\psi \circ f = \psi \circ p_1$, where p_1 is the natural projection of $E_G \times G$ to E_G,
- for each point $x \in X$, the action of G on the reduced fiber $\psi^{-1}(x)_{\mathrm{red}}$ is transitive,
- the restriction of ψ to $\psi^{-1}(X \setminus D)$ makes $\psi^{-1}(X \setminus D)$ a principal G-bundle over $X \setminus D$,
- for each irreducible component $D_i \subset D$, the reduced inverse image $\psi^{-1}(D_i)_{\mathrm{red}}$ is a smooth divisor and

  $$\widehat{D} := \sum_{i=1}^{\ell} \psi^{-1}(D_i)_{\mathrm{red}}$$

  is a normal crossing divisor on E_G,

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- for any smooth point x of D, and any point $z \in \psi^{-1}(x)$, the isotropy group $G_z \subset G$, for the action of G on E_G, is a finite cyclic group that acts faithfully on the quotient line $T_z E_G / T_z \psi^{-1}(D)_{\text{red}}$.

Parabolic principal G-bundles were defined in [2]. We recall that a parabolic principal G-bundle on X is a functor from the category of rational G-representations to the category of parabolic vector bundles on X satisfying certain conditions. The conditions in question say that the functor is compatible with standard operations like direct sum, tensor product, taking dual, etc. (see [2, 7] for the details). There is a natural bijective correspondence between the ramified principal G-bundles with ramification over D and the parabolic G-bundles with D as the parabolic divisor (see [3,8]); in [3] this correspondence was established under the assumption that the base is a curve, but in [8], this assumption is removed. We recall below this correspondence.

Let $\operatorname{Rep}(G)$ be the category of finite dimensional complex G-modules. Let $\psi\colon E_G \to X$ be a ramified principal G-bundle with ramification over D. Take any finite-dimensional complex G-module V_0. Recall that

$$E_G^0 := \psi^{-1}(X \setminus D) \longrightarrow X \setminus D$$

is a usual principal G-bundle. Let $E_V^0 := E_G^0(V) \to X \setminus D$ be the associated vector bundle. This vector bundle has a natural extension to X as a parabolic vector bundle (its construction is similar to the construction of a parabolic vector bundle from an orbifold vector bundle; see [6]). Therefore, we get a functor from $\operatorname{Rep}(G)$ to the category of parabolic vector bundles over X with parabolic structure over D. The parabolic principal G-bundle corresponding to E_G is defined by this functor.

We will give an alternative description of the correspondence.

There is a natural bijective correspondence between parabolic vector bundles and orbifold vector bundles [6]. Let $\mathcal{E}_G$ be a parabolic principal G-bundle over X given by a functor $\mathcal{F}$ from $\operatorname{Rep}(G)$ to the parabolic vector bundles over X with D as the parabolic divisor. Using the above mentioned bijection between parabolic vector bundles and orbifold vector bundles, there is a finite (ramified) Galois covering

$$\eta\colon Y \longrightarrow X$$

such that the functor $\mathcal{F}$ defines a functor from $\operatorname{Rep}(G)$ to the category of orbifold vector bundles over Y. Such a functor gives a principal G-bundle $F_G \to Y$ equipped with a lift of the action of the Galois group $\operatorname{Gal}(\eta)$ on Y [12, 13]. The quotient $F_G/\operatorname{Gal}(\eta)$ is a ramified principal G-bundle over $X = Y/\operatorname{Gal}(\eta)$.

Conversely, if $F_G \to X$ is a ramified principal G-bundle, then there is a finite (ramified) Galois covering $\eta\colon Y \to X$ such that the normalizer $\widetilde{F_G \times_X Y}$ of the fiber product $F_G \times_X Y$ is smooth. The projection $\widetilde{F_G \times_X Y} \to Y$ is a principal G-bundle equipped with an action of $\operatorname{Gal}(\eta)$. Let $\mathcal{F}_0$ be the functor from $\operatorname{Rep}(G)$ to the category of orbifold vector bundles over Y that sends any G-module V_0 to the associated vector bundle $\widetilde{F_G \times_X Y}(V_0)$. But an orbifold vector bundle over Y gives a parabolic vector bundle over X [6]. Therefore, the functor $\mathcal{F}_0$ gives a functor from $\operatorname{Rep}(G)$ to the category of parabolic vector bundles over X. This functor defines the parabolic principal G-bundle corresponding to the ramified principal G-bundle F_G.

Let $\psi\colon E_G \to X$ be a ramified principal G-bundle with ramification over D. Let $H \subset G$ be a Zariski closed subgroup. Let $U \subset X$ be a Zariski open subset. The inverse image $\psi^{-1}(U)$ will also be denoted by $E_G|_U$.

A *reduction of structure group of* E_G *to* H *over* U is a subvariety $E_H \subset E_G|_U$ satisfying the following conditions:

(i) the action of H on E_G preserves E_H, and for each point $x \in U$, the action of H on $\psi^{-1}(x) \cap E_H$ is transitive,

(ii) for each point $z \in E_H$, the isotropy group of z for the action of G on E_G is contained in H.

Note that any E_H satisfying the above conditions is a ramified principal H-bundle over X with ramification over D.

3 Polystable Ramified Principal G-Bundles

In this section we assume the group G to be connected and reductive. We also fix a polarization on X in order to define the parabolic degree of a parabolic vector bundle (see [11] for parabolic degree).

Let P be a parabolic subgroup of the reductive group G. Therefore, G/P is a complete variety. A *Levi subgroup* of P is a maximal connected reductive subgroup of P; any two Levi subgroups of P are conjugate. For any character λ of P, let $L_\lambda \to G/P$ be the associated line bundle. So L_λ is a quotient of $G \times \mathbb{C}$, where two points (z_1, c_1) and (z_2, c_2) of $G \times \mathbb{C}$ are identified if there is an element $g \in P$ such that $z_2 = z_1 g$ and $c_2 = \lambda(g)^{-1} c_1$. Let $Z_0(G) \subset G$ be the connected component of the center of G containing the identity element. It is known that $Z_0(G) \subset P$. A character λ of P which is trivial on $Z_0(G)$ is called *strictly antidominant* if the corresponding line bundle L_λ over G/P is ample.

Let E_G be a ramified principal G-bundle over X with ramification over D. Consider quadruples of the form (H, λ, U, E_H), where

- $H \subset G$ is a proper parabolic subgroup,
- λ is a strictly antidominant character of H,
- $U \subset X$ is a nonempty Zariski open subset such that the codimension of the complement $X \setminus U$ is at least two,
- $E_H \subset E_G$ is a reduction of structure group of E_G to H over U.

Let $E_H(\lambda) \to X$ be the ramified principal $\mathbb{C}^*$-bundle obtained by extending the structure group of E_H using the character λ. This ramified principal $\mathbb{C}^*$-bundle defines a parabolic line bundle over X with parabolic structure over D (see [9, p. 179]). Let $E_H(\lambda)_*$ be the parabolic line bundle corresponding to $E_H(\lambda)$.

The ramified principal G-bundle E_G is stable (respectively, semistable) if and only if for every quadruple (H, λ, U, E_H) of the above type, par-deg$(E_H(\lambda)_*) > 0$ (respectively, par-deg$(E_H(\lambda)_*) \geq 0$).

Let E_G be a ramified principal G-bundle over X. A reduction of structure group

$$E_H \subset E_G$$

to some parabolic subgroup $H \subset G$ over X is called *admissible* if for each character λ of H trivial on $Z_0(G)$, the associated parabolic line bundle $E_H(\lambda)_*$ over X satisfies the following condition:

$$\text{par-deg}(E_H(\lambda)_*) = 0.$$

A ramified principal G-bundle E_G over X is called *polystable* if either E_G is stable or there is a proper parabolic subgroup H and a reduction of structure group

$$E_{L(H)} \subset E_G$$

to a Levi subgroup $L(H)$ of H over X such that the following conditions hold:

- the ramified principal $L(H)$-bundle $E_{L(H)}$ is stable,
- the reduction of structure group of E_G to H, obtained by extending the structure group of $E_{L(H)}$ using the inclusion of $L(H)$ in H, is admissible.

(See [3, 8, 9].)

The bijective correspondence between the parabolic principal G-bundles and the ramified principal G-bundles preserves polystability.

4 Hermitian–Einstein Connection on Ramified Principal G-Bundles

4.1 Connections on a Ramified Principal G-Bundle

Let

$$\psi\colon E_G \to X \tag{4.1}$$

be a ramified principal G-bundle with ramification over D, where G is a linear algebraic group defined over $\mathbb{C}$.

Let $\mathcal{K} \subset TE_G$ be the algebraic subbundle defined by the action of G (see (2.1)). So $\mathcal{K}$ is the tangent bundle of the orbits. Let

$$\mathcal{Q} := TE_G/\mathcal{K} \tag{4.2}$$

be the quotient bundle. Tensoring the obvious short exact sequence

$$0 \longrightarrow \mathcal{K} \longrightarrow TE_G \longrightarrow \mathcal{Q} \longrightarrow 0$$

with $\mathcal{Q}^*$, we get the short exact sequence

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{Q}^* \longrightarrow TE_G \otimes \mathcal{Q}^* \xrightarrow{q_0} \mathcal{Q} \otimes \mathcal{Q}^* \longrightarrow 0 \tag{4.3}$$

over E_G. Let $\mathcal{O}_{E_G} \hookrightarrow \mathcal{Q} \otimes \mathcal{Q}^*$ be the homomorphism that sends any function g to $g \cdot \mathrm{Id}_{\mathcal{Q}}$. Define

$$\mathcal{V}_{E_G} := q_0^{-1}(\mathcal{O}_{E_G}),$$

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where q_0 is the projection in (4.3). So we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{K} \otimes \mathcal{Q}^* \longrightarrow \mathcal{V}_{E_G} \xrightarrow{q_0} \mathcal{O}_{E_G} \longrightarrow 0 \tag{4.4}$$

over E_G obtained from (4.3).

We note that the action of G on E_G has natural lift to all three vector bundles in the exact sequence in (4.4), and all the homomorphisms there commute with the actions of G. Therefore, the direct image on X of any of the vector bundles in (4.4) is equipped with an action of G. Define the holomorphic vector bundles

$$\mathcal{A}_{E_G} := (\psi_*(\mathcal{K} \otimes \mathcal{Q}^*))^G \longrightarrow X \quad \text{and} \quad \mathcal{B}_{E_G} := (\psi_* \mathcal{V}_{E_G})^G \longrightarrow X,$$

where ψ is the projection in (4.1). (By W^G, where W is any sheaf on X equipped with an action of G, we mean the G-invariant part of W.) From (4.4) we have the short exact sequence of holomorphic vector bundles

$$0 \longrightarrow \mathcal{A}_{E_G} \longrightarrow \mathcal{B}_{E_G} \longrightarrow \mathcal{O}_X \longrightarrow 0 \tag{4.5}$$

over X.

A *complex connection on* E_G is defined to be a C^∞ splitting of the short exact sequence in (4.5) (see [9, Definition 4.3]). See [8] for an alternative definition of connection; that the two definitions are equivalent is proved in [9, Theorem 4.4].

4.2 Hermitian Structure on a Ramified Principal G-Bundle

Henceforth, we will always assume that the group G is connected and reductive. Fix a maximal compact subgroup

$$K \subset G. \tag{4.6}$$

Let $\psi\colon E_G \to X$ be a ramified principal G-bundle with ramification over D. A Hermitian structure on E_G is a C^∞ reduction of structure group of E_G to the subgroup K in (4.6). More precisely, a *Hermitian structure on* E_G is a C^∞ submanifold $E_K \subset E_G$ satisfying the following conditions:

(i) the action of K on E_G preserves E_K,
(ii) for each point $x \in X$, the action of K on $\psi^{-1}(x) \cap E_K$ is transitive,
(iii) for each point $z \in E_K$, the isotropy group of z, for the action of G on E_G, is contained in K.

Compare the above definition with the definition in Section 2. We note that the third condition in the above definition holds if for each point $x \in U$, there exists a point $z \in \psi^{-1}(x) \cap E_H$ such that $\Gamma_z \subset H$, where $\Gamma_z \subset G$ is the isotropy group of z for the action of G on E_G.

Let $E_K \subset E_G$ be a Hermitian structure, and let ∇ be a complex connection on E_G. Let

$$\nabla'\colon \mathcal{Q} \longrightarrow TE_G \tag{4.7}$$

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be the C^∞ homomorphism associated with ∇ (see (4.2) for $\mathcal{Q}$). The connection ∇ is said to *preserve* E_K if the image $\nabla'(\mathcal{Q})|_{E_K}$ is contained in $T^{\mathbb{C}}E_K = (T^{\mathbb{R}}E_K) \otimes_{\mathbb{R}} \mathbb{C}$, where ∇' is the homomorphism in (4.7).

Lemma 4.1 *Let $E_K \subset E_G$ be a Hermitian structure on a ramified principal G-bundle $E_G \to X$. Then there is a unique complex connection on E_G that preserves E_K.*

Proof Consider the principal G-bundle $E'_G := E_G|_{X\setminus D} \to X \setminus D$. There is a unique complex connection on E'_G that preserves the Hermitian structure $E_K|_{X\setminus D} \subset E'_G$ (it is known as the *Chern connection*). Therefore, E_G can have at most one complex connection preserving E_K.

To prove that there is complex connection preserving E_K, we first recall that there is a ramified finite Galois covering

$$\phi\colon Y \longrightarrow X \tag{4.8}$$

and an algebraic principal G-bundle

$$F_G \longrightarrow Y \tag{4.9}$$

such that the action of the Galois group $\Gamma := \mathrm{Gal}(\phi)$ on Y lifts to an action of G on F_G that commutes with the action of G on F_G, and $E_G = F_G/\Gamma$. To explain this, we recall that in [2] it was shown that for any parabolic G-bundle E_* over X, there is a covering Y and a Γ-linearized principal G-bundle F_G on Y that gives E_*; also, any ramified principal G-bundle corresponds to a parabolic G-bundle. Combining these, it follows that there is a pair (Y, F_G) satisfying the above conditions.

Let $q\colon F_G \to F_G/\Gamma = E_G$ be the quotient map. Let $\widetilde{E}_K := q^{-1}(E_K) \subset F_G$ be the inverse image of E_K. Clearly, $\widetilde{E}_K$ is a C^∞ reduction of the structure group of F_G to K. Therefore, there is a unique complex connection $\widetilde{\nabla}$ on F_G that preserves $\widetilde{E}_K$.

The action of Γ on F_G clearly preserves $\widetilde{E}_K$. Therefore, the connection $\widetilde{\nabla}$ is preserved by the action of Γ on F_G. This immediately implies that the connection $\widetilde{\nabla}$ defines a complex connection on the ramified principal G-bundle E_G. This connection on E_G given by $\widetilde{\nabla}$ preserves E_K because $\widetilde{\nabla}$ preserves $\widetilde{E}_K$. ■

4.3 Hermitian–Einstein Equation

Let $D = \sum_{i=1}^{\ell} D_i$ be the decomposition of the divisor D into irreducible components. For each $i \in [1, \ell]$, fix a Hermitian structure on the holomorphic line bundle $\mathcal{O}_X(D_i)$. Let f_i be the continuous function on X given by the norm of the holomorphic section of $\mathcal{O}_X(D_i)$ defined by the constant function 1. So f_i is C^∞ and nowhere vanishing on the complement $X \setminus D_i$, and it vanishes on D_i. Note the $f_i(z)$ can be taken to be the distance of z from D_i with respect to some Kähler metric on X of diameter less than one.

Fix a Kähler form ω on X. For any real number $\alpha \in (0, 2)$, let

$$\omega_\alpha := \frac{2\sqrt{-1}}{2-\alpha} \sum_{i=1}^{\ell} \partial\overline{\partial} f_i^{2-\alpha} + C_\alpha \cdot \omega \tag{4.10}$$

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be the Kähler form on X, where C_α is a sufficiently large positive real number such that ω_α is positive. (See [10, p. 451] for the details.)

Let $E_K \subset E_G$ be a Hermitian structure on E_G. Let ∇ be the unique connection on E_G that preserves E_K (see Lemma 4.1). Let ∇' be the connection on the principal G-bundle $E'_G := E_G|_{X\setminus D}$. The curvature of ∇' will be denoted by $K(\nabla')$. Let Λ_{ω_α} be the adjoint of multiplication by the Kähler form ω_α in (4.10). So

$$\Lambda_{\omega_\alpha} K(\nabla') \in C^\infty(X \setminus D, \mathrm{ad}(E'_G)), \tag{4.11}$$

where $\mathrm{ad}(E'_G) \to X \setminus D$ is the adjoint vector bundle. We recall that $\mathrm{ad}(E'_G)$ is the vector bundle associated with the principal G-bundle E'_G for the adjoint action of G on the Lie algebra $\mathfrak{g} = \mathrm{Lie}(G)$. More precisely, $\mathrm{ad}(E'_G)$ is a quotient of $E'_G \times \mathfrak{g}$, and two points (z_1, v_1) and (z_2, v_2) of $E'_G \times \mathfrak{g}$ are identified in $\mathrm{ad}(E'_G)$ if there is an element $g \in G$ such that $z_2 = z_1 g$ and $v_2 = \mathrm{Ad}(g^{-1})(v_1)$.

Let $\mathfrak{z}(\mathfrak{g}) \subset \mathfrak{g}$ be the center. Since the adjoint action of G on $\mathfrak{g}$ fixes $\mathfrak{z}(\mathfrak{g})$ pointwise, each element of $\mathfrak{z}(\mathfrak{g})$ defines a smooth section of the vector bundle $\mathrm{ad}(E'_G)$. So we have

$$\mathfrak{z}(\mathfrak{g}) \subset C^\infty(X \setminus D, \mathrm{ad}(E'_G)). \tag{4.12}$$

The connection ∇ on E_G is called *Hermitian–Einstein* if there is an element $v_0 \in \mathfrak{z}(\mathfrak{g})$ such that $\Lambda_{\omega_\alpha} K(\nabla') = v_0$ (see (4.11) and (4.12)).

For parabolic vector bundles the above definition of a Hermitian–Einstein connection coincides with the one in [10] (see [10, Definition 6.1]).

5 Hermitian–Einstein Connection and Polystable Ramified Principal G-Bundles

5.1 Tensor Product and Semistability

A nonempty Zariski open subset U of a variety Z will be called *big* if the codimension of the complement $Z \setminus U$ is at least two.

Let $U \subset X$ be a big Zariski open subset. Let $E_{\mathrm{GL}_m(\mathbb{C})} \to U$ and $F_{\mathrm{GL}_n(\mathbb{C})} \to U$ be a ramified principal $\mathrm{GL}_m(\mathbb{C})$-bundle and $\mathrm{GL}_n(\mathbb{C})$-bundle, respectively. Let E_* and F_* be the parabolic vector bundles over U associated with $E_{\mathrm{GL}_m(\mathbb{C})}$ and $F_{\mathrm{GL}_n(\mathbb{C})}$, respectively for the standard representation (see [2] for parabolic vector bundles associated to parabolic principal bundles). So the ranks of E_* and F_* are m and n, respectively. Consider the parabolic tensor product $E_* \otimes F_*$ (see [2] for tensor product of parabolic vector bundles).

Lemma 5.1 *If both $E_{\mathrm{GL}_m(\mathbb{C})}$ and $F_{\mathrm{GL}_n(\mathbb{C})}$ are polystable, then the parabolic vector bundle $E_* \otimes F_*$ over U is also polystable.*

Proof Recall the correspondence between ramified principal bundles and Γ-linearized principal bundles for a suitable Γ (see the proof of Lemma 4.1). Also recall that ramified principal G-bundles are identified with the parabolic G-bundles. A parabolic G-bundle defined over a big open subset is polystable if and only if the corresponding principal G-bundle over the covering is polystable [2, Theorem 4.3].

Let Y be a complex projective manifold with a Kähler form such that the corresponding class in $H^2(Y,\mathbb{R})$ lies in $H^2(Y,\mathbb{Q})$. Let $\iota\colon U_0 \hookrightarrow Y$ be a big Zariski open subset, and let $V_i \to U_0$, $i = 1, 2$, be polystable vector bundles. Consider the direct image $\iota_* V_i \to Y$ which is a polystable reflexive sheaf. Therefore, $\iota_* V_i$ has an admissible Hermitian–Einstein connection [4, Theorem 3]. The admissible Hermitian–Einstein connections on $\iota_* V_1$ and $\iota_* V_2$ together induce an admissible Hermitian–Einstein connection on the reflexive sheaf $((\iota_* V_1) \otimes (\iota_* V_2))^*$. Therefore, the torsionfree part of the tensor product $(\iota_* V_1) \otimes (\iota_* V_2)$ is polystable [4, Theorem 3]. ■

Proposition 5.2 *If both $E_{\mathrm{GL}_m(\mathbb{C})}$ and $F_{\mathrm{GL}_n(\mathbb{C})}$ are semistable, then the parabolic vector bundle $E_* \otimes F_* \to U$ is also semistable.*

Proof A parabolic G-bundle defined over a big open subset is semistable if and only if the corresponding principal G-bundle over the covering is semistable [2, Theorem 4.3]. In view of Lemma 5.1, the proposition follows from [1, Lemma 2.7]. ■

5.2 The Main Theorem

Let $\psi\colon E_G \to X$ be a ramified principal G-bundle. For any smooth point smooth x of D, let n_x be the order of the isotropy group of any point $z \in \psi^{-1}(x) \subset E_G$ for the action of G (note that n_x is independent of the choice of z in the fiber over x). Let

$$\delta := \operatorname{lcm}\{n_x\}_{x \in D^{\mathrm{sm}}} \tag{5.1}$$

be the least common multiple, where D^{sm} is the smooth locus of D.

Theorem 5.3 *Any polystable ramified principal G-bundle E_G admits a Hermitian structure satisfying the Hermitian–Einstein equation for all $\alpha \in (2(1-\delta), 2)$.*

If a ramified principal G-bundle E_G over X admits a Hermitian structure satisfying the Hermitian–Einstein equation for some $\alpha \in (2(1-\delta), 2)$, then E_G is polystable.

Proof Let E_G be a ramified principal G-bundle over X admitting a Hermitian structure satisfying the Hermitian–Einstein equation for some $\alpha \in (2(1-\delta), 2)$. Fix a Hermitian structure

$$E_K \subset E_G \tag{5.2}$$

that satisfies the Hermitian–Einstein equation for some $\alpha \in (2(1-\delta), 2)$.

Consider the adjoint representation

$$\rho\colon G \longrightarrow \mathrm{GL}(\mathfrak{g}). \tag{5.3}$$

Fix a maximal compact subgroup $\widetilde{K} \subset \mathrm{GL}(\mathfrak{g})$ containing $\rho(K)$. Let

$$\widetilde{\psi}\colon E_{\mathrm{GL}(\mathfrak{g})} \longrightarrow X \tag{5.4}$$

be the ramified principal $\mathrm{GL}(\mathfrak{g})$-bundle obtained by extending the structure group of E_G using the homomorphism ρ in (5.3). Let

$$E_K(\widetilde{K}) = E_K \times^K \widetilde{K} \subset E_{\mathrm{GL}(\mathfrak{g})} \tag{5.5}$$

be the reduction of structure group of $E_{\mathrm{GL}(\mathfrak{g})}$ to $\widetilde{K}$ given by the reduction in (5.2). Since E_K in (5.2) satisfies the Hermitian–Einstein equation, the corresponding reduction $E_K(\widetilde{K})$ in (5.5) also satisfies the Hermitian–Einstein equation.

For any smooth point x of D, let m_x be the order of the isotropy group of any point $z \in \widetilde{\psi}^{-1}(x) \subset E_{\mathrm{GL}(\mathfrak{g})}$ for the action of $\mathrm{GL}(\mathfrak{g})$ (note that m_x is independent of the choice of z in the fiber over x). The integer m_x clearly divides the integer n_x in (5.1). Therefore,

$$\widetilde{\delta} := \mathrm{lcm}\{m_x\}_{x \in D^{\mathrm{sm}}} \le \delta, \tag{5.6}$$

where δ is defined in (5.1). Using (5.6) and the fact that the reduction $E_K(\widetilde{K})$ in (5.5) satisfies the Hermitian–Einstein equation, it follows that the ramified principal $\mathrm{GL}(\mathfrak{g})$-bundle $E_{\mathrm{GL}(\mathfrak{g})}$ is polystable [10, Theorem 6.3]. Since $E_{\mathrm{GL}(\mathfrak{g})}$ is polystable, we conclude that the ramified principal G-bundle E_G is polystable [2, Corollary 4.6]. This proves the second statement of the theorem.

To prove the first statement, let $\psi\colon E_G \to X$ be a polystable ramified principal G-bundle. Fix a ramified finite Galois covering $\phi\colon Y \to X$ and a Γ-linearized principal G-bundle $F_G \to Y$ corresponding to E_G, where $\Gamma = \mathrm{Gal}(\phi)$ (see (4.8) and (4.9)). Since E_G is polystable, it follows that the principal G-bundle F_G is polystable with respect to the pullback of the polarization on X (see [2, Theorem 4.3]). The adjoint vector bundle $\mathrm{ad}(F_G)$ is polystable because F_G is polystable [1, Corollary 3.8]. Since $\mathrm{ad}(F_G)$ is polystable, the ramified principal $\mathrm{GL}(\mathfrak{g})$-bundle $\widetilde{\psi}\colon E_{\mathrm{GL}(\mathfrak{g})} \to X$ in (5.4) is polystable [2, Theorem 4.3]. (We note that polystability of $E_{\mathrm{GL}(\mathfrak{g})}$ can also be deduced using Lemma 5.1 and Proposition 5.2.)

Since $E_{\mathrm{GL}(\mathfrak{g})}$ is polystable, and (5.6) holds, we know that $E_{\mathrm{GL}(\mathfrak{g})}$ admits a Hermitian structure satisfying the Hermitian–Einstein equation for all $\alpha \in (2(1-\delta), 2)$ [10, Theorem 6.3]. A Hermitian structure on $E_{\mathrm{GL}(\mathfrak{g})}$ satisfying the Hermitian–Einstein equation produces a Hermitian structure on E_G satisfying the Hermitian–Einstein equation; see the proof of Theorem 3.7 in [1] for the details. ■

References

[1] B. Anchouche and I. Biswas, *Einstein-Hermitian connections on polystable principal bundles over a compact Kähler manifold.* Amer. J. Math. **123**(2001), no. 2, 207–228. http://dx.doi.org/10.1353/ajm.2001.0007

[2] V. Balaji, I. Biswas, and D. S. Nagaraj, *Principal bundles over projective manifolds with parabolic structure over a divisor.* Tohoku Math. J. **53**(2001), no. 3, 337–367. http://dx.doi.org/10.2748/tmj/1178207416

[3] ———, *Ramified G-bundles as parabolic bundles.* J. Ramanujan Math. Soc. **18**(2003), no. 2, 123–138.

[4] S. Bando and Y.-T. Siu, *Stable sheaves and Einstein-Hermitian metrics.* In: Geometry and Analysis on Complex Manifolds. World Sci. Publishing, River Edge, NJ, 1994, pp. 39–50.

[5] O. Biquard, *Sur les fibrés paraboliques sur une surface complexe.* J. Lond. Math. Soc. **53**(1996), no. 2, 302–316.

[6] I. Biswas, *Parabolic bundles as orbifold bundles.* Duke Math. J. **88**(1997), no. 2, 305–325. http://dx.doi.org/10.1215/S0012-7094-97-08812-8

[7] ———, *On the principal bundles with parabolic structure.* J. Math. Kyoto Univ. **43**(2003), no. 2, 305–332.

[8] ———, *Connections on a parabolic principal bundle over a curve.* Canad. J. Math. **58**(2006), no. 2, 262–281. http://dx.doi.org/10.4153/CJM-2006-011-4

[9] ______, *Connections on a parabolic principal bundle. II.* Canad. Math. Bull. **52**(2009), no. 2, 175–185. http://dx.doi.org/10.4153/CMB-2009-020-2

[10] J. Li, *Hermitian-Einstein metrics and Chern number inequalities on parabolic stable bundles over Kähler manifolds.* Comm. Anal. Geom. **8**(2000), no. 3, 445–475.

[11] M. Maruyama and K. Yokogawa, *Moduli of parabolic stable sheaves.* Math. Ann. **293**(1992), no. 1, 77–99. http://dx.doi.org/10.1007/BF01444704

[12] M. V. Nori, *On the representations of the fundamental group.* Compositio Math. **33**(1976), no. 1, 29–41.

[13] ______, *The fundamental group-scheme.* Proc. Indian Acad. Sci. Math. Sci. **91**(1982), no. 2, 73–122. http://dx.doi.org/10.1007/BF02967978

[14] C. S. Seshadri, *Moduli of vector bundles on curves with parabolic structures.* Bull. Amer. Math. Soc. **83**(1977), no. 1, 124–126. http://dx.doi.org/10.1090/S0002-9904-1977-14210-9

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