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SMOOTHNESS OF NOETHERIAN RINGS

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Introduction

In [16] we studied the following problems which had been asked by H. Matsumura (cf. [11]):

(I) What is the difference between smoothness and I-smoothness? In particular, concerning the characterization of smoothness,

(II) When is a ring $A[\![X_1, \cdots, X_n]\!]/\mathfrak{a}$ smooth over A?

In this paper, according to these problems, we will study I-smoothness further when rings are noetherian as in [16].

Now if A is a noetherian ring and B is a quotient ring of a noetherian smooth A-algebra, I-smoothness is fairly easy to handle and some strong results are known (cf. [10, (29. E) Theorem 64] or [16, Proposition (5.2)]). But in general, since all noetherian A-algebras are not necessarily quotient rings of noetherian smooth A-algebras, it is difficult to deal with I-smoothness. For example, for a noetherian ring A and an ideal I of A, $A[\![X_1, \cdots, X_n]\!]$ is $(X_1, \cdots, X_n)A[\![X_1, \cdots, X_n]\!]$-smooth over A and the I-adic completion $(A, I)^\wedge$ is $\hat{I}$-smooth over A. But since these rings are not quotient rings of noetherian smooth A-algebras in general, it is hard to show whether these are smooth over A or not.

In Section 1, we will state some preliminary results.

In Section 2, restricting ourselves to the case of local rings containing a field, we will characterize noetherian smooth local rings over a "D-finite" subfield of A (for the definition, see Section 2). As a result, when the residue field and the dimension of the ring are fixed, we can give the "maximal" noetherian smooth local ring over a "D-finite" subfield of A.

In Section 3, we will deal with Problem (II). Concerning this problem, we listed up the following problems in [16]:

(A) When is $A[\![X_1, \cdots, X_n]\!]$ smooth over A?

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(B) When is $A[\![X_1, \cdots, X_n]\!]/\mathfrak{a}$ smooth over A in the case that $\mathfrak{a} \neq 0$? In particular

(C) when is $(A, I)^\wedge$ smooth over A?

And when A contains a field, some results were given. In this section, we will deal with these problems, mainly Problem (A), when A does not necessarily contain a field or the topology is not necessarily discrete.

In Section 4, we study the set of prime ideals

$$\mathscr{L}_R(R[\![\underline{X}]\!]) = \{P \in \operatorname{Spec}(R[\![\underline{X}]\!]) \mid R[\![\underline{X}]\!] \text{ is } P\text{-smooth over } R\}$$

where R is a noetherian local ring of dimension 1 and $\underline{X}$ are variables over R. This set is stable under specialization, but it is not necessarily a closed subset of $\operatorname{Spec}(R[\![\underline{X}]\!])$ (cf. [16, § 5]). In this section, we will show some results about this set. Some of them are closely related to the smoothness of $\hat{R}$ over R. So [16, § 4] will be useful to study $\mathscr{L}_R(R[\![\underline{X}]\!])$.

The notation and the terminology of [10] and [16] are freely used. If the topology is discrete, we say simply smooth (or unramified, or etale) instead of saying 0-smooth (or 0-unramified, or 0-etale, resp.). For the fundamental properties of André's homology, see [1] or [4].

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§ 1. Preliminary results

First we state the criteria about I-smoothness and I-unramifiedness.

Proposition (1.1) (cf. [3, 1.1]). *Let A be a ring and B be a noetherian A-algebra. For an ideal I of B, B is I-smooth over A if and only if $\Omega_{B/A}$ is formally projective and $H_1(A, B, B/J) = 0$ for all the open ideals J of B.*

On the other hand, B is I-unramified over A if and only if $\Omega_{B/A} \otimes_B (B/J) = 0$ for all the open ideals J of B.

If $I = 0$, we have the following:

Proposition (1.2) (cf. [16, Proposition (1.5)]). *When A is a noetherian ring and B is a noetherian A-algebra, B is smooth over A if and only if $\Omega_{B/A}$ is a projective B-module and the ring homomorphism $A \to B$ is regular.*

For I-etaleness, we have only to notice that "I-etale $= I$-smooth $+$ I-unramified".

Now let $(A, \mathfrak{m}, K)$ be a noetherian local ring containing a field k. Then it is well-known that A is $\mathfrak{m}$-smooth over k if and only if A is geometrically regular over k (cf. [10, (39.C) Theorem 93]). In addition to this, we will notice the following:

LEMMA (1.3). *Consider the following conditions:*

(1) *A is $\mathfrak{m}$-smooth over k;*

(2) *A is regular and K is separable over k;*

(3) *A is regular and $\mathfrak{m}$-smooth over $k[\underline{x}]$ where $\underline{x} = \{x_1, \cdots, x_n\}$ is a regular system of parameters of A.*

Then $(1) \Leftarrow (2) \Leftrightarrow (3)$. *If A has a quasi-coefficient field containing k* (cf. [10, (38.F)]), *then* $(1) \Rightarrow (2)$ *holds.*

Proof. $(2) \Rightarrow (3)$ follows easily by [6, 0_{IV} (19.7.1.)]. $(3) \Rightarrow (2)$ is obvious. $(2) \Rightarrow (1)$ follows by [10, (28.M) Proposition]. Now we assume that A has a quasi-coefficient field containing k. We will show $(1) \Rightarrow (2)$. By [6, 0_{IV} (22.5.8)], A is regular. Hence we have only to show that K is separable over k. Now by [10, (38.F), Theorem 91] and our assumption, $\hat{A}$ has a coefficient field K' containing k. Then $\hat{A} = K'[\![\underline{x}]\!]$ where $\underline{x} = \{x_1, \cdots, x_n\}$ is a regular system of parameters of A. Since $k \to \hat{A}$ is regular by [6, 0_{IV} (22.5.8)], $Q(\hat{A}) = K'((\underline{x}))$ is separable over k. Since K' contains k, K' is separable over k. Q.E.D.

Remark (1.4). (1) If we assume (2), A has a quasi-coefficient field containing k by [9, Theorem 3].

(2) We cannot prove $(1) \Rightarrow (2)$ without the condition that A has a quasi-coefficient field containing k. We can find such an example in [11, § 28, Exercise]: Let k be an imperfect field of characteristic $p > 0$, and let $a \in k - k^p$. Put $A = k[X]_{(X^p - a)}$ where X is a variable over k. Then A has no quasi-coefficient fields containing k and A is smooth over k. Moreover the residue field of A is not separable over k.

We quote the following lemma from [3].

LEMMA (1.5) (cf. [3, 3.3 Remark]). *Let A be a noetherian ring, B be a noetherian A-algebra and C be an A-algebra. Assume that C is essentially of finite type and faithfully flat over A. Then $B \otimes_A C$ is etale (or regular) over C if and only if B is etale (or regular, resp.) over A.*

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§ 2. Smooth local rings over subfields

Throughout this section, $(A, \mathfrak{m}, K)$ means a noetherian local ring where $\mathfrak{m}$ is the maximal ideal of A and K is the residue field. We assume that A contains a field k. Put $n = \dim A$. If K is smooth over k and $\operatorname{rank}_K \Omega_{K/k} < \infty$, we say that k is a D-finite subfield of A. Recall that if K is etale over k, k is called a quasi-coefficient field of A (cf. [10, (38.F)]). We notice that a quasi-coefficient field of A is a D-finite subfield of A. Now the smoothness of A over k is characterized as follows:

THEOREM (2.1). *Let k be a D-finite subfield of A. Assume that A is smooth over k. Then A is an excellent regular local ring. Moreover such a local ring is characterized as follows:*

(1) *If* $\operatorname{ch}(k) = 0$, *we have* $A^h \cong K\langle \underline{X} \rangle$ *where* $\underline{X} = \{X_1, \cdots, X_n\}$ *are variables over K and $K\langle \underline{X} \rangle$ means $(K[\underline{X}], (\underline{X}))^h$. Conversely, if $A^h = K\langle \underline{X} \rangle$ where $\underline{X}$ are not necessarily contained in A, A is smooth over all the D-finite subfields of A contained in K.*

(2) *If* $\operatorname{ch}(k) = p > 0$, *we have $A^p[l, \underline{x}] = A$ where l is a quasi-coefficient field of A containing k (cf.* [9, *Theorem* 3]) *and $\underline{x} = \{x_1, \cdots, x_n\}$ is a regular system of parameters of A. Conversely, if A is a regular local ring and $A^p[l, \underline{x}] = A$, A is smooth over all the D-finite subfields of A contained in l.*

Proof. By [6, 0_{IV} (22.5.8)], A is regular. In particular, A is universally catenary. Now by [6, 0_{IV} (22.5.8)], $k \to \hat{A}$ is regular. Moreover since A is smooth over k, $\Omega_{A/k}$ is a free A-module. So by [4, Theorem 2.1], it follows that $A \to \hat{A}$ is regular. Therefore A is excellent.

To prove the remaining assertions, we will first show the following lemma:

LEMMA (2.2). *Let $\underline{x} = \{x_1, \cdots, x_n\}$ be a regular system of parameters of A, and let l be a quasi-coefficient field of A containing k* (cf. [9, Theorem 3]). *Then if A is smooth over k, A is etale over $l[\underline{x}]$.*

Proof. Since K is separable over l, A is $\mathfrak{m}$-smooth over $l[\underline{x}]$ by Lemma (1.3). So by André's theorem (cf. [2]), $l[\underline{x}] \to A$ is regular. So we have only to show that $\Omega_{A/l[\underline{x}]} = 0$. Now we have the following exact sequence:

$$\Omega_{l[x]/k} \otimes_{l[x]} A \xrightarrow{\varphi} \Omega_{A/k} \longrightarrow \Omega_{A/l[x]} \longrightarrow 0 .$$

Since k is a D-finite subfield of A, the first term of the sequence is a finitely generated A-module. And $\Omega_{A/k}$ is a free A-module by our assumption. Since $\Omega_{A/l[x]} \otimes_A (A/\mathfrak{m}) = 0$, $\varphi \otimes_A (A/\mathfrak{m})$ is surjective. Therefore $\operatorname{rank}_A \Omega_{A/k} < \infty$ and $\Omega_{A/l[x]}$ is a finitely generated A-module. Since $\Omega_{A/l[x]} \otimes_A (A/\mathfrak{m}) = 0$, we have $\Omega_{A/l[x]} = 0$ by *NAK*. Q.E.D.

Case I. $\operatorname{ch}(k) = 0$. Since A^h is etale over A, A^h is etale over $l[\underline{x}]$ by Lemma (2.2). Hence A^h is unramified over $l\langle \underline{x} \rangle$, that is, $\Omega_{A^h/l\langle x \rangle} = 0$. Therefore since A^h is a domain of characteristic zero, A^h is algebraic over $l\langle \underline{x} \rangle$. Denoting a coefficient field of $\hat{A}$ containing l by K' (cf. [10, (38.F) Theorem 91]), we have the following commutative diagram:

$$\begin{array}{ccc} K'\langle \underline{x} \rangle & \longrightarrow & \hat{A} \\ \uparrow & & \uparrow \\ l\langle \underline{x} \rangle & \longrightarrow & A^h \, . \end{array}$$

Since $K'\langle \underline{x} \rangle$ and A^h are excellent and $\widehat{K'\langle \underline{x} \rangle} \cong \widehat{A^h} \cong \hat{A}$, both $K'\langle \underline{x} \rangle$ and A^h are algebraically closed in $\hat{A}$ by [12, (44.1) Theorem]. Since $K'\langle \underline{x} \rangle$ is algebraic over $l\langle \underline{x} \rangle$, we have $A^h = K'\langle \underline{x} \rangle$. Conversely, assume that $A^h = K\langle \underline{X} \rangle$, and let k be a D-finite subfield of A contained in K. Let l be a quasi-coefficient field of A containing k, and let K' be a coefficient field of $\hat{A}$ containing l (cf. [9, Theorem 3]). Now since A^h is regular, A is also regular. Hence we denote a regular system of parameters of A by $\underline{y} = \{y_1, \cdots, y_n\}$. Then we have the following commutative diagram:

$$\begin{array}{ccccccc} & & A & \longrightarrow & A^h = K\langle \underline{X} \rangle & \longrightarrow & \hat{A} \\ & & \uparrow & & \uparrow & & \uparrow \\ k & \longrightarrow & l[\underline{y}] & \longrightarrow & l\langle \underline{y} \rangle & \longrightarrow & K'\langle \underline{y} \rangle \, . \end{array}$$

Since $\operatorname{tr.deg}_k K = \operatorname{tr.deg}_k K' = \operatorname{tr.deg}_k l$, we have

$$\operatorname{tr.deg}_k K\langle \underline{X} \rangle = \operatorname{tr.deg}_k l\langle \underline{y} \rangle \, .$$

Therefore $K\langle \underline{X} \rangle$ is algebraic over $l\langle \underline{y} \rangle$. Since $K'\langle \underline{y} \rangle$ is algebraic over $l\langle \underline{y} \rangle$ and since $\hat{A} = K[\![\underline{X}]\!] = K'[\![\underline{y}]\!]$, we have $K\langle \underline{X} \rangle = K'\langle \underline{y} \rangle$ by [12, (44.1) Theorem]. So we may assume that $l \subseteq K$ and $A^h = K\langle \underline{y} \rangle$. Now A^h is etale over A. Hence it follows that $\Omega_{A/l[y]} \otimes_A A^h \simeq \Omega_{A^h/l[y]}$. Since $A^h = K\langle \underline{y} \rangle$ is etale over $l[\underline{y}]$, we have $\Omega_{A^h/l[y]} = 0$. Therefore $\Omega_{A/l[y]} = 0$. Since $l[\underline{y}] \to A$ is regular by the proof of Lemma (2.2), A is etale over $l[\underline{y}]$. Since $l[\underline{y}]$ is smooth over k, A is smooth over k.

Case II. $\operatorname{ch}(k) = p > 0$. By Lemma (2.2), we have $\Omega_{A/l[x]} = 0$. Thus $\Omega_{Q(A)/l(x)} = 0$. Since $Q(A)$ is a field of characteristic p, we have $Q(A) =$

$Q(A^p[l, \underset{\sim}{x}])$. Now since A is a regular local ring and A is etale over $l[\underset{\sim}{x}]$, we have that $A^p \otimes_{l^p[\underset{\sim}{x}^p]} l[\underset{\sim}{x}] \cong A^p[l, x]$ (cf. [16, Lemma (4.5)]) and this ring is integrally closed in its total quotient field by [6, (6.14.1)]. Since A is integral over $A^p[l, \underset{\sim}{x}]$, we have $A = A^p[l, \underset{\sim}{x}]$. Conversely, let A be a regular local ring and $\underset{\sim}{x} = \{x_1, \cdots, x_n\}$ be a regular system of parameters of A. Assume that $A^p[l, \underset{\sim}{x}] = A$ for a quasi-coefficient field l of A. Since $l[\underset{\sim}{x}]$ is smooth over all the D-finite subfields of A contained in l, we have only to show that A is smooth over $l[\underset{\sim}{x}]$. By the proof of Lemma (2.2), $l[\underset{\sim}{x}] \to A$ is regular. Moreover since $A^p[l, \underset{\sim}{x}] = A$, we have $\Omega_{A/l[x]} = 0$. Therefore A is etale over $l[\underset{\sim}{x}]$. Q.E.D.

Remark (2.3). In the case of characteristic p, A has a p-basis $x_1, \cdots, x_n$ over l.

COROLLARY (2.4). *Let $(A, \mathfrak{m}, K)$ be a regular local ring containing a field k and let $\underset{\sim}{x} = \{x_1, \cdots, x_n\}$ be a regular system of parameters of A. Assume that* $\mathrm{ch}(k) = p > 0$ *and k is a D-finite subfield of A. Then A is smooth over k if and only if $Q(A) = Q(A^p[l, \underset{\sim}{x}])$ where l is a quasi-coefficient field of A containing k.*

EXAMPLE (2.5). Let K be a field of characteristic $p > 0$. Then $K^{p^\infty} = \bigcap_n K^{p^n}$ is a maximal perfect field contained in K. Let X be a variable over K and put $A = K^{p^\infty}[\![X]\!][K]_{(X)}$. Then it is easy to see that $(A, (X), K)$ is a DVR containing K and that if K is not perfect, $A \not\cong \hat{A}$. Now $(K^{p^\infty}[\![X]\!][K])^p[K, X] = K^{p^\infty}[\![X]\!][K]$. So A is smooth over K by Corollary (2.4).

COROLLARY (2.6). *Let $(A, \mathfrak{m}, K)$, k and $\underset{\sim}{x}$ satisfy the same conditions as Corollary* (2.4). *Let K be a coefficient field of $\hat{A}$ containing k. Assume that A is smooth over k. Then we have the following*:

(1) *If* $\mathrm{ch}(k) = 0$, *we have* $A \subseteq K\langle \underset{\sim}{x} \rangle$;

(2) *If* $\mathrm{ch}(k) = p > 0$, *we have* $A \subseteq \bigcap_m K^{p^m}[\![\underset{\sim}{x}]\!][K]$.

Therefore in the case $\mathrm{ch}(k) = 0$, if we fix the dimension and the residue field, $K\langle \underset{\sim}{x} \rangle$ is maximal among the regular local subrings of $K[\![\underset{\sim}{x}]\!]$ with completion $K[\![\underset{\sim}{x}]\!]$ which are smooth over D-finite subfields. Our next interest is in the case $\mathrm{ch}(k) = p > 0$. The answer is the following:

THEOREM (2.7). *Let K be a field of characteristic $p > 0$ and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over K. Put $A = \bigcap_m K^{p^m}[\![\underline{X}]\!][K]$. Then $(A, (\underline{X}), K)$ is a regular local ring containing K and $\hat{A} \cong K[\![\underline{X}]\!]$. And A is smooth* *over K. Therefore $\bigcap_m K^{p^m}[\![\underline{X}]\!][K]$ plays a similar rôle as $K\langle\underline{X}\rangle$ in the case of characteristic zero.*

Proof. First we characterize the elements of A. For $f \in K[\![\underline{X}]\!]$,

($\sharp$) $f \in A$ if and only if $K^{p^m}(\{\text{all the coefficients of } f\})$ is a finite extension of K^{p^m} for all $m \geq 0$ (K^{p^0} means K).

Step I. We will show that A is noetherian. For the purpose, we have only to show that all the ideals of A are finitely generated. Let $\mathfrak{a}$ be an ideal of A. Put $I = \mathfrak{a}K[\![\underline{X}]\!]$. Since $K[\![\underline{X}]\!]$ is noetherian, $I = (f_1, \cdots, f_t)K[\![\underline{X}]\!]$ for some $f_1, \cdots, f_t \in \mathfrak{a}$. Let $g \in I \cap A$. Put $k_m = K^{p^m}(\{\text{all the coefficients of } f_1, \cdots, f_t, g\})$. Then by ($\sharp$), k_m is a finite extension over K^{p^m} for all $m \geq 0$. Hence $k_m[\![\underline{X}]\!] \subseteq K^{p^m}[\![\underline{X}]\!][K]$. Put $k = \bigcap_m k_m$ and $J = I \cap k[\![\underline{X}]\!]$. Then since $K[\![\underline{X}]\!]$ is faithfully flat over $k[\![\underline{X}]\!]$ and $f_1, \cdots, f_t$, $g \in k[\![\underline{X}]\!]$, we have $J = (f_1, \cdots, f_t)k[\![\underline{X}]\!]$ and $g \in J$. On the other hand, $k[\![\underline{X}]\!] = \bigcap_m k_m[\![\underline{X}]\!] \subseteq \bigcap_m K^{p^m}[\![\underline{X}]\!][K] = A$. Therefore we have $g \in JA = (f_1, \cdots, f_t)A$. Hence $\mathfrak{a} = (f_1, \cdots, f_t)A$. Thus A is noetherian. It follows easily that $(A, (\underline{X}), K)$ is a regular local ring and $\hat{A} \cong K[\![\underline{X}]\!]$.

Step II. We will show that A is smooth over K. By Step I, A is regular. So we have only to show that $A = A^p[K, \underline{X}]$ by Theorem (2.1). Thus it suffices to prove that $A \subseteq A^p[K, \underline{X}]$. Let $f \in A$. Then f can be written as follows:

$$f = \sum_{0 \leq \alpha_i < p} X_1^{\alpha_1} \cdots X_n^{\alpha_n} F_{\alpha_1 \cdots \alpha_n}(X_1^p, \cdots, X_n^p)$$
$$\text{where all } F_{\alpha_1 \cdots \alpha_n}(X_1^p, \cdots, X_n^p) \in A\,.$$

Now by ($\sharp$), $l_m = K^{p^m}(\{\text{all the coefficients of } f\})$ is a finite extension of K^{p^m} for all $m \geq 0$. Let Γ be a p-basis of K over K^p. Then there is a family of increasing subsets $\Gamma_1 \subseteq \Gamma_2 \subseteq \cdots \subseteq \Gamma_m \subseteq \cdots$ of Γ such that $\sharp(\Gamma_m) < \infty$ and $l_m \subseteq K^{p^m}(\Gamma_m)$ for all $m \geq 1$. So if $\Gamma_1 = \{\gamma_1, \cdots, \gamma_t\}$, each $F_{\alpha_1 \cdots \alpha_n}(X_1^p, \cdots, X_n^p)$ can be written as follows:

$$F_{\alpha_1 \cdots \alpha_n}(X_1^p, \cdots, X_n^p) = \sum_{0 \leq \beta_i < p} \gamma_1^{\beta_1} \cdots \gamma_t^{\beta_t} G_{\beta_1 \cdots \beta_t}(X_1^p, \cdots, X_n^p)$$
$$\text{where } G_{\beta_1 \cdots \beta_t}(X_1^p, \cdots, X_n^p) \in K^p[\![X_1^p, \cdots, X_n^p]\!]\,.$$

Then since $l_m \subseteq K^{p^m}(\Gamma_m)$ and $\Gamma_1 \subseteq \Gamma_m$, it follows easily that $L_m = K^{p^m}(\{\text{all the coefficients of } G_{\beta_1 \cdots \beta_t}\}) \subseteq K^{p^m}(\Gamma_m^p)$. So L_m is a finite extension of K^{p^m}. Therefore $L_m^{p^{-1}} = K^{p^{m-1}}(\{\text{all the coefficients of } G_{\beta_1 \cdots \beta_t}^{p^{-1}}\})$ is a finite extension of $K^{p^{m-1}}$ for all $m \geq 1$. Therefore by ($\sharp$), each $G_{\beta_1 \cdots \beta_t}^{p^{-1}}$ is an element of A. So $f \in A^p[K, \underline{X}]$. Q.E.D.

Remark (2.8). (1) In the above theorem, if $[K: K^p] < \infty$, $A = K[\![\underline{X}]\!]$. Of course, A is smooth over K. On the other hand, if $[K: K^p] = \infty$, $K^{p^m}[\![\underline{X}]\!][K]$ is not a Nagata ring for all $m \geq 1$ (cf. [10, (34.B)]). But $\bigcap_m K^{p^m}[\![\underline{X}]\!][K]$ is excellent by Theorem (2.1). Moreover $K[\![\underline{X}]\!]$ is not smooth over $\bigcap_m K^{p^m}[\![\underline{X}]\!][K]$ by [16, Theorem (4.7)] and Theorem (2.7).

(2) It is easy to show that if $\mathrm{ch}(K) = p > 0$ and K is not a perfect field, $(K^{p^\infty}[\![X]\!][K])_{(X)} \neq \bigcap_m K^{p^m}[\![X]\!][K]$ where X is a variable over K (cf. Example (2.5)).

§3. Problems (A), (B) and (C)

In this section, we mainly deal with Problem (A). First we show the following lemma:

LEMMA (3.1). *Let A be a noetherian ring with* $\mathrm{ch}(A) > 0$ *and B be a noetherian flat A-algebra. Then if B/pB is etale over A/pA for all the prime divisors p of* $\mathrm{ch}(A)$, *B is etale over A.*

Proof. Since it follows easily that $A \to B$ is regular, we have only to show that $\Omega_{B/A} = 0$. Put $\mathrm{ch}(A) = p_1 \cdots p_t$ where $p_1, \cdots, p_t$ are prime numbers not necessarily distinct. Then in order to prove $\Omega_{B/A} = 0$, we use induction on t. If $t = 1$, the assertion is obvious. Suppose $t > 1$. Put $p = p_1$ and $q = p_2 \cdots p_t$. From the exact sequence $0 \to pA \to A \to A/pA \to 0$ we have the following exact sequence:

$$\Omega_{B/A} \otimes_A pA \longrightarrow \Omega_{B/A} \longrightarrow \Omega_{B/A} \otimes_A (A/pA) \longrightarrow 0\ .$$

Now $\Omega_{B/A} \otimes_A pA \cong (\Omega_{B/A} \otimes_A (A/qA)) \otimes_A pA \cong \Omega_{(B/qB)/(A/qA)} \otimes_A pA$. So by the induction hypothesis, we have $\Omega_{B/A} \otimes_A pA = 0$. On the other hand, by the assumption we have $\Omega_{B/A} \otimes_A (A/pA) = 0$. Therefore $\Omega_{B/A} = 0$. Q.E.D.

Now we will give an answer to Problem (A) when the ring A does not necessarily contain a field (cf. [16, Theorem (2.2)]).

THEOREM (3.2). *Let A be a noetherian ring and $X_1, \cdots, X_n$ be variables over A. Then the following conditions are equivalent*:

(1) *$A[\![X_1, \cdots, X_n]\!]$ is smooth over A for all $n \geq 1$;*

(2) *$A[\![X_1, \cdots, X_n]\!]$ is smooth over A for some $n \geq 1$;*

(3) $\mathrm{ch}(A) > 0$, *and for all prime divisors p of* $\mathrm{ch}(A)$, *A/pA is a finite $(A/pA)^p$-module.*

Proof. (1) $\Rightarrow$ (2). Obvious.

(2) $\Rightarrow$ (3). We assume that $\mathrm{ch}(A) = 0$. Then there exists $\mathfrak{p} \in \mathrm{Spec}(A)$ such that $\mathrm{ch}(A/\mathfrak{p}) = 0$. So we assume that A is a domain with $\mathrm{ch}(A) = 0$. Then in the same way as in the proof of Lemma (2.2), we can show $\Omega_{A[[X]]/A[X]} = 0$ where $\underline{X} = \{X_1, \cdots, X_n\}$. So since $\mathrm{ch}(A) = 0$, $Q(A[\![\underline{X}]\!])$ is algebraic over $Q(A[\underline{X}])$. But $\mathrm{tr.deg}_{Q(A[X])} Q(A[\![\underline{X}]\!]) > 0$. This is a contradiction. So we have $\mathrm{ch}(A) > 0$. Therefore we have the assertion by [16, Theorem (2.2)].

(3) $\Rightarrow$ (1) follows easily from [16, Theorem (2.2)] and Lemma (3.1). Q.E.D.

COROLLARY (3.3). *Let A be a noetherian ring and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over A. Assume that $A[\![\underline{X}]\!]$ is smooth over A for some $n \geq 1$. Then A is an excellent ring, and for all the multiplicatively closed subsets S of A, $(S^{-1}A)[\![\underline{X}]\!]$ is smooth over $S^{-1}A$.*

Proof. The former follows from Theorem (3.2) and Kunz' theorem (cf. [7, Theorem 2.5]). The latter is obvious. Q.E.D.

Now concerning the above corollary, we will consider the following conditions:

(1) $A[\![X_1, \cdots, X_n]\!]$ is smooth over A;

(2) $A_\mathfrak{m}[\![X_1, \cdots, X_n]\!]$ is smooth over $A_\mathfrak{m}$ for all $\mathfrak{m} \in \mathrm{Max}(A)$.

By Theorem (3.2), if one of (1) and (2) holds, it follows that A is a G-ring such that $\mathrm{ch}(A) > 0$ and $[A/\mathfrak{m} : (A/\mathfrak{m})^p] < \infty$ for all $\mathfrak{m} \in \mathrm{Max}(A)$ where $p = \mathrm{ch}(A/\mathfrak{m})$. Now by Corollary (3.3), (1) $\Rightarrow$ (2) holds. But the converse seems difficult. For the equivalence of (1) and (2), we will show the following result:

PROPOSITION (3.4). *Let A be a noetherian ring satisfying the condition (2). Then the condition (1) holds if and only if $A[\![\underline{X}]\!]$ is a G-ring where $\underline{X} = \{X_1, \cdots, X_n\}$.*

Proof. The "only if" part follows easily from Theorem (3.2) and Kunz' theorem (cf. [7, Theorem 2.5]). We show the "if" part. For $\mathfrak{m} \in \mathrm{Max}(A)$, put $M = (\mathfrak{m}, \underline{X})A[\![\underline{X}]\!]$. Consider the following sequence of ring homomorphisms:

$$A_\mathfrak{m}[\underline{X}] \longrightarrow A[\![\underline{X}]\!]_M \longrightarrow A_\mathfrak{m}[\![\underline{X}]\!] .$$

Then we have the following exact sequence:

$$H_1(A[\![\underline{X}]\!]_M, A_\mathfrak{m}[\![\underline{X}]\!], A_\mathfrak{m}[\![\underline{X}]\!]) \longrightarrow \Omega_{A[[X]]_M/A_\mathfrak{m}[X]} \otimes_{A[[X]]_M} A_\mathfrak{m}[\![\underline{X}]\!]$$
$$\longrightarrow \Omega_{A_\mathfrak{m}[[X]]/A_\mathfrak{m}[X]} .$$

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Since $A[\![\underline{X}]\!]$ is a G-ring by our assumption, $A[\![\underline{X}]\!]_M \to A_{\mathfrak{m}}[\![\underline{X}]\!]$ is regular. Hence the first term of the sequence is equal to zero. On the other hand, in the same way as in the proof of Lemma (2.2), we can show that the last term is equal to zero. Therefore it follows easily that $\Omega_{A[[X]]/A[X]} = 0$. Now since A is a G-ring, $A[\underline{X}] \to A[\![\underline{X}]\!]$ is regular. Hence $A[\![\underline{X}]\!]$ is etale over $A[\underline{X}]$. Thus $A[\![\underline{X}]\!]$ is smooth over A. Q.E.D.

Therefore, in order to show (2) under the condition (1), it is enough to prove the following problem which is the special case of the problem asked by A. Grothendieck (cf. [6, (7.4.8)]):

PROBLEM (3.5). Let A be a G-ring containing a field of characteristic $p > 0$ and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over A. Assume that

$$[A/\mathfrak{m}:(A/\mathfrak{m})^p] < \infty \qquad \text{for all } \mathfrak{m} \in \mathrm{Max}(A)\,.$$

Then is $A[\![\underline{X}]\!]$ a G-ring?

Remark (3.6). (1) If Problem (3.5) is true, the following property holds by Theorem (3.2) and Proposition (3.4): Let A be a noetherian ring of characteristic $p > 0$. Assume that $A_{\mathfrak{m}}$ is a finite $A_{\mathfrak{m}}^p$-module for all $\mathfrak{m} \in \mathrm{Max}(A)$. Then A is a finite A^p-module.

(2) By J. Nishimura (cf. [13]), it was shown that the completion of a G-ring is not necessarily a G-ring. But his example does not satisfy our assumption.

Next concerning Problem (A), we will study I-smoothness of $A[\![\underline{X}]\!]$ over A for a non-zero ideal I of $A[\![\underline{X}]\!]$. If I is generated by the elements of A, we have the following result.

THEOREM (3.7). *Let A be a noetherian ring with* $\mathrm{ch}(A) > 0$ *and* $\underline{X} = \{X_1, \cdots, X_n\}$ *be variables over A, and let $\mathfrak{a}$ be an ideal of A such that* $\mathfrak{a} \subseteq \mathrm{rad}(A)$. *Assume that A is an N-ring* (cf. [13, Definition (0.1)]). *Then the following are equivalent*:

(1) *$A[\![\underline{X}]\!]$ is $\mathfrak{a}A[\![\underline{X}]\!]$-smooth over A;*

(2) *$A[\![\underline{X}]\!]$ is smooth over A.*

Proof. (2) $\Rightarrow$ (1). Obvious.

(1) $\Rightarrow$ (2). If $A[\![\underline{X}]\!]$ is $\mathfrak{a}A[\![\underline{X}]\!]$-smooth over A, $(A/\mathfrak{a})[\![\underline{X}]\!]$ is smooth over $A/\mathfrak{a}$. So by Corollary (3.3) and Marot's theorem (cf. [10, (41.D) Theorem 106]), $\hat{A} = (A, \mathfrak{a})^{\wedge}$ is a Nagata ring. Since A is an N-ring and $\mathfrak{a} \subseteq \mathrm{rad}(A)$, $A \to \hat{A}$ is faithfully flat and reduced. So by [8, Lemma (4.9)], A is also a

Nagata ring. On the other hand, since $\mathfrak{a} \subseteq \operatorname{rad}(A)$, $(A/\mathfrak{m})[\![\underline{X}]\!]$ is smooth over $A/\mathfrak{m}$ for all $\mathfrak{m} \in \operatorname{Max}(A)$. So by Theorem (3.2) we have $[A/\mathfrak{m}: (A/\mathfrak{m})^p] < \infty$ where $p = \operatorname{ch}(A/\mathfrak{m})$. Therefore since A is a Nagata ring, by [14, Theorem (1.2)] we have $[Q(A/\mathfrak{p}): Q(A/\mathfrak{p})^p] < \infty$ for all $\mathfrak{p} \in \operatorname{Min}(A)$ where $p = \operatorname{ch}(A/\mathfrak{p})$. So $A/\mathfrak{p}$ is a finite $(A/\mathfrak{p})^p$-module. Then it follows easily (cf. the proof of (3.1)) that A/pA is a finite $(A/pA)^p$-module for all the prime divisors p of $\operatorname{ch}(A)$. Therefore $A[\![\underline{X}]\!]$ is smooth over A by Theorem (3.2).

Q.E.D.

We cannot prove (1) $\Rightarrow$ (2) without the condition N-ring. We will construct such an example.

Example (3.8). Let K be a field of characteristic $p > 0$ such that $[K: K^p] = \infty$, and let $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over K. Put $A = K[\![\underline{X}]\!][K^{p^{-\infty}}]$. Then by [15, Lemma 9], $(A, (\underline{X}), K^{p^{-\infty}})$ is a regular local ring of dimension n and $\hat{A} \cong K^{p^{-\infty}}[\![\underline{X}]\!]$. Since $Q(\hat{A})$ is not separable over $Q(A)$, A is not a N-ring by [10, (31. F) Theorem 71]. In particular, A is not a Nagata ring. Now let Y be a variable over A. Then since $K^{p^{-\infty}}$ is a perfect field, $A[\![Y]\!]$ is $(\underline{X})A[\![Y]\!]$-smooth over A by Theorem (3.2). But since A is not a Nagata ring, $A[\![Y]\!]$ is not smooth over A by Corollary (3.3).

Now we will consider the case that the ideal I of $A[\![\underline{X}]\!]$ is not necessarily generated by the elements of A.

We will recall the following definition:

Definition (3.9) (cf. [16, § 3]). For a ring A and $\mathfrak{m} \in \operatorname{Max}(A)$, we say that A satisfies SC at $\mathfrak{m}$ if one of the following is satisfied:

(1) $\operatorname{ch}(A/\mathfrak{m}) = 0$;

(2) $\operatorname{ch}(A/\mathfrak{m}) = p > 0$ and $[A/\mathfrak{m}: (A/\mathfrak{m})^p] = \infty$.

In particular, we say that A satisfies SC if for all $\mathfrak{m} \in \operatorname{Max}(A)$, A satisfies SC at $\mathfrak{m}$. We notice that it may happen that $\operatorname{ch}(A/\mathfrak{m}) \neq \operatorname{ch}(A/\mathfrak{m}')$ for $\mathfrak{m}$, $\mathfrak{m}' \in \operatorname{Max}(A)$. If A contains a rational field or $(A, \mathfrak{m}, K)$ is a local ring such that $\operatorname{ch}(K) = p > 0$ and $[K: K^p] = \infty$, A satisfies SC.

First we show the following lemma which gives an answer when A is a field.

Lemma (3.10). *Let K be a field satisfying SC and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over K. For an ideal I of $K[\![X]\!]$, we assume that $K[\![\underline{X}]\!]$ is I-smooth over K. Then $\sqrt{I} = (\underline{X})$.*

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Proof. Let P be a minimal prime divisor of I. We have only to show $P = (\underline{X})$. So we assume $P \neq (\underline{X})$. Put $A = K[\![\underline{X}]\!]/P$. Now $K[\![\underline{X}]\!]$ is P-smooth over K. Hence in the same way as in the proof of Lemma (2.2), it follows that $\Omega_{K[[X]]/K} \otimes_{K[[X]]} A$ is a finitely generated A-module. Therefore $\Omega_{A/K}$ is also a finitely generated A-module. Thus $\operatorname{rank}_L \Omega_{L/K} < \infty$ where $L = Q(A)$. On the other hand, since $P \neq (\underline{X})$, A is a complete local domain with $\dim A > 0$. Hence there are elements $\underline{y} = \{y_1, \cdots, y_r\}$ of A such that $\underline{y}$ are analytically independent over K and A is a finite $K[\![\underline{y}]\!]$-module. Now by the following lemma, we have $\operatorname{rank}_{K((y))} \Omega_{K((y))/K} = \infty$. Therefore by the following exact sequence:

$$H_1(K((\underline{y})), L, L) \longrightarrow \Omega_{K((y))/K} \otimes_{K((y))} L \longrightarrow \Omega_{L/K} \longrightarrow \Omega_{L/K((y))} \longrightarrow 0\,,$$

we have $\operatorname{rank}_L \Omega_{L/K} = \infty$. This is a contradiction. Q.E.D.

LEMMA (3.11). *Let K be a field and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over K. Then if K satisfies SC, we have* $\operatorname{rank}_{K((X))} \Omega_{K((X))/K} = \infty$.

The proof is left to the reader.

THEOREM (3.12). *Let A be a noetherian ring and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over A. Assume that A satisfies SC. Then, for an ideal I of $A[\![\underline{X}]\!]$, if $A[\![\underline{X}]\!]$ is I-smooth over A, we have* $\dim A[\![\underline{X}]\!]/I \leq \dim A$.

Proof. Let $M \in \operatorname{Max}(A[\![\underline{X}]\!])$ such that $I \subseteq M$, and put $\mathfrak{m} = M \cap A$. Then $\mathfrak{m} \in \operatorname{Max}(A)$ and $M = (\mathfrak{m}, \underline{X})A[\![\underline{X}]\!]$. Now $(A/\mathfrak{m})[\![\underline{X}]\!]$ is $I(A/\mathfrak{m})[\![\underline{X}]\!]$-smooth over $A/\mathfrak{m}$. Since $A/\mathfrak{m}$ satisfies SC, $\sqrt{I + \mathfrak{m}A[\![\underline{X}]\!]} = M$ by Lemma (3.10). So for a positive integer t, there are elements $g_i = X_i^t + f_i$ $(i = 1, \cdots, n)$ of I where $f_1, \cdots, f_n \in \mathfrak{m}A[\![\underline{X}]\!]$. Now let $a_1, \cdots, a_m$ be a system of parameters of $A_\mathfrak{m}$. Then $\sqrt{(a_1, \cdots, a_m, g_1, \cdots, g_n)A[\![\underline{X}]\!]_M} = MA[\![\underline{X}]\!]_M$. So $a_1, \cdots, a_m, g_1, \cdots, g_n$ is a system of parameters of $A[\![\underline{X}]\!]_M$. Since $g_1, \cdots, g_n \in I$, we have $\dim A[\![\underline{X}]\!]_M/IA[\![\underline{X}]\!]_M \leq \dim A_\mathfrak{m}$. Q.E.D.

COROLLARY (3.13). *With A and $\underline{X}$ as in Theorem (3.12), assume that A satisfies SC. Then for a prime ideal P of $A[\![\underline{X}]\!]$ such that $P \subseteq (\underline{X})$, if $A[\![\underline{X}]\!]$ is P-smooth over A, then we have $P = (\underline{X})$.*

For Problems (B) and (C), we will show some results. First, for Problem (B), we have:

PROPOSITION (3.14). *With notation as in* [16, *Proposition* (3.3)], *let I be an ideal of A contained in* rad(A). *Assume that A satisfies SC. If*

$R = A[\![\underline{X}]\!]/\mathfrak{a}$ *is IR-smooth over* A, *then* $R \cong (A, \pi(\mathfrak{a}))^\wedge$.

The proof is similar to that of [16, Proposition (3.3)].

For Problem (C) in the case that the ring A does not necessarily contain a field, by [16, Theorem (4.7)] and the fact that the henselization A^h is etale over A, it is enough to show the following results which can be proved easily.

PROPOSITION (3.15). *Let A be a noetherian ring and B be a noetherian flat A-algebra. Then if $B/\mathfrak{p}B$ is etale over $A/\mathfrak{p}$ for all $\mathfrak{p} \in \mathrm{Min}(A)$, B is etale over A.*

The proof is similar to that of Lemma (3.1).

PROPOSITION (3.16) (cf. [16, Theorem (4.4)]). *With notation as above, assume that (A, I) is a henselian couple and A is a domain with* $\mathrm{ch}(A) = 0$ *which does not necessarily contain a field. Then $\hat{A} = (A, I)^\wedge$ is smooth over A if and only if $\hat{A} \cong A$.*

§4. The formal power series rings over noetherian local rings of dimension 1

Let R be a ring and A be an R-algebra. Recall the following notation (cf. [16, §5]):

$$\mathscr{L}_R(A) = \{P \in \mathrm{Spec}(A) \mid A \text{ is } P\text{-smooth over } R\}.$$

Now let $(R, \mathfrak{m}, K)$ be a noetherian local ring and let $A = R[\![\underline{X}]\!]$ where $\underline{X} = \{X_1, \cdots, X_n\}$ are variables over R. Then we want to determine $\mathscr{L}_R(R[\![\underline{X}]\!])$. But it seems very difficult. Here we will give several results on the problems in some special cases.

First we have the following proposition:

PROPOSITION (4.1). *Let $(R, \mathfrak{m}, K)$ be a noetherian local ring. Then*

(1) $\mathscr{L}_R(R[\![\underline{X}]\!]) = \mathrm{Spec}(R[\![\underline{X}]\!])$ *if and only if* $\mathrm{ch}(R) = p^t > 0$ *where p is a prime number and $t \in N$, and R/pR is a finite $(R/pR)^p$-module;*

(2) *suppose that R is a domain of dimension 1, then $P \cap R = 0$ for all $P \in \mathscr{L}_R(R[\![\underline{X}]\!]) - \{(\mathfrak{m}, \underline{X})\}$ if and only if R satisfies SC, that is,* $\mathrm{ch}(K) = 0$ *or* $\mathrm{ch}(K) = p > 0$, $[K : K^p] = \infty$;

(3) *under the same assumption as in (2), if $\hat{R}$ is smooth over R and R is a DVR, $\mathscr{L}_R(R[\![\underline{X}]\!]) \supset \{(X_1 - a_1, \cdots, X_n - a_n) \mid a_1, \cdots, a_n \in \mathfrak{m}\}$. Conversely, if $(X_1 - a_1, \cdots, X_n - a_n) \in \mathscr{L}_R(R[\![\underline{X}]\!])$ for some $a_1, \cdots, a_n \in \mathfrak{m}$ such*

that $a_i \neq 0$ *for some* i, *then* $\hat{R}$ *is smooth over* R.

Proof. (2) follows easily from Theorem (3.2) and Lemma (3.10), and (3) follows easily from [16, Proposition (5.4)]. Thus we will show (1). By Theorem (3.2), we have only to show the "only if" part. Assume $\mathscr{L}_R(R[\![\underline{X}]\!]) = \operatorname{Spec}(R[\![\underline{X}]\!])$. Then by the following lemma, $R[\![\underline{X}]\!]$ is P-etale over $R[\underline{X}]$ for all $P \in \operatorname{Spec}(R[\![\underline{X}]\!])$. Since $\mathfrak{p}R[\![\underline{X}]\!] \in \operatorname{Spec}(R[\![\underline{X}]\!])$ for all $\mathfrak{p} \in \operatorname{Min}(R[\underline{X}])$, $R[\![\underline{X}]\!]/\mathfrak{p}R[\![\underline{X}]\!]$ is etale over $R[\underline{X}]/\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Min}(R[\underline{X}])$. Therefore by Proposition (3.15) $R[\![\underline{X}]\!]$ is etale over $R[\underline{X}]$. Hence $R[\![\underline{X}]\!]$ is smooth over R. So we have the conclusion by Theorem (3.2). Q.E.D.

LEMMA (4.2). *Let* R *be a noetherian ring and* $\underline{X} = \{X_1, \cdots, X_n\}$ *be variables over* R, *and let* I *be an ideal of* $R[\![\underline{X}]\!]$. *Then if* $R[\![\underline{X}]\!]$ *is* I*-smooth over* R, $R[\![\underline{X}]\!]$ *is* I*-etale over* $R[\underline{X}]$.

Proof. The situation is similar to that of Lemma (2.2). But we need more careful study. First we have the following exact sequence by our assumption:

$$0 \longrightarrow H_1(R[\underline{X}], R[\![\underline{X}]\!], A) \longrightarrow \Omega_{R[X]/R} \otimes_{R[X]} A$$
$$\xrightarrow{\varphi} \Omega_{R[[X]]/R} \otimes_{R[[X]]} A \longrightarrow \Omega_{R[[X]]/R[X]} \otimes_{R[[X]]} A \longrightarrow 0\,,$$

where $A = R[\![\underline{X}]\!]/I$. By the proof of Lemma (2.2), it follows that $\Omega_{R[[X]]/R[X]} \otimes_{R[[X]]} A = 0$. On the other hand, for all $M \in \operatorname{Max}(R[\![\underline{X}]\!])$ such that $M \supseteq I$, we have the following exact sequence:

$$0 \longrightarrow H_1(R[\underline{X}], R[\![\underline{X}]\!], B) \longrightarrow \Omega_{R[X]/R} \otimes_{R[X]} B \xrightarrow{\psi} \Omega_{R[[X]]/R} \otimes_{R[[X]]} B \longrightarrow 0\,,$$

where $B = R[\![\underline{X}]\!]/M$. Since $H_1(R[\underline{X}], R[\![\underline{X}]\!], B) \cong H_1(R, R, B) = 0$, and since $\psi = \varphi \otimes B$, it follows easily that φ is injective. Hence $H_1(R[\underline{X}], R[\![\underline{X}]\!], A) = 0$. Therefore, by Proposition (1.1), $R[\![\underline{X}]\!]$ is I-etale over $R[\underline{X}]$. Q.E.D.

From now on, we assume that $\dim R = 1$.

PROPOSITION (4.3). *Let* $(R, \mathfrak{m})$ *be a henselian local domain of dimension* 1 *and* $\underline{X} = \{X_1, \cdots, X_n\}$ *be variables over* R. *Let* S *be a subset of* $\operatorname{Spec}(R[\![\underline{X}]\!])$ *whose element* P *satisfies the following conditions*: (1) $P \not\subseteq (\underline{X})$, (2) $P \cap R = 0$, (3) P *is generated by polynomials over* R *and* (4) $R[\underline{X}]/\mathfrak{p}$ *is integral over* R *where* $\mathfrak{p} = P \cap R[\underline{X}]$. *Then if* $P \in \mathscr{L}_R(R[\![\underline{X}]\!])$ *for some* $P \in S$, R *is excellent. Moreover if* $\operatorname{ch}(R) = 0$, $\hat{R}$ *is smooth over* R, *and if* $\operatorname{ch}(R) = p > 0$, $\hat{\tilde{R}}$ *is smooth over* $\tilde{R}$ *where* $\tilde{R}$ *is the derived normal ring of* R. *Conversely, if* $\hat{R}$ *is smooth over* R, *then* $S \subset \mathscr{L}_R(R[\![\underline{X}]\!])$.

Proof. Assume that $P \in \mathscr{L}_R(R[\![\underline{X}]\!])$ for some $P \in S$. Put $\mathfrak{p} = P \cap R[\underline{X}]$, then $P = \mathfrak{p}R[\![\underline{X}]\!]$ by (3). We denote $R[\underline{X}]/\mathfrak{p}$ by A. Since R is henselian, A is also a henselian local domain with maximal ideal $\sqrt{\mathfrak{m}A}$ by (4). Moreover as $\dim A = 1$ and $\mathfrak{p} \not\subset (\underline{X})$, we have $\sqrt{\mathfrak{m}A} = \sqrt{(\underline{X})A}$. Since $R[\![\underline{X}]\!]/\mathfrak{p}R[\![\underline{X}]\!]$ is the $(\underline{X})A$-adic completion of A, $R[\![\underline{X}]\!]/P \cong \hat{A}$. Then we have the following commutative diagram:

$$\begin{array}{ccc} A & \longrightarrow & \hat{A} \\ \uparrow & & \uparrow \\ R & \longrightarrow & \hat{R}\,. \end{array}$$

By Lemma (4.2), $\hat{A}$ is etale over A. Hence A is a G-ring. Since A is finite over R, R is also a G-ring by [5, 1.3 Proposition]. Since $\dim R = 1$, R is universally catenary. Therefore R is excellent.

Now we assume that $\operatorname{ch}(R) = 0$. Then since $\hat{A}$ is etale over A, we have $A = \hat{A}$ by Proposition (3.16). Since A is finite over R, $\hat{R}$ is finite over R. Thus $R = \hat{R}$. Therefore $\hat{R}$ is smooth over R.

Next we assume that $\operatorname{ch}(R) = p > 0$. Then since R is excellent, $\tilde{R}$ is a finite R-module. So $\tilde{R}$ is an excellent regular local ring of dimension 1. Hence we have only to show that $Q(\hat{\tilde{R}}^p[\tilde{R}]) = Q(\hat{\tilde{R}})$ by [16, Corollary (4.8)]. Now since $\hat{A}$ is etale over A and $\hat{A} \cong A \otimes_R \hat{R}$, it follows that $\Omega_{\hat{A}/A} \cong \Omega_{\hat{R}/R} \otimes_R A = 0$. Hence $\Omega_{Q(\hat{R})/Q(R)} = 0$, and $Q(\hat{R}^p[R]) = Q(\hat{R})$. On the other hand, from the finiteness of $\tilde{R}$ over R we have $Q(\hat{\tilde{R}}) = Q(\hat{R})$. Therefore $Q(\hat{\tilde{R}}^p[\tilde{R}]) = Q(\hat{\tilde{R}})$.

The converse follows easily. Q.E.D.

Remark (4.4). We assume that R satisfies SC. Then for $P \in \mathscr{L}_R(R[\![\underline{X}]\!])$, we have $P \not\subset (\underline{X})$ by Corollary (3.13), and if $P \cap R \neq 0$, then $P = (\mathfrak{m}, \underline{X})$ by Lemma (3.10). So the conditions (1) and (2) of Proposition (4.3) are not so strong.

If R is a regular local ring of dimension 1, we obtain a stronger result as follows:

PROPOSITION (4.5). *Let $(R, \mathfrak{m})$ be a regular local ring of dimension* 1 *and $\underline{X} = \{X_1, \cdots, X_n\}$ be variables over R. Let P be a prime ideal of $R[\![\underline{X}]\!]$ such that $P \not\subset (\underline{X})$, $P \cap R = 0$ and P is generated by polynomials. Then if $P \in \mathscr{L}_R(R[\![\underline{X}]\!])$, $\hat{R}$ is smooth over R. In particular, R is excellent.*

Proof. Let $M = (\mathfrak{m}, \underline{X})R[\underline{X}]$ and $\mathfrak{p} = P \cap R[\underline{X}]_M$. Then $P = \mathfrak{p}R[\![\underline{X}]\!]$ by our assumption. Put $A = R[\underline{X}]_M/\mathfrak{p}$. Then $(A, \underline{x}A)^\wedge \cong R[\![\underline{X}]\!]/P$ where $\underline{x}$ are the homomorphic images of $\underline{X}$ in A. We denote the ring by $\hat{A}$. Since $\dim R = 1$ and $P \not\subseteq (\underline{X})$, $\hat{A}$ is a complete local ring with respect to the maximal ideal. Then there exists a canonical homomorphism $\psi\colon A \otimes_R \hat{R} \to \hat{A}$. Since A is finitely generated over R, $A \otimes_R \hat{R}$ is noetherian. Put $Q = (\mathfrak{m}, \underline{x})\hat{A} \cap (A \otimes_R \hat{R})$ and $B = (A \otimes_R \hat{R})_Q$. Then we have the following commutative diagram:

$$\begin{tikzcd} & \hat{A} \\ A \arrow[ur, "\varphi"] \arrow[r, "v"] & B = (A \otimes_R \hat{R})_Q \arrow[u, "\psi"'] \\ R \arrow[u] \arrow[r, "u"] & \hat{R} \arrow[u] \end{tikzcd}$$

Now since $R[\![\underline{X}]\!]$ is P-smooth over R, $R[\![\underline{X}]\!]$ is P-etale over $R[\underline{X}]$ by Lemma (4.2). Therefore $\hat{A}$ is etale over A, in particular, φ is regular. Since $\hat{B} \cong \hat{A}$, ψ is faithfully flat. Hence v is regular. On the other hand, since $\hat{R}$ is a G-ring and A is essentially of finite type over R, $B = (A \otimes_R \hat{R})_Q$ is also a G-ring. Hence ψ is regular. Therefore we have the following exact sequence:

$$0 \longrightarrow \Omega_{B/A} \otimes_B \hat{A} \longrightarrow \Omega_{\hat{A}/A}\,.$$

Since $\hat{A}$ is etale over A, $\Omega_{\hat{A}/A} = 0$. Hence $\Omega_{B/A} \otimes_B \hat{A} = 0$. Since $\hat{B} \cong \hat{A}$, we have $\Omega_{B/A} = 0$. Thus $B = (A \otimes_R \hat{R})_Q$ is etale over A. Now R is a DVR and A is a torsion-free R-module. Hence A is faithfully flat over R. Therefore by Lemma (1.5), $\hat{R}$ is etale over R. Q.E.D.

In a special case, $\mathscr{L}_R(R[\![\underline{X}]\!])$ is completely determined as follows:

PROPOSITION (4.6). *Let $(R, \mathfrak{m})$ be a noetherian local domain of dimension* 1 *containing a field. Assume that R satisfies SC and* $\operatorname{rank}_{Q(R)} \Omega_{Q(R)/k} < \infty$ *for a quasi-coefficient field k of R. Then we have* $\mathscr{L}_R(R[\![\underline{X}]\!]) = \{(\underline{X}), (\mathfrak{m}, \underline{X})\}$.

Proof. Assume the contrary. Then there exists $P \in \mathscr{L}_R(R[\![\underline{X}]\!]) - \{(\underline{X}), (\mathfrak{m}, \underline{X})\}$. Put $A = R[\![\underline{X}]\!]/P$. Now $R[\![\underline{X}]\!]$ is P-smooth over R. Hence in the same way as in the proof of Lemma (3.10), it follows that $\Omega_{A/R}$ is a finitely generated A-module. Thus $\operatorname{rank}_{Q(A)} \Omega_{Q(A)/k} < \infty$ by our assumption. On the other hand, since $P \not\subseteq (\underline{X})$ by Corollary (3.13), A is a complete local domain with $\dim A > 0$. Hence in the same way as in the proof of Lemma (3.10), we have $\operatorname{rank}_{Q(A)} \Omega_{Q(A)/k} = \infty$. This is a contradiction. Q.E.D.

References

[1] M. André, Homologie des algèbres commutatives, Springer, 1974.
[2] ——, Localisation de la lissité formelle, Manuscripta Math., **13** (1974), 297–307.
[3] A. Brezuleanu, Smoothness and regularity, Compositio Math., **24** (1972), 1–10.
[4] A. Brezuleanu and N. Radu, Excellent rings and good separation of the module of differentials, Rev. Roummaine Math. Pures Appl., **23** (1978), 1455–1470.
[5] S. Greco, Two theorems on excellent rings, Nagoya Math. J., **60** (1976), 139–149.
[6] A. Grothendieck, Éléments de Géométrie Algébrique, Ch. IV, Publ. IHES no. **20** (1964), no. **24** (1965), no. **32** (1967).
[7] E. Kunz, On noetherian rings of characteristic p, Amer. J. Math., **98** (1976), 999–1013.
[8] J. Marot, Limite inductive plate de P-anneaux, J. Algebra, **57** (1979), 484–496.
[9] H. Matsumura, Quasi-coefficient rings of a local ring, Nagoya Math. J., **68** (1977), 123–130.
[10] ——, Commutative Algebra, second edition, Benjamin, 1980.
[11] ——, Commutative Rings (in Japanese), Kyoritsu, Tokyo 1980.
[12] M. Nagata, Local Rings, Interscience, 1962.
[13] J. Nishimura, On ideal-adic completion of noetherian rings, J. Math. Kyoto Univ., **21** (1981), 153–169.
[14] H. Seydi, Sur la théorie des anneaux excellents en caractéristique p, Bull. Soc. Math. France, **96** (1972), 193–198.
[15] H. Tanimoto, On the base field chane of P-rings and P-2 rings, Nagoya Math. J., **90** (1983), 77–83.
[16] ——, Some characterizations of smoothness, J. Math. Kyoto Univ., **23** (1983), 695–706.

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