Math. Proc. Camb. Phil. Soc. (2023),**175**, 423–432
doi:10.1017/S0305004123000221
First published online 13 April 2023

The Picard group of vertex affinoids in the first Drinfeld covering

BY JAMES TAYLOR
*Mathematical Institute, Andrew Wiles Building, University of Oxford, Radcliffe Observatory Quarter, Woodstock Road, Oxford, OX*2 6*GG.*
e-mail: james.taylor@maths.ox.ac.uk

(*Received* 23 *March* 2022*; revised* 20 *March* 2023*; accepted* 15 *March* 2023)

Abstract

Let F be a finite extension of $\mathbb{Q}_p$. Let Ω be the Drinfeld upper half plane, and Σ^1 the first Drinfeld covering of Ω. We study the affinoid open subset Σ_v^1 of Σ^1 above a vertex of the Bruhat–Tits tree for $\mathrm{GL}_2(F)$. Our main result is that $\mathrm{Pic}\left(\Sigma_v^1\right)[p]=0$, which we establish by showing that $\mathrm{Pic}(\mathbf{Y})[p]=0$ for $\mathbf{Y}$ the Deligne–Lusztig variety of $\mathrm{SL}_2\left(\mathbb{F}_q\right)$. One formal consequence is a description of the representation $H^1_{\text{ét}}\left(\Sigma_v^1, \mathbb{Z}_p(1)\right)$ of $\mathrm{GL}_2(\mathcal{O}_F)$ as the p-adic completion of $\mathcal{O}\left(\Sigma_v^1\right)^\times$.

2020 Mathematics Subject Classification: 11S37, 14G22 (Primary); 14C22 (Secondary)

1. *Introduction*

Let p be a prime, F a finite extension of $\mathbb{Q}_p$, and K the completion of the maximal unramified extension of F. Let $\mathcal{M}_0$ be the disjoint union of $\mathbb{Z}$ copies of Ω, where Ω is the *Drinfeld upper half plane*: the rigid analytic space over K defined by removing all F-rational points from $\mathbb{P}_K^{1,\mathrm{an}}$. The work of Drinfeld [**14**] implies the existence of a tower of finite étale coverings $(\mathcal{M}_n)_{n\geq 0}$ of $\mathcal{M}_0$ equipped with compatible actions of $\mathrm{GL}_2(F)$, which has been shown to realise both the local Langlands and Jacquet–Langlands correspondence in its étale cohomology [**4**, **5**, **18**, **19**]. On the other hand, there is at present no formulated p-adic local Langlands correspondence for $\mathrm{GL}_2(F)$ for general finite extensions F. The Drinfeld tower is expected to be of importance in yielding natural representations of $\mathrm{GL}_2(F)$ that should appear in any such correspondence. For example, the geometric p-adic étale cohomology of the Drinfeld tower has been shown to encode the p-adic local Langlands correspondence for $F=\mathbb{Q}_p$ [**7**].

The preimage of the index zero piece $\Omega \hookrightarrow \mathcal{M}_0$ in the tower $(\mathcal{M}_n)_{n\geq 0}$ defines a tower $(\Sigma^n)_{n\geq 0}$ of finite étale coverings of $\Sigma^0=\Omega$. The transition morphisms are equivariant for the action of the stabilising subgroup $\mathrm{GL}_2(F)^+=\left\{g\in \mathrm{GL}_2(F) \mid \det(g)\in \mathcal{O}^\times\right\}$. Let $\mathcal{T}$ be the Bruhat–Tits tree for $\mathrm{GL}_2(F)$, v the central vertex of $\mathcal{T}$, and $r:\Sigma^1\to\Omega\to\mathcal{T}$ the retraction map. In this paper we study the open affinoid subset $\Sigma_v^1:=r^{-1}(v)$ of Σ^1. This is stable under the action of $\mathrm{GL}_2(\mathcal{O}_F)$ and after a finite extension of K, Σ_v^1 splits up into $q-1$ geometrically connected components, each isomorphic to $\mathrm{Sp}(B)$, where,

$$B=A[z]/\left(z^{q+1}-\left(x^q-x\right)\right), \quad \text{for } A=K\left\langle x, \frac{1}{x^q-x}\right\rangle.$$

The group $\mathrm{GL}_2(F)^+$ acts with two orbits on the set of vertices of $\mathcal{T}$, and one can show that for any vertex w adjacent to v, $\Sigma^1_w \cong \Sigma^1_v$. As any such w will be in the other orbit from v, $\Sigma^1_w \cong \Sigma^1_v$ for all vertices $w \in \mathcal{T}$, and consequently this open subset often determines global properties of Σ^1. For example, the first de-Rham cohomology $H^1_{\mathrm{dR}}(\Sigma^1)$ as a representation of $\mathrm{GL}_2(F)$ is determined by $H^1_{\mathrm{dR}}(\Sigma^1_v)$ [**21**, theorem 6·1].

Our main result is that $\mathrm{Pic}(\Sigma^1_v)[p] = 0$ (Theorem 3·2). The p-adic étale cohomology groups of Drinfeld spaces are of considerable interest [**3**, **6–9**, **24**], and one immediate consequence of Theorem 3·2 is a description of the $\mathrm{GL}_2(\mathcal{O}_F)$-representation $H^1_{\text{ét}}(\Sigma^1_v, \mathbb{Z}_p(1))$, as the p-adic completion of $\mathcal{O}(\Sigma^1_v)^\times$ (Theorem 3·4). This description is very explicit, as the unit group $\mathcal{O}(\Sigma^1_v)^\times$ has been described by Junger [**22**, theorem 5·1].

Our main interest in Theorem 3·2 is the following. A precise statement of the p-adic local Langlands correspondence is formulated when $F = \mathbb{Q}_p$ [**10**], and Dospinescu and Le Bras [**13**] have used this to show that for $F = \mathbb{Q}_p$ and all $n \geq 1$, the representation $\mathcal{O}(\Sigma^n)$ is naturally a coadmissible module over $D(G, K)$, the distribution algebra of G.

In an effort to remove the restriction on F, Ardakov and Wadsley show in their forthcoming work [**1**] using p-adic $\mathcal{D}$-modules that the representation $\mathcal{O}(\Sigma^1)$ splits up naturally into a direct sum of coadmissible $D(G, K)$-modules. This decomposition contains $\mathcal{O}(\Omega)$, and all other components are shown to be topologically irreducible $D(G, K)$-modules. The benefits of this approach over that of [**13**], are that it holds for general field extensions F, is purely local, and establishes topological irreducibility. The obvious disadvantage is that it describes $\mathcal{O}(\Sigma^n)$ only for $n = 1$. One would like to establish similar results for $\mathcal{O}(\Sigma^n)$ for $n \geq 2$, where the situation is significantly more complicated. This is partially due to the fact that $\Sigma^n \to \Sigma^{n-1}$ has degree a power of p, whearas the degree of $\Sigma^1 \to \Omega$ is coprime to p. The methods of [**1**] use the standard result that $\mathrm{Pic}(\Omega) = 0$, and in attempting to transfer these methods to $\mathcal{O}(\Sigma^2)$, one considers the group $\mathrm{Pic}(\Sigma^1)[p]$ instead. Almost nothing is known about $\mathrm{Pic}(\Sigma^1)[p]$, which is strongly expected to be non-zero. Our result that $\mathrm{Pic}(\Sigma^1_v)[p] = 0$ is therefore slightly suprising. It also provides the first steps towards computing $\mathrm{Pic}(\Sigma^1)[p]$ (by choosing an appropriate Čech cover), and allows one the possibility of using similar methods to [**1**] locally.

In order to prove Theorem 3·2, we consider the affine curve $\mathbf{Y}$ defined by,

$$xy^q - yx^q = 1,$$

over the residue field of K, where $\mathbb{F}_q$ is the residue field of F. This curve was first considered by Drinfeld, who showed that all the discrete series representations of $\mathrm{SL}_2(\mathbb{F}_q)$ can be realised in the cohomology of $\mathbf{Y}$ [**2**, preface]. Inspired by this, these ideas were generalised to all reductive groups $\mathbb{G}$ by Deligne and Lusztig in their landmark paper [**12**]. They introduce what are now called *Deligne–Lusztig* varieties, which assign to $\mathbb{G}(\mathbb{F}_q)$ and $w \in W$, the Weyl group, a base space $X(w)$ and a finite covering $Y(w)$, and it is in the étale cohomology of $Y(w)$ that the cuspidal representations are realised. These are spaces of considerable interest, and the Picard groups of the base spaces $X(w)$ have been considered in [**17**]. Here we consider $\mathbf{Y} = Y(w)$ in the special case of $\mathbb{G} = \mathrm{SL}_2$, and $w \neq 1$. It would be interesting to study the Picard groups of $Y(w)$ more generally.

2. *Deligne–Lusztig curves*

Throughout this section, let $\mathbb{F}$ be an algebraic field extension of $\mathbb{F}_q$. We consider the affine curve,

$$\mathbf{Y} = \mathrm{Spec}\left(\frac{\mathbb{F}[x, y]}{xy^q - yx^q = 1}\right),$$

and its projective closure,

$$\mathbf{Z} = \mathrm{Proj}\left(\frac{\mathbb{F}[X, Y, Z]}{XY^q - YX^q = Z^{q+1}}\right).$$

We also consider the projective curve,

$$\mathbf{W} = \mathrm{Proj}\left(\frac{\mathbb{F}[U, V, W]}{UV^q + VU^q = W^{q+1}}\right).$$

We would first like to show that $\mathrm{Pic}(\mathbf{Z})[p] = 0$.

LEMMA 2·1. *$\mathbf{Z}$ is a smooth integral projective curve over $\mathbb{F}$. Furthermore, if $\mathbb{F}_{q^4} \subset \mathbb{F}$, then $\mathbf{W} \cong \mathbf{Z}$.*

Proof. The polynomial $P(X, Y, Z) = Z^{q+1} - (XY^q - YX^q) \in \mathbb{F}[X, Y, Z]$ is prime, which follows from Eisenstein's criterion for $P \in \mathbb{F}[X, Y][Z]$, at the prime ideal (X). Therefore $\mathbf{Z}$ is integral. Furthermore, $\mathbf{Z}$ is smooth, because the system $\partial_X P = \partial_Y P = \partial_Z P = 0$ has no solutions over $\mathbf{Z}(\overline{\mathbb{F}})$. For the isomorphism, let $\lambda \in \mathbb{F}_{q^2}$ with $\lambda^{q-1} = -1$, and let $\mu \in \overline{\mathbb{F}}$ with $\mu^{q+1} = \lambda^q$. The element μ lies in $\mathbb{F}_{q^4}$, as,

$$\mu^{q^2} = (\lambda^q)^{q-1}\mu = -\mu,$$

so,

$$\mu^{q^4} = (-\mu)^{q^2} = -(-\mu) = \mu.$$

Then the claimed isomorphism is given by,

$$U = X, \quad V = \lambda Y, \quad W = \mu Z.$$

Indeed,

$$\begin{aligned} X(\lambda Y)^q + (\lambda Y)X^q &= \lambda^q\left(XY^q - YX^q\right), \\ &= \lambda^q Z^{q+1} = (\mu Z)^{q+1}, \end{aligned}$$

and similarly $U(\lambda^{-1}V)^q - (\lambda^{-1}V)U^q = (\mu^{-1}W)^{q+1}$.

PROPOSITION 2·2. $\mathrm{Pic}(\mathbf{Z})[p] = 0$.

Proof. By Lemma 2·1, $\mathbf{Z}_{\overline{\mathbb{F}}} \cong \mathbf{W}_{\overline{\mathbb{F}}}$, and thus the group $\mathrm{Pic}\left(\mathbf{Z}_{\overline{\mathbb{F}}}\right)[p] \cong \mathrm{Pic}\left(\mathbf{W}_{\overline{\mathbb{F}}}\right)[p] \cong J(\overline{\mathbb{F}})[p]$, where J is the Jacobian of $\mathbf{W}$. $\mathbf{W}$ is known as the Hermitian curve, defined by affine equation $w^{q+1} = v^q + v$, and is maximal over $\mathbb{F}_{q^2}$ [26, lemma 6·4·4], hence $J(\overline{\mathbb{F}})[p] = 0$ by [15, corollary 2·5]. Then, because pullback induces an exact sequence $0 \to \mathrm{Pic}(\mathbf{Z}) \to \mathrm{Pic}\left(\mathbf{Z}_{\overline{\mathbb{F}}}\right)$ [25, Tag 0CC5], and p-torsion is left exact, $\mathrm{Pic}(\mathbf{Z})[p] = 0$.

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Our next goal is to establish that $\mathrm{Pic}(\mathbf{Y})[p] = 0$.

LEMMA 2·3. $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}})$ *consists of the* $q+1$ *points,*

$$\mathcal{P} := \{(a:b:0) \mid (a:b) \in \mathbb{P}^1\left(\mathbb{F}_q\right)\}.$$

Furthermore, $\mathcal{P} = \mathbf{Z}\left(\mathbb{F}_q\right)$.

Proof. If $(a:b:c) \in \mathbf{Z}(\overline{\mathbb{F}})$ with $c=0$, then $b^q a - a^q b = 0$, so $b^q a = a^q b$. If $a \neq 0$, then $(b/a)^q = b/a$, so $b/a \in \mathbb{F}_q$, and $(a:b) \in \mathbb{P}^1\left(\mathbb{F}_q\right)$. Similarly, if $b \neq 0$, $(a:b) \in \mathbb{P}^1\left(\mathbb{F}_q\right)$. Thus $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}}) = \mathcal{P}$. To see $\mathbf{Z}(\overline{\mathbb{F}}) \setminus \mathbf{Y}(\overline{\mathbb{F}}) = \mathbf{Z}\left(\mathbb{F}_q\right)$, there are no points $(a:b:c) \in \mathbf{Z}\left(\mathbb{F}_q\right)$ with $c=1$, because if so then $1 = ab^q - ba^q = ab - ba = 0$, as $a, b \in \mathbb{F}_q$.

Therefore the closed points of $\mathbf{Z} \setminus \mathbf{Y}$ are $\mathcal{P}$ [16, proposition 5·4], which we enumerate by $\mathcal{P} = \{P_0, \dots, P_q\}$. From [27, exercise 5·12 (a)] we have an exact sequence,

$$\mathbb{Z}^{q+1} \longrightarrow \mathrm{Cl}(\mathbf{Z}) \longrightarrow \mathrm{Cl}(\mathbf{Y}) \longrightarrow 0,$$

where the first map sends,

$$(m_0, \dots, m_q) \longmapsto \sum_{i=0}^{q} m_i [P_i],$$

and the second sends, for I a finite set of closed points of $\mathbf{Z}$,

$$\sum_{P \in I} n_P [P] \longmapsto \sum_{P \in I \setminus \mathcal{P}} n_P [P].$$

Let $\Gamma = \langle [P_0], \dots, [P_q] \rangle \subset \mathrm{Cl}(\mathbf{Z})$ be the image of $\mathbb{Z}^{q+1}$ in $\mathrm{Cl}(\mathbf{Z})$. The resulting exact sequence,

$$0 \longrightarrow \Gamma \longrightarrow \mathrm{Cl}(\mathbf{Z}) \longrightarrow \mathrm{Cl}(\mathbf{Y}) \longrightarrow 0,$$

yields the long exact sequence,

$$\begin{aligned} 0 \longrightarrow \Gamma[p] \longrightarrow \mathrm{Cl}(\mathbf{Z})[p] \longrightarrow \mathrm{Cl}(\mathbf{Y})[p] \\ \longrightarrow \Gamma/p\Gamma \longrightarrow \mathrm{Cl}(\mathbf{Z})/p\mathrm{Cl}(\mathbf{Z}) \longrightarrow \mathrm{Cl}(\mathbf{Y})/p\mathrm{Cl}(\mathbf{Y}) \longrightarrow 0, \end{aligned}$$

from the right derived functors of $\mathrm{Hom}_{\mathbb{Z}}(\mathbb{Z}/p\mathbb{Z}, -)$. Then from Proposition 2·2 and the above discussion we have the following.

PROPOSITION 2·4. *There is an exact sequence*

$$0 \longrightarrow \mathrm{Cl}(\mathbf{Y})[p] \longrightarrow \Gamma/p\Gamma \longrightarrow \mathrm{Cl}(\mathbf{Z})/p\mathrm{Cl}(\mathbf{Z}),$$

where the map $\Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\mathrm{Cl}(\mathbf{Z})$ *is that induced by the inclusion* $\Gamma \hookrightarrow \mathrm{Cl}(\mathbf{Z})$.

Remark. We note that if $\mathbf{Z} \setminus \mathbf{Y}$ contained exactly one degree 1 closed point Q, then we could establish that $\mathrm{Pic}(\mathbf{Y})[p] = 0$ almost immediately in the following way. In the exact sequence,

$$\mathbb{Z} \longrightarrow \mathrm{Cl}(\mathbf{Z}) \longrightarrow \mathrm{Cl}(\mathbf{Y}) \longrightarrow 0,$$

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the map $\mathbb{Z} \to \mathrm{Cl}(\mathbf{Z})$ is actually injective and split by the degree homomorphism, hence $\mathrm{Cl}(\mathbf{Z}) \cong \mathbb{Z} \times \mathrm{Cl}(\mathbf{Y})$ so,

$$0 = \mathrm{Cl}(\mathbf{Z})[p] \cong \mathbb{Z}[p] \times \mathrm{Cl}(\mathbf{Y})[p] = \mathrm{Cl}(\mathbf{Y})[p].$$

In particular, this can be applied to show that the class groups of affine dehomogenisations of $\mathbf{Z}$ with respect to both X and Y both have no p-torsion.

We want to show that $\mathrm{Cl}(\mathbf{Y})[p] = 0$, and so in light of Proposition 2·4, we want to show that,

$$\Gamma/p\Gamma \longrightarrow \mathrm{Cl}(\mathbf{Z})/p\mathrm{Cl}(\mathbf{Z}),$$

is injective. In order to do so, we now examine the structure of Γ. First we compute the principal divisors of some rational functions on $\mathbf{Z}$.

Definition 2·5. For $(a:b) \in \mathbb{P}^1\left(\mathbb{F}_q\right)$, we let $P_{(a:b)}$ be the closed point of $\mathbf{Z}$ defined by $(a:b:0) \in \mathbb{P}^1(\overline{\mathbb{F}})$.

LEMMA 2·6. *Let $(a:b), (c:d) \in \mathbb{P}^1\left(\mathbb{F}_q\right)$ with $(a:b) \neq (c:d)$. Then the rational function,*

$$f := \frac{bX - aY}{dX - cY},$$

has associated principal divisor,

$$(f) = (q+1)\left[P_{(a:b)}\right] - (q+1)\left[P_{(c:d)}\right].$$

Proof. Consider the morphism $\zeta : \mathbf{Z} \to \mathbb{P}^1$ corresponding to the extension of function fields $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$, which sends,

$$\frac{S}{T} \longmapsto \frac{bX - aY}{dX - cY},$$

where $\mathbb{P}^1 = \mathrm{Proj}(\mathbb{F}[S, T])$, and $\mathbb{F}(\mathbb{P}^1) = \mathbb{F}(S/T)$. On $\overline{\mathbb{F}}$-points, $\zeta : \mathbf{Z} \to \mathbb{P}^1$ is given by,

$$\zeta(x:y:z) = (bx - ay : dx - cy).$$

This extension $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$ has degree $q+1$ because it differs by an automorphism of $\mathbb{P}^1$ from the extension $\mathbb{F}(\mathbb{P}^1) \to \mathbb{F}(\mathbf{Z})$, defined by,

$$\frac{S}{T} \longmapsto \frac{X}{Y},$$

which clearly has degree $q+1$. Let Q_0, Q_∞ be the closed points of $\mathbb{P}^1$ defined by $(0:1), (1:0) \in \mathbb{P}^1(\overline{\mathbb{F}})$ respectively. By [23, corollary 3·9], we have that,

$$(f) = \zeta^*(S/T) = \zeta^*([Q_0]) - \zeta^*([Q_\infty]),$$

and $\deg(\zeta^*([Q_0])) = \deg(\zeta^*([Q_\infty])) = [\mathbb{F}(\mathbb{P}^1) : \mathbb{F}(\mathbf{Z})] = q+1$. But $\zeta^*([Q_0])$ is some integer multiple of $\left[P_{(a:b)}\right]$ and $\zeta^*([Q_\infty])$ some integer multiple of $\left[P_{(c:d)}\right]$, hence,

$$(f) = (q+1)\left[P_{(a:b)}\right] - (q+1)\left[P_{(c:d)}\right].$$

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Let $\Gamma^0 \subset \Gamma$ be the degree 0 subgroup of Γ, and $\mathrm{Cl}^0(\mathbf{Z}) \subset \mathrm{Cl}(\mathbf{Z})$ the degree 0 subgroup of $\mathrm{Cl}(\mathbf{Z})$.

LEMMA 2·7. *The function* $\phi : \mathbb{Z} \times (\mathbb{Z}/(q+1)\mathbb{Z})^q \to \Gamma$,

$$\phi : \left(n_0, \ldots n, n_q\right) \longmapsto n_0[P_0] + n_1([P_1] - [P_0]) + \ldots + n_q([P_q] - [P_0])),$$

is a surjective homomorphism. In particular, Γ^0 *is a quotient of* $(\mathbb{Z}/(q+1)\mathbb{Z})^q$.

Proof. For each $P_k \in \mathcal{P}$, we can write $P_k = P_{(a_k\,:\,b_k)}$ for some $a_k, b_k \in \mathbb{F}_q$. For each $0 \leq i \neq j \leq q$, consider the rational function,

$$f = \frac{b_i X - a_i Y}{b_j X - a_j Y}.$$

Taking the divisor of f,

$$0 = (f) = (q+1)[P_i] - (q+1)[P_j],$$

in Γ, by Lemma 2·6. Therefore, ϕ is a well-defined homomorphism, which is surjective because $\{[P_0], \ldots, [P_q]\}$ generate Γ. Finally, as $\Gamma^0 = \langle [P_1] - [P_0], \ldots, [P_q] - [P_0] \rangle$, then Γ^0 is a quotient of $(\mathbb{Z}/(q+1)\mathbb{Z})^q$.

We are finally in a position to prove the main result of this section.

THEOREM 2·8. $\mathrm{Pic}(\mathbf{Y})[p] = 0$.

Proof. We can split the degree homomorphism with $[P_0]$, as $[P_0]$ has degree 1 and $\langle [P_0] \rangle$ is free [27, exercise 5·12 (b)]. Then,

$$\begin{aligned} \psi : \mathrm{Cl}(\mathbf{Z}) &\longrightarrow \mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z}, \\ Q &\longmapsto (Q - \deg(Q)[P_0], \deg(Q)), \end{aligned}$$

is an isomorphism, which restricts to,

$$\Gamma \cong \Gamma^0 \times \mathbb{Z}.$$

We then obtain the following commutative diagram,

$$\begin{array}{ccccc} \dfrac{\Gamma}{p\Gamma} & \xrightarrow{\sim} & \dfrac{\Gamma^0 \times \mathbb{Z}}{p(\Gamma^0 \times \mathbb{Z})} & \xrightarrow{\sim} & \dfrac{\Gamma^0}{p\Gamma^0} \times \dfrac{\mathbb{Z}}{p\mathbb{Z}} \\ \downarrow & & \downarrow & & \downarrow \\ \dfrac{\mathrm{Cl}(\mathbf{Z})}{p\,\mathrm{Cl}(\mathbf{Z})} & \xrightarrow{\sim} & \dfrac{\mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z}}{p(\mathrm{Cl}^0(\mathbf{Z}) \times \mathbb{Z})} & \xrightarrow{\sim} & \dfrac{\mathrm{Cl}^0(\mathbf{Z})}{p\,\mathrm{Cl}^0(\mathbf{Z})} \times \dfrac{\mathbb{Z}}{p\mathbb{Z}}. \end{array}$$

Here, the vertical maps are induced from the inclusions of Γ into $\mathrm{Cl}(\mathbf{Z})$ and of Γ^0 into $\mathrm{Cl}^0(\mathbf{Z})$, the left horizontal maps are induced by ψ, and the right horizontal maps are the standard identifications.

Now, by Lemma 2·7, Γ^0 is a quotient of $(\mathbb{Z}/(q+1)\mathbb{Z})^q$, thus $\Gamma^0/p\Gamma^0 = 0$. Consequently, $\Gamma/p\Gamma \to \mathrm{Cl}(\mathbf{Z})/p\mathrm{Cl}(\mathbf{Z})$ is an injection. Therefore, $\mathrm{Cl}(\mathbf{Y})[p] = 0$, by the exact sequence of Proposition 2·4.

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3. *Rigid curves*

Let F be a finite extension of $\mathbb{Q}_p$ with uniformiser π and residue field $\mathbb{F}_q$. Let K be a complete field extension of F with residue field $\mathbb{F}$, such that $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$. Let R be the ring of integers of K and $\varpi \in K$ an element with $0 < |\varpi| < 1$.

Let A be the affinoid algebra,

$$A = K\left\langle x, \frac{1}{x^q - x}\right\rangle,$$

for which the associated rigid space Sp(A) has admissible formal model Spf(A_0), where,

$$A_0 = R\left\langle x, \frac{1}{x^q - x}\right\rangle.$$

Let $u := x^q - x \in A_0^\times \subset A^\times$, and let B be the affinoid algebra,

$$B := A[z]/\left(z^{q+1} - u\right).$$

Consider the ring extension,

$$B_0 := A_0[z]/\left(z^{q+1} - u\right).$$

B_0 is ϖ-torsion free, and the natural map,

$$B_0 = A_0[z]/\left(z^{q+1} - u\right) \longrightarrow R\left\langle x, \frac{1}{x^q - x}, z\right\rangle \Big/ \left(z^{q+1} - u\right),$$

is an isomorphism (because $u \in A_0^\times$ is a unit), hence B_0 is an admissible R-algebra. The special fibre of Spf(B_0) is,

$$\mathrm{Spec}(B_0 \otimes_R \mathbb{F}) = \mathrm{Spec}\left(\mathbb{F}\left[y, 1/v, t\right] / \left(t^{q+1} - v\right)\right),$$

where $v = y^q - y$, and the generic fibre of Spf(B_0) is $\mathrm{Sp}(B_0 \otimes_R K) = \mathrm{Sp}(B)$.

LEMMA 3·1. $\mathrm{Pic}(\mathrm{Sp}(B)) \cong \mathrm{Pic}(\mathbf{Y})$.

Proof. First note that there is an isomorphism of $\mathbb{F}$-algebras,

$$\mathbb{F}[r, s]/(rs^q - sr^q - 1) \xrightarrow{\sim} \mathbb{F}\left[y, 1/v, t\right] / \left(t^{q+1} - v\right),$$

given by $r \mapsto 1/t$, $s \mapsto y/t$, with inverse $y \mapsto s/r$, $t \mapsto 1/r$. Thus $\mathrm{Spec}(B_0 \otimes_R \mathbb{F}) \cong \mathbf{Y}$, and Spf($B_0$) is a smooth admissible formal model of Sp(B). Therefore by [20, lemma 3·6], the natural maps,

$$\mathrm{Pic}(\mathrm{Sp}(B)) \xleftarrow{\sim} \mathrm{Pic}(\mathrm{Spf}(B_0)) \xrightarrow{\sim} \mathrm{Pic}(\mathrm{Spec}(B_0 \otimes_R \mathbb{F})),$$

are isomorphisms and we're done.

We can now state our main results. If K contains $\breve{F}$ the completion of the maximal unramified extension of F, then we can consider the rigid analytic space Σ^1 defined over any such K. For an overview of the construction and properties of Σ^1 see [22, section 2]. If $v \in \mathcal{T}$ is

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the central vertex of the Bruhat-Tits tree, then the open affinoid subset $\Sigma_v^1 := r^{-1}(v) \subset \Sigma^1$ has coordinate ring isomorphic to,

$$\mathcal{O}\left(\Sigma_v^1\right) \cong A[z]/\left(z^{q^2-1} - \left(\pi u^{q-1}\right)\right),$$

by [22, theorem 2·7].

Let ω be a primitive (q^2-1)st root of π in $\overline{F}$. From now on we strengthen our assumption on the complete field extension K of F and assume that,

K contains $\breve{F}(\omega)$ and $\mathbb{F}$ is an algebraic extension of $\mathbb{F}_q$.

We note that this forces $\mathbb{F}$ to be an algebraic closure of $\mathbb{F}_q$, and that this assumption holds for any complete field extension K of $\breve{F}(\omega)$ which is contained in $\mathbb{C}_p$.

THEOREM 3·2. $\operatorname{Pic}\left(\Sigma_v^1\right)[p] = 0$.

Proof. Because K contains ω,

$$\mathcal{O}\left(\Sigma_v^1\right) \cong B^{q-1},$$

and therefore,

$$\operatorname{Pic}\left(\Sigma_v^1\right) \cong \operatorname{Pic}\left(\operatorname{Sp}\left(B^{q-1}\right)\right) = \operatorname{Pic}(\operatorname{Sp}(B))^{q-1} \cong \operatorname{Pic}(\mathbf{Y})^{q-1},$$

by Lemma 3·1. But then $\operatorname{Pic}\left(\Sigma_v^1\right)[p] \cong \operatorname{Pic}(\mathbf{Y})[p]^{q-1}$, which is zero by Theorem 2·8.

Recall that $\Sigma_v^1 = r^{-1}(v)$ is the pre-image of v, the central vertex of the Bruhat–Tits tree. The vertex v is fixed by $\mathrm{GL}_2(\mathcal{O}_F)$, and because r is equivariant, $\mathrm{GL}_2(\mathcal{O}_F)$ acts on Σ_v^1.

COROLLARY 3·3. *The natural map,*

$$\mathcal{O}\left(\Sigma_v^1\right)^{\times} / \mathcal{O}\left(\Sigma_v^1\right)^{\times p^n} \longrightarrow H^1_{\text{ét}}\left(\Sigma_v^1, \mu_{p^n}\right),$$

arising from the Kummer exact sequence is an isomorphism of $\mathrm{GL}_2(\mathcal{O}_F)$*-modules.*

Proof. Because K has characteristic 0, we can consider the Kummer exact sequence for rigid analytic spaces [11, section 3·2]. Then the result follows from Theorem 3·2 after taking the long exact sequence in étale cohomology, using that $\operatorname{Pic}\left(\Sigma_v^1\right) \cong H^1_{\text{ét}}\left(\Sigma_v^1, \mathbb{G}_m\right)$ [11, proposition 3·2·4].

As a consequence, we may now compute $H^1_{\text{ét}}\left(\Sigma_v^1, \mathbb{Z}_p(1)\right)$ as the p-adic completion of $\mathcal{O}\left(\Sigma_v^1\right)^{\times}$. This is completely explicit, as the group $\mathcal{O}\left(\Sigma_v^1\right)^{\times}$ has been computed by Junger [22, theorem 5·1].

THEOREM 3·4. *There is an isomorphism of $\mathbb{Z}_p$-linear representations of* $\mathrm{GL}_2(\mathcal{O}_F)$*,*

$$H^1_{\text{ét}}\left(\Sigma_v^1, \mathbb{Z}_p(1)\right) \cong \varprojlim_{n \geq 1} \mathcal{O}\left(\Sigma_v^1\right)^{\times} / \mathcal{O}\left(\Sigma_v^1\right)^{\times p^n}.$$

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Proof. For all $n \geq 1$ the diagram,

$$\begin{array}{ccc} \mathcal{O}(\Sigma_v^1)^\times/\mathcal{O}(\Sigma_v^1)^{\times p^{n+1}} & \longrightarrow & H^1_{\text{ét}}(\Sigma_v^1, \mu_{p^{n+1}}) \\ \downarrow & & \downarrow \\ \mathcal{O}(\Sigma_v^1)^\times/\mathcal{O}(\Sigma_v^1)^{\times p^n} & \longrightarrow & H^1_{\text{ét}}(\Sigma_v^1, \mu_{p^n}) \end{array}$$

commutes. Then by the definition of $H^1_{\text{ét}}\big(\Sigma_v^1, \mathbb{Z}_p(1)\big)$ and Corollary 3·3,

$$H^1_{\text{ét}}\left(\Sigma_v^1, \mathbb{Z}_p(1)\right) = \varprojlim_{n\geq 1} H^1_{\text{ét}}\left(\Sigma_v^1, \mu_{p^n}\right) \xleftarrow{\sim} \varprojlim_{n\geq 1} \mathcal{O}\left(\Sigma_v^1\right)^\times / \mathcal{O}\left(\Sigma_v^1\right)^{\times p^n}.$$

Acknowledgements. The author would like to thank Konstantin Ardakov, Damien Junger and the referee for their comments on this paper. This research was financially supported by the EPSRC.

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