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Principal bundle structure of the space of metric measure spaces

Daisuke Kazukawa
Faculty of Mathematics, Kyushu University, Fukuoka-shi, Fukuoka-ken 819-0395, Japan (kazukawa@math.kyushu-u.ac.jp) (corresponding author)

Hiroki Nakajima
Mathematical Sciences Course, Ehime University, Matsuyama-shi, Ehime-ken 790-8577, Japan (nakajima.hiroki.nz@ehime-u.ac.jp)

Takashi Shioya
Mathematical Institute, Tohoku University, Sendai-shi, Miyagi-ken 980-8578, Japan (shioya@math.tohoku.ac.jp)

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We study the topological structure of the space $\mathcal{X}$ of isomorphism classes of metric measure spaces equipped with the box or concentration topologies. We consider the scale-change action of the multiplicative group $\mathbb{R}_+$ of positive real numbers on $\mathcal{X}$, which has a one-point metric measure space, say $*$, as only one fixed-point. We prove that the $\mathbb{R}_+$-action on $\mathcal{X}_* := \mathcal{X} \setminus \{*\}$ admits the structure of non-trivial and locally trivial principal $\mathbb{R}_+$-bundle over the quotient space. Our bundle $\mathbb{R}_+ \to \mathcal{X}_* \to \mathcal{X}_*/\mathbb{R}_+$ is a curious example of a non-trivial principal fibre bundle with contractible fibre. A similar statement is obtained for the pyramidal compactification of $\mathcal{X}$, where we completely determine the structure of the fixed-point set of the $\mathbb{R}_+$-action on the compactification.

Keywords: Box distance; concentration topology; metric measure space; principal bundle; pyramid

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1. Introduction

It is a challenging problem to study the structure of the space $\mathcal{X}$ of isomorphism classes of metric measure spaces, where we assume a metric measure space to be a complete separable metric space with a Borel probability measure. Denote by $\mathbb{R}_+$ the multiplicative group of positive real numbers. We have the natural group action of $\mathbb{R}_+$ on $\mathcal{X}$ defined as

$$\mathbb{R}_+ \times \mathcal{X} \ni (t, X) \longmapsto tX \in \mathcal{X},$$

where tX is the space X with the t-scaled metric of X. Note that the isomorphism class of a one-point metric measure space, denoted as $*$, is the only fixed-point of this action. Let $\mathcal{X}_* := \mathcal{X} \setminus \{*\}$ and let Σ denote the quotient space $\mathcal{X}_*/\mathbb{R}_+$ equipped with the quotient topology.

As for the structure of the space $\mathcal{X}$, Sturm [19] obtained the remarkable result that the subspace $\mathcal{X}_{pq}$ of $\mathcal{X}$ with finite L^{pq}-size and equipped with the $L^{p,q}$-distortion metric is a non-negatively curved Alexandrov space isometric to a Euclidean cone for $p = 2$ and $q \in [1, +\infty)$. He also determined geodesics in $\mathcal{X}_{pq}$ for $p, q \in [1, +\infty)$ and proved that any orbit of the $\mathbb{R}_+$-action is a geodesic ray, which implies that $\mathcal{X}_{pq}$ is homeomorphic to the cone over Σ_{pq} for any $p, q \in [1, +\infty)$, where Σ_{pq} is the subspace of Σ corresponding to $\mathcal{X}_{pq}$.

Also, Ivanov and Tuzhilin [8] pointed out that the Gromov–Hausdorff space is homeomorphic to the cone over the quotient space by the $\mathbb{R}_+$-action.

In this article, we study the topological structure of $\mathcal{X}$ with the box and observable metrics and also of the pyramidal compactification of $\mathcal{X}$. Those metrics and the pyramidal compactification are fundamental concepts in the study of metric measure spaces, originally introduced by Gromov [6] (see also [18]). The box metric is closely related to the $L^{0,q}$-distortion metric (see [19]) and coincides with the Gromov–Prokhorov metric (see [5, 12]). The observable metric is defined by the idea of the concentration of measure phenomenon established by Lévy and Milman [13, 15] (see also [14]) and is useful to study convergence of spaces with dimensions divergent to infinity. The topologies induced from the box and observable metrics are called the box and concentration topologies, respectively. In our previous article [11], we have proved that the box metric is geodesic and that $\mathcal{X}$ is locally path-connected and contractible with respect to the box and concentration topologies, which are the same properties as those of $\mathcal{X}_{pq}$ with the $L^{p,q}$-distortion metric studied by Sturm [19]. However, the box and observable metrics are much different from the $L^{p,q}$-distortion metric. One of the essential differences is that any orbit of the $\mathbb{R}_+$-action is never a geodesic ray for the box and observable metrics. Also, there are intricately branching geodesics with respect to the box metric, and the Alexandrov curvature is not bounded neither from below nor from above (see [11, remark 6.7]). The question that arises here is:

- Is $\mathcal{X}$ with the box and/or concentration topologies homeomorphic to the cone over Σ?

Surprisingly the answer is negative.

THEOREM 1.1 *For neither of the box nor concentration topologies, $\mathcal{X}_* = \mathcal{X} \setminus \{*\}$ is not homeomorphic to $\mathbb{R}_+ \times \Sigma$, and $\mathcal{X}$ is not homeomorphic to the cone over Σ.*

One of our main theorems is stated as follows.

THEOREM 1.2 *For the box and concentration topologies, the action of $\mathbb{R}_+$ on $\mathcal{X}_*$ admits the structure of a non-trivial and locally trivial principal $\mathbb{R}_+$-bundle over Σ.*

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In general, a locally trivial principal bundle with contractible fibre over a paracompact Hausdorff base space is trivial (see [3, corollary 2.8] and [7, 8.1 theorem of Chapter 4] for example), which is not necessarily true if the base space is not paracompact. It is remarkable that our principal bundle $\mathbb{R}_+ \to \mathcal{X}_* \to \Sigma$ presents such a counterexample.

The action of $\mathbb{R}_+$ on $\mathcal{X}$ naturally extends to the pyramidal compactification of $\mathcal{X}$, say Π (see § 2.3 for the definition of Π). Denote by $\mathrm{Fix}(\Pi)$ the set of fixed-points of the action of $\mathbb{R}_+$ on Π, and put $\Pi_* := \Pi \setminus \mathrm{Fix}(\Pi)$. We also have the following.

THEOREM 1.3 *The action of $\mathbb{R}_+$ on Π_* admits the structure of a non-trivial and locally trivial principal $\mathbb{R}_+$-bundle over the quotient space $\Pi_*/\mathbb{R}_+$.*

We investigate the structure of $\mathrm{Fix}(\Pi)$ and have the following theorem. Denote by $\mathcal{A}$ the set of all monotone non-increasing sequences of non-negative real numbers with total sum not greater than 1.

THEOREM 1.4 *The fixed-point set $\mathrm{Fix}(\Pi)$ is homeomorphic to $\mathcal{A}$ with the l^2-weak topology.*

As for the topology of $\mathcal{X}_*$ and Π_*, we have the following.

PROPOSITION 1.5

(i) $\mathcal{X}_*$ is contractible with respect to the box and concentration topologies.
(ii) Π_* is contractible.

Since the fibres of our bundles in theorems 1.2 and 1.3 are contractible, we have the following.

COROLLARY 1.6.

(i) For the box and concentration topologies, all homotopy groups of Σ vanish.
(ii) All homotopy groups of $\Pi_*/\mathbb{R}_+$ vanish.

Let us mention the ideas of our proofs.

A key point of the proof of theorem 1.1 is to prove that Σ is not a Urysohn space (lemma 3.2). If $\Sigma \times \mathbb{R}_+$ were to be homeomorphic to $\mathcal{X}_*$, then Σ would be metrizable, which is contrary to the non-Urysohn property of Σ. It is quite delicate that Σ is a Hausdorff space (proposition 3.10).

For theorems 1.2 and 1.3, the local triviality of the bundles is a core of the proof.

For theorem 1.2 with the box topology, we construct an $\mathbb{R}_+$-invariant open covering $\{\mathcal{X}_\Delta\}_{\Delta \in (0,1)}$ of $\mathcal{X}$ and continuous 1-homogeneous functions $r_\Delta \colon \mathcal{X}_\Delta \to \mathbb{R}_+$ for $\Delta \in (0,1)$. We define $\mathcal{X}_\Delta$ to be the set of mm-spaces such that any atom has measure less than Δ and define $r_\Delta(X)$, $X \in \mathcal{X}_\Delta$, as the integral of the partial diameter $\mathrm{diam}(X; s)$ with respect to the parameter $s \in [0, (\Delta+1)/2]$. The reason why we take the integral is for the sake of the continuity of r_Δ. Using r_Δ, we obtain a local trivialization $\mathcal{X}_\Delta \simeq \mathcal{X}_\Delta/\mathbb{R}_+ \times \mathbb{R}_+$.

Theorem 1.2 for the concentration topology is derived from theorem 1.3 just by restricting the base space $\Pi_*/\mathbb{R}_+$ to $\mathcal{X}_*/\mathbb{R}_+$.

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For the proof of theorem 1.3, we construct an $\mathbb{R}_+$-invariant open covering of Π_* and continuous 1-homogeneous functions on each open set in the covering. This time, for an $(N+1)$-tuple $\kappa = (\kappa_0, \ldots, \kappa_N)$ of positive real numbers with $\sum_{i=0}^N \kappa_i < 1$, we define Π_κ to be the set of all $\mathcal{P} \in \Pi$ such that $\mathrm{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N) < +\infty$ and $\mathrm{Sep}(\mathcal{P}; \kappa_0+\delta, \ldots, \kappa_N+\delta) > 0$ for some $\delta > 0$, and we define $r_\kappa(\mathcal{P})$, $\mathcal{P} \in \Pi_\kappa$, to be the integral of $\mathrm{Sep}(\mathcal{P}; \kappa_0+s, \ldots, \kappa_N+s)$ with respect to $s \in [0,1]$ (see definition 2.17 for the definition of $\mathrm{Sep}(\cdots)$). These induce a local trivialization $\Pi_\kappa \simeq \Pi_\kappa/\mathbb{R}_+ \times \mathbb{R}_+$. However, it is not easy to prove that the union of all Π_κ coincides with Π_*. For the proof, we need to investigate the structure of pyramids in $\mathrm{Fix}(\Pi)$ as follows. For $A = \{a_i\}_{i=1}^\infty \in \mathcal{A}$, we define

$$\mathcal{P}_A := \left\{ X \in \mathcal{X} \;\middle|\; \text{There exists a sequence } \{x_i\}_{i=1}^\infty \subset X \text{ such that } \sum_{i=1}^\infty a_i \delta_{x_i} \le \mu_X \right\}.$$

THEOREM 1.7 *For a given $\mathcal{P} \in \Pi$, the following (i)–(iv) are equivalent to each other.*

(i) $\mathcal{P} \in \mathrm{Fix}(\Pi)$.
(ii) *$t\mathcal{P} = \mathcal{P}$ for some $t \in \mathbb{R}_+$ with $t \neq 1$.*
(iii) *For any $\kappa_0, \ldots, \kappa_N > 0$ with $\sum_{i=0}^N \kappa_i < 1$, the separation distance $\mathrm{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N)$ is either 0 or $+\infty$.*
(iv) *There exists $A \in \mathcal{A}$ such that $\mathcal{P} = \mathcal{P}_A$.*

In theorem 1.7, the implication '(iii) ⇒ (iv)' is highly non-trivial, and we need a delicate discussion to prove it. Theorem 1.7 with a little discussion implies that the union of Π_κ coincides with Π_*.

Theorem 1.4 is derived from theorem 1.7 and the following.

THEOREM 1.8 *The map $\mathcal{A} \ni A \longmapsto \mathcal{P}_A \in \Pi$ is a topological embedding.*

Theorem 1.8 is also proved by Esaki–Kazukawa–Mitsuishi [4] independently. Our proof is simpler than [4]. It is proved in [4] that the weak topology on $\mathcal{A}$ coincides with the l^∞-topology.

The organization of this article is as follows. After the preliminaries section, we study in § 3 the scale-change action of $\mathbb{R}_+$ on $\mathcal{X}_*$. We prove that Σ is not Urysohn, which leads to theorem 1.1. We also prove theorems 1.2 and 1.3 with the help of theorem 1.7. In § 4, we determine the structure of pyramids in $\mathrm{Fix}(\Pi)$ and prove theorems 1.7 and 1.8 to obtain theorem 1.4. In § 5, we prove proposition 1.5. In § 6, we present several questions.

2. Preliminaries

In this section, we describe the definitions and some properties of metric measure space, the box distance, the observable distance, the pyramid, and the weak topology. We use most of these notions along [18]. As for more details, we refer to [18] and [6, Chapter $3\frac{1}{2}{}_+$].

2.1. Metric measure spaces

Let (X, d_X) be a complete separable metric space and μ_X a Borel probability measure on X. We call the triple (X, d_X, μ_X) a *metric measure space*, or an *mm-space* for short. We sometimes say that X is an mm-space, in which case the metric and the measure of X are, respectively, indicated by d_X and μ_X.

DEFINITION 2.1 (mm-Isomorphism). *Two mm-spaces X and Y are said to be mm-isomorphic to each other if there exists an isometry $f\colon \operatorname{supp}\mu_X \to \operatorname{supp}\mu_Y$ such that $f_*\mu_X = \mu_Y$, where $f_*\mu_X$ is the push-forward measure of μ_X by f. Such an isometry f is called an mm-isomorphism. Denote by $\mathcal{X}$ the set of mm-isomorphism classes of mm-spaces.*

Note that an mm-space X is mm-isomorphic to $(\operatorname{supp}\mu_X, d_X, \mu_X)$. We assume that an mm-space X satisfies

$$X = \operatorname{supp}\mu_X,$$

unless otherwise stated.

DEFINITION 2.2 (Lipschitz order). *Let X and Y be two mm-spaces. We say that X (Lipschitz) dominates Y and write $Y \prec X$ if there exists a 1-Lipschitz map $f\colon X \to Y$ satisfying $f_*\mu_X = \mu_Y$. We call the relation $\prec$ on $\mathcal{X}$ the Lipschitz order.*

The Lipschitz order $\prec$ is a partial order relation on $\mathcal{X}$.

2.2. Box distance and observable distance

For a subset A of a metric space (X, d_X) and for a real number $r > 0$, we set

$$U_r(A) := \{x \in X \mid d_X(x, A) < r\},$$

where $d_X(x, A) := \inf_{a\in A} d_X(x, a)$.

DEFINITION 2.3 (Prokhorov distance). *The Prokhorov distance $d_{\mathrm{P}}(\mu, \nu)$ between two Borel probability measures μ and ν on a metric space X is defined to be the infimum of $\varepsilon > 0$ satisfying*

$$\mu(U_\varepsilon(A)) \geq \nu(A) - \varepsilon$$

for any Borel subset $A \subset X$.

The Prokhorov metric d_{P} is a metrization of the weak convergence of Borel probability measures on X provided that X is a separable metric space.

DEFINITION 2.4 (Ky Fan metric). *Let (X, μ) be a measure space and (Y, d_Y) a metric space. For two μ-measurable maps $f, g\colon X \to Y$, we define $d^{\mu}_{\mathrm{KF}}(f, g)$ to be the infimum of $\varepsilon \geq 0$ satisfying*

$$\mu(\{x \in X \mid d_Y(f(x), g(x)) > \varepsilon\}) \leq \varepsilon.$$

The function d^{μ}_{KF} is a metric on the set of μ-measurable maps from X to Y by identifying two maps if they are equal to each other μ-almost everywhere. We call d^{μ}_{KF} the Ky Fan metric.

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Definition 2.5 (Parameter). *Let $I := [0,1)$ and let X be an mm-space. A map $\varphi\colon I \to X$ is called a parameter of X if φ is a Borel measurable map such that*

$$\varphi_* \mathcal{L}^1 = \mu_X,$$

where $\mathcal{L}^1$ is the one-dimensional Lebesgue measure on I.

Note that any mm-space has a parameter (see [18, lemma 4.2]).

Definition 2.6 (Box distance). *We define the box distance $\square(X,Y)$ between two mm-spaces X and Y to be the infimum of $\varepsilon \geq 0$ satisfying that there exist parameters $\varphi\colon I \to X$, $\psi\colon I \to Y$, and a Borel subset $I_0 \subset I$ with $\mathcal{L}^1(I_0) \geq 1-\varepsilon$ such that*

$$|d_X(\varphi(s),\varphi(t)) - d_Y(\psi(s),\psi(t))| \leq \varepsilon$$

for any $s,t \in I_0$.

Theorem 2.7 ([18, theorem 4.10]). *The box distance function $\square$ is a complete separable metric on $\mathcal{X}$.*

Various distances equivalent to the box distance are defined and studied, for example, the Gromov–Prokhorov distance introduced by Greven–Pfaffelhuber–Winter [5].

Theorem 2.8 ([12, theorem 3.1], [18, remark 4.16]). *For any two mm-spaces X and Y, we have*

$$\square(X,Y) = d_{\mathrm{GP}}((X, 2d_X, \mu_X), (Y, 2d_Y, \mu_Y)),$$

where $d_{\mathrm{GP}}(X,Y)$ is the Gromov–Prokhorov metric defined to be the infimum of $d_{\mathrm{P}}(\mu_X, \mu_Y)$ for all metrics on the disjoint union of X and Y that are extensions of d_X and d_Y. In particular,

$$d_{\mathrm{GP}}(X,Y) \leq \square(X,Y) \leq 2d_{\mathrm{GP}}(X,Y).$$

The topology induced from the box distance has historically various names, for example, the weak-Gromov topology. However, we call it the *box topology* in this article.

The total variation distance is useful for estimating the box distance.

Definition 2.9 (Total variation distance). *The total variation distance $d_{\mathrm{TV}}(\mu,\nu)$ of two Borel probability measures μ and ν on a topological space X is defined by*

$$d_{\mathrm{TV}}(\mu,\nu) := \sup_A |\mu(A) - \nu(A)|,$$

where A runs over all Borel subsets of X.

If μ and ν are both absolutely continuous with respect to a Borel measure ω on X, then

$$d_{\mathrm{TV}}(\mu,\nu) = \frac{1}{2}\int_X \left| \frac{d\mu}{d\omega} - \frac{d\nu}{d\omega} \right| d\omega,$$

where $\frac{d\mu}{d\omega}$ is the Radon–Nikodym derivative of μ with respect to ω.

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Proposition 2.10 ([18, proposition 4.12]). *For any two Borel probability measures μ and ν on a complete separable metric space X, we have*

$$\square((X,\mu),(X,\nu)) \le 2d_{\mathrm{P}}(\mu,\nu) \le 2d_{\mathrm{TV}}(\mu,\nu).$$

Given an mm-space X and a parameter $\varphi\colon I \to X$ of X, we set

$$\varphi^*\mathcal{L}ip_1(X) := \{f \circ \varphi \mid f\colon X \to \mathbb{R} \text{ is 1-Lipschitz}\},$$

which consists of Borel measurable functions on I.

Definition 2.11 (Observable distance). *We define the observable distance $d_{\mathrm{conc}}(X,Y)$ between two mm-spaces X and Y by*

$$d_{\mathrm{conc}}(X,Y) := \inf_{\varphi,\psi} d_{\mathrm{H}}(\varphi^*\mathcal{L}ip_1(X), \psi^*\mathcal{L}ip_1(Y)),$$

where $\varphi\colon I \to X$ and $\psi\colon I \to Y$ run over all parameters of X and Y, respectively, and d_{H} is the Hausdorff distance with respect to the metric $d_{\mathrm{KF}}^{\mathcal{L}^1}$.

Theorem 2.12 ([18, proposition 5.5 and theorem 5.13]). *The observable distance function d_{conc} is a metric on $\mathcal{X}$. Moreover, for any two mm-spaces X and Y,*

$$d_{\mathrm{conc}}(X,Y) \le \square(X,Y).$$

We call the topology on $\mathcal{X}$ induced from d_{conc} the *concentration topology*. We say that a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces *concentrates* to an mm-space X if X_n d_{conc}-converges to X as $n \to \infty$.

2.3. Pyramid

Definition 2.13 (Pyramid). *A subset $\mathcal{P} \subset \mathcal{X}$ is called a pyramid if it satisfies the following (i) – (iii).*

(i) If $X \in \mathcal{P}$ and $Y \prec X$, then $Y \in \mathcal{P}$.
(ii) For any $Y, Y' \in \mathcal{P}$, there exists $X \in \mathcal{P}$ such that $Y \prec X$ and $Y' \prec X$.
(iii) $\mathcal{P}$ is non-empty and $\square$-closed.

We denote the set of all pyramids by Π. Note that Gromov's definition of a pyramid is only by (i) and (ii). The condition (iii) is added in [18].

For an mm-space X, we define

$$\mathcal{P}_X := \{Y \in \mathcal{X} \,|\, Y \prec X\},$$

which is a pyramid. We call $\mathcal{P}_X$ the pyramid associated with X.

We observe that $Y \prec X$ if and only if $\mathcal{P}_Y \subset \mathcal{P}_X$. Note that $\mathcal{X}$ itself is a pyramid.

We define the weak convergence of pyramids as follows. This is exactly the Kuratowski–Painlevé convergence as closed subsets of $(\mathcal{X}, \square)$ (see [11, §8]).

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Definition 2.14 (Weak convergence). *Let $\mathcal{P}$ and $\mathcal{P}_n$, $n = 1, 2, \ldots$, be pyramids. We say that $\mathcal{P}_n$ converges weakly to $\mathcal{P}$ as $n \to \infty$ if the following (i) and (ii) are both satisfied.*

(i) For any mm-space $X \in \mathcal{P}$, we have

$$\lim_{n\to\infty} \square(X, \mathcal{P}_n) = 0.$$

(ii) For any mm-space $X \in \mathcal{X} \setminus \mathcal{P}$, we have

$$\liminf_{n\to\infty} \square(X, \mathcal{P}_n) > 0.$$

Theorem 2.15 ([18, Section 6]). *There exists a metric ρ on Π such that the following (i)–(iv) hold.*

(i) ρ is compatible with weak convergence.
(ii) Π is ρ-compact.
(iii) The map $\mathcal{X} \ni X \mapsto \mathcal{P}_X \in \Pi$ is a 1-Lipschitz topological embedding map with respect to d_{conc} and ρ.
(iv) The image of $\mathcal{X}$ is ρ-dense in Π.

In particular, (Π, ρ) is a compactification of $(\mathcal{X}, d_{\mathrm{conc}})$. We call (Π, ρ) the *pyramidal compactification* of $(\mathcal{X}, d_{\mathrm{conc}})$. We often identify X with $\mathcal{P}_X$, and we say that a sequence of mm-spaces *converges weakly* to a pyramid if the associated pyramid converges weakly.

Definition 2.16 (Approximation of a pyramid). *A sequence $\{Y_m\}_{m=1}^\infty$ of mm-spaces is called an approximation of a pyramid $\mathcal{P}$ provided that it satisfies*

$$Y_1 \prec Y_2 \prec \cdots \prec Y_m \prec \cdots \quad \text{and} \quad \overline{\bigcup_{m=1}^\infty \mathcal{P}_{Y_m}}^{\square} = \mathcal{P}.$$

In particular, $\{Y_m\}_{m=1}^\infty$ converges weakly to $\mathcal{P}$ as $m \to \infty$ and $Y_m \in \mathcal{P}$ for all m.

It is known that any pyramid $\mathcal{P}$ admits an approximation (see [18, lemma 7.14]).

2.4. Separation distance

The separation distance is one of the most fundamental invariants of an mm-space and a pyramid.

Definition 2.17 (Separation distance). *Let X be an mm-space. For any real numbers $\kappa_0, \kappa_1, \ldots, \kappa_N > 0$ with $N \geq 1$, we define the separation distance*

$$\mathrm{Sep}(X; \kappa_0, \kappa_1, \ldots, \kappa_N)$$

of X as the supremum of $\min_{i\neq j} d_X(A_i, A_j)$ over all sequences of $N+1$ Borel subsets $A_0, A_1, \ldots, A_N \subset X$ satisfying $\mu_X(A_i) \geq \kappa_i$ for all $i = 0, 1, \ldots, N$. If $\kappa_i > 1$

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for some i*, then we define* $\mathrm{Sep}(X;\kappa_0,\kappa_1,\ldots,\kappa_N) := 0$*. Moreover, we define the separation distance of a pyramid* $\mathcal{P}$ *by*

$$\mathrm{Sep}(\mathcal{P};\kappa_0,\kappa_1,\ldots,\kappa_N) := \lim_{\delta\to 0+}\sup_{X\in\mathcal{P}}\mathrm{Sep}(X;\kappa_0-\delta,\kappa_1-\delta,\ldots,\kappa_N-\delta)\ (\le +\infty).$$

The separation distance for mm-spaces is an invariant under mm-isomorphism. Note that

$$\mathrm{Sep}(\mathcal{P}_X;\kappa_0,\kappa_1,\ldots,\kappa_N) = \mathrm{Sep}(X;\kappa_0,\kappa_1,\ldots,\kappa_N)$$

for any $\kappa_0,\kappa_1,\ldots,\kappa_N > 0$ and that $\mathrm{Sep}(\mathcal{P};\kappa_0,\kappa_1,\ldots,\kappa_N)$ is monotone non-increasing and left-continuous in κ_i for each $i = 0,1,\ldots,N$, and that

$$\mathrm{Sep}(\mathcal{P};\kappa_0,\kappa_1,\ldots,\kappa_N) \le \mathrm{Sep}(\mathcal{P}';\kappa_0,\kappa_1,\ldots,\kappa_N) \quad \text{if } \mathcal{P}\subset\mathcal{P}'.$$

THEOREM 2.18 ([16, theorem 1.1], Limit formula for separation distance). *Let* $\mathcal{P}$ *and* $\mathcal{P}_n$*,* $n = 1,2,\ldots,$ *be pyramids. If* $\mathcal{P}_n$ *converges weakly to* $\mathcal{P}$ *as* $n\to\infty$*, then*

$$\begin{aligned}\mathrm{Sep}(\mathcal{P};\kappa_0,\kappa_1,\ldots,\kappa_N) &= \lim_{\varepsilon\to 0+}\liminf_{n\to\infty}\mathrm{Sep}(\mathcal{P}_n;\kappa_0-\varepsilon,\kappa_1-\varepsilon,\ldots,\kappa_N-\varepsilon)\\ &= \lim_{\varepsilon\to 0+}\limsup_{n\to\infty}\mathrm{Sep}(\mathcal{P}_n;\kappa_0-\varepsilon,\kappa_1-\varepsilon,\ldots,\kappa_N-\varepsilon)\end{aligned}$$

for any $\kappa_0,\kappa_1,\ldots,\kappa_N > 0$*.*

3. Scale-change action

In this section, we prove theorems 1.1–1.3.

Let $\mathbb{R}_+ := (0,+\infty)$ be the multiplicative group of positive real numbers. We consider the scale-change action on $\mathcal{X}$;

$$\mathbb{R}_+\times\mathcal{X}\ni(t,X)\mapsto tX := (X,td_X,\mu_X)\in\mathcal{X}.$$

The one-point space $*$ is the only fixed-point of this action and the set $\mathcal{X}_* := \mathcal{X}\setminus\{*\}$ is invariant. The $\mathbb{R}_+$-action on $\mathcal{X}_*$ is free. Let $\Sigma := \mathcal{X}_*/\mathbb{R}_+$ be the quotient space of $\mathcal{X}_*$ and $\pi\colon\mathcal{X}_*\to\Sigma$ the quotient map. We denote the orbit $\pi(X)$ by $[X]$.

Simultaneously, we consider the scale-change action on Π;

$$\mathbb{R}_+\times\Pi\ni(t,\mathcal{P})\mapsto t\mathcal{P} := \{tX \mid X\in\mathcal{P}\}\in\Pi,$$

which is a natural extension of the action on $\mathcal{X}$. Denote by $\mathrm{Fix}(\Pi)$ the set of fixed-points of this action, and put $\Pi_* := \Pi\setminus\mathrm{Fix}(\Pi)$. Then, $\mathbb{R}_+$ acts on Π_* freely.

For the proof of theorem 1.1, we need a lemma.

LEMMA 3.1. *Let* Y_ε *be the mm-space defined by* $Y_\varepsilon := (\{0,1\},|\cdot|,(1-\varepsilon)\delta_0+\varepsilon\delta_1)$ *for* $0<\varepsilon<1$*. Then, for any closed subset* V *of* Σ *with non-empty interior with respect to the box topology, there exists a* $\delta(V) > 0$ *such that* $[Y_\varepsilon]$ *belongs to* V *for any* ε *with* $0<\varepsilon\le\delta(V)$*.*

Proof. Let V be a closed subset $V\subset\Sigma$ with non-empty interior. Since any mm-space can be approximated by an mm-space with finite diameter, there is an mm-space X with finite diameter such that $[X]$ is an interior point of V. Suppose that

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$[Y_\varepsilon]$ does not belong to V. Then, since Y_ε is an element in the open set $\pi^{-1}(\Sigma \setminus V)$, there is a large number $r_\varepsilon > 0$ such that the mm-space Z_ε defined by

$$Z_\varepsilon := X \sqcup \{z\}, \quad d_{Z_\varepsilon}|_{X\times X} := d_X, \quad d_{Z_\varepsilon}(z, X) := r_\varepsilon, \quad \mu_{Z_\varepsilon} := (1-\varepsilon)\mu_X + \varepsilon\delta_z$$

satisfies $r_\varepsilon^{-1} Z_\varepsilon \in \pi^{-1}(\Sigma \setminus V)$. Indeed, $\square(Y_\varepsilon, r_\varepsilon^{-1} Z_\varepsilon)$ is sufficiently small since $X \subset Z_\varepsilon$ is close to a one-point by scaling down Z_ε with $r_\varepsilon^{-1} d_{Z_\varepsilon}(z, X) = 1$. On the other hand, Z_ε $\square$-converges to X as $\varepsilon \to 0+$, which implies $[Z_\varepsilon] \in V$ for $\varepsilon > 0$ small enough. This is a contradiction. Thus, $[Y_\varepsilon]$ belongs to V for every sufficiently small $\varepsilon > 0$. This completes the proof. □

Lemma 3.1 implies the following.

Lemma 3.2. *For the quotient of the box topology on Σ, any two distinct points in Σ cannot be separated by any closed neighbourhoods. In particular, Σ is not a Urysohn space.*

Proof. We take two distinct points $[X], [X'] \in \Sigma$ and take any closed neighbourhoods V, V' of $[X]$, $[X']$, respectively. Lemma 3.1 proves that $[Y_\varepsilon]$ belongs to both V and V' for $0 < \varepsilon \leq \min\{\delta(V), \delta(V')\}$. This completes the proof. □

Remark 3.3. As is proved in proposition 3.10, Σ is Hausdorff. In lemmas 3.1 and 3.2, to consider closed neighbourhoods is essential.

Corollary 3.4. *For the quotient of the concentration topology, Σ is not Urysohn. Moreover, $\Pi_*/\mathbb{R}_+$ is not Urysohn.*

Proof. Since the quotient of the concentration topology is coarser than that of the box topology on Σ, lemma 3.2 implies the first statement of the corollary. Since Σ is contained in $\Pi_*/\mathbb{R}_+$ as a subspace, we obtain the second. This completes the proof. □

Proof of theorem 1.1. Suppose that $\mathcal{X}_*$ is homeomorphic to $\Sigma \times \mathbb{R}_+$. Since $\mathcal{X}_*$ is a metric space, $\Sigma \simeq \Sigma \times \{1\}$ is metrizable, which contradicts lemma 3.2 for the box topology and corollary 3.4 for the concentration topology. In the same way, $\mathcal{X}$ is not homeomorphic to the cone over Σ. This completes the proof. □

In the same way as above, we see the following.

Theorem 3.5 *Π_* is not homeomorphic to $(\Pi_*/\mathbb{R}_+) \times \mathbb{R}_+$.*

The rest of this section is devoted to prove theorems 1.2 and 1.3.

We first assume that $\mathcal{X}$ is equipped with the box topology and prove theorem 1.2 for the box topology.

Proposition 3.6. *$\pi\colon \mathcal{X}_* \to \Sigma$ is a principal $\mathbb{R}_+$-bundle.*

Proof. We verify that if sequences $\{X_n\}_{n=1}^\infty \subset \mathcal{X}_*$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ satisfy that X_n and $t_n X_n$ $\square$-converge to $X \in \mathcal{X}_*$ and tX, respectively, as $n \to \infty$, then t_n converges

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to t. Since $\mathcal{X} \in \mathcal{X}_*$, there exist real numbers $\kappa_0, \ldots, \kappa_N > 0$ with $\sum_{i=0}^{N} \kappa_i < 1$ such that

$$0 < \mathrm{Sep}(X; \kappa_0, \ldots, \kappa_N) < +\infty.$$

Suppose that $t_n \not\to t$. There exist a real number $\delta > 0$ and a subsequence $\{t_{n_k}\}_{k=1}^{\infty}$ such that either $t_{n_k} > t + \delta$ for any k or $t_{n_k} < t - \delta$ for any k. By applying theorem 2.18, if $t_{n_k} > t + \delta$, then we have

$$\begin{aligned} t\ \mathrm{Sep}(X; \kappa_0, \ldots, \kappa_N) &= \mathrm{Sep}(tX; \kappa_0, \ldots, \kappa_N) \\ &= \lim_{\varepsilon \to 0+} \limsup_{k \to \infty} \mathrm{Sep}(t_{n_k} X_{n_k}; \kappa_0 - \varepsilon, \ldots, \kappa_N - \varepsilon) \\ &\geq (t + \delta) \lim_{\varepsilon \to 0+} \limsup_{k \to \infty} \mathrm{Sep}(X_{n_k}; \kappa_0 - \varepsilon, \ldots, \kappa_N - \varepsilon) \\ &= (t + \delta)\, \mathrm{Sep}(X; \kappa_0, \ldots, \kappa_N), \end{aligned}$$

which implies the contradiction $t \geq t + \delta$. Similarly, if $t_{n_k} < t - \delta$, then the contradiction $t \leq t - \delta$ holds. Thus, we obtain $t_n \to t$. The proof is completed. □

Let us prove the local triviality of the principal fibre bundle $\pi \colon \mathcal{X}_* \to \Sigma$. Let Δ be a real number with $0 < \Delta < 1$ and put

$$\mathcal{X}_\Delta := \{X \in \mathcal{X} \mid \mu_X(\{x\}) < \Delta \text{ for all } x \in X\}.$$

We remark that $\sup_{x \in X} \mu_X(\{x\}) < \Delta$ if and only if $\mathrm{diam}(X; \Delta) > 0$, where $\mathrm{diam}(X; \alpha)$ is the *partial diameter*, which is a fundamental invariant of an mm-space, given by

$$\mathrm{diam}(X; \alpha) := \inf \{\mathrm{diam}\, A \mid A \text{is a Borel subset with } \mu_X(A) \geq \alpha\}.$$

We see that $\mathcal{X}_\Delta$ is open. We have $\mathcal{X}_\Delta \subset \mathcal{X}_{\Delta'}$ for $\Delta \leq \Delta'$, and

$$\mathcal{X}_* = \bigcup_{0 < \Delta < 1} \mathcal{X}_\Delta.$$

Since $X \in \mathcal{X}_\Delta$ implies $tX \in \mathcal{X}_\Delta$ for any $t > 0$, the set $\mathcal{X}_\Delta$ is invariant with respect to the $\mathbb{R}_+$-action. Put $\Sigma_\Delta := \mathcal{X}_\Delta / \mathbb{R}_+$. Then, for the local triviality, it is sufficient to prove that $\mathcal{X}_\Delta$ is homeomorphic to $\Sigma_\Delta \times \mathbb{R}_+$ for every $\Delta \in (0, 1)$.

We define a map $r_\Delta \colon \mathcal{X}_\Delta \to \mathbb{R}_+$ by

$$r_\Delta(X) := \int_0^{\frac{\Delta + 1}{2}} \mathrm{diam}(X; s)\, ds, \quad X \in \mathcal{X}_\Delta.$$

Note that, for any $X \in \mathcal{X}_\Delta$,

$$r_\Delta(X) \geq \frac{1 - \Delta}{2}\, \mathrm{diam}(X; \Delta) > 0$$

and that $r_\Delta(tX) = t\, r_\Delta(X)$ for any $t > 0$ since $\mathrm{diam}(tX; s) = t\, \mathrm{diam}(X; s)$ for any $s > 0$.

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Lemma 3.7. ([18, lemma 5.43]). *If a sequence $\{X_n\}_{n=1}^\infty$ of mm-spaces $\square$-converges to an mm-space X, then we have*

$$\operatorname{diam}(X;s) \le \liminf_{n\to\infty} \operatorname{diam}(X_n;s) \le \limsup_{n\to\infty} \operatorname{diam}(X_n;s) \le \lim_{\delta\to 0+} \operatorname{diam}(X;s+\delta)$$

for any $s > 0$.

Lemma 3.8. *The map r_Δ is continuous on $\mathcal{X}_\Delta$.*

Proof. Take any sequence $\{X_n\}_{n=1}^\infty \subset \mathcal{X}_\Delta$ $\square$-converging to an mm-space $X \in \mathcal{X}_\Delta$. Let

$$f_n(s) := \operatorname{diam}(X_n;s) \quad \text{and} \quad f(s) := \operatorname{diam}(X;s)$$

for $s \in [0, \frac{\Delta+1}{2}]$. Since f is non-decreasing on $[0, \frac{\Delta+1}{2}]$, the discontinuous points of f are at most countable. Thus, lemma 3.7 implies f_n converges almost everywhere to f and

$$\limsup_{n\to\infty} \sup_{s\in[0,\frac{\Delta+1}{2}]} f_n(s) \le \operatorname{diam}(X; \frac{\Delta+3}{4}) < +\infty.$$

Therefore, by the dominated convergence theorem,

$$\lim_{n\to\infty} r_\Delta(X_n) = \lim_{n\to\infty} \int_0^{\frac{\Delta+1}{2}} f_n(s)\,ds = \int_0^{\frac{\Delta+1}{2}} f(s)\,ds = r_\Delta(X).$$

The proof is completed. □

Proof of theorem 1.2 for the box topology. The continuous 1-homogeneous map $r_\Delta \colon \mathcal{X}_\Delta \to \mathbb{R}_+$ induces the homeomorphism $\Phi \colon \mathcal{X}_\Delta \to \Sigma_\Delta \times \mathbb{R}_+$ defined by

$$\Phi(X) := ([X], r_\Delta(X)) \quad \text{for } X \in \mathcal{X}_\Delta.$$

Indeed, the inverse map Φ^{-1} is given by

$$\Phi^{-1}([X], t) = r_\Delta(X)^{-1} tX.$$

In other words, the map r_Δ produces the continuous section

$$\Sigma_\Delta \ni [X] \mapsto r_\Delta(X)^{-1} X \in \mathcal{X}_\Delta,$$

so that $\mathcal{X}_\Delta \to \Sigma_\Delta$ is trivial. This implies the local triviality of our principal fibre bundle $\mathbb{R}_+ \to \mathcal{X}_* \to \Sigma$.

Theorem 1.1 proves that the fibre bundle $\mathbb{R}_+ \to \mathcal{X}_* \to \Sigma$ is globally non-trivial. This completes the proof of theorem 1.2 for the box topology. □

The following corollary is a by-product of theorem 1.2.

Corollary 3.9. *There is no continuous 1-homogeneous map $r \colon \mathcal{X}_* \to \mathbb{R}_+$ with respect to the box topology on $\mathcal{X}_*$.*

The following proposition is compared with lemma 3.2.

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PROPOSITION 3.10. *Σ is a Hausdorff space.*

Proof. For any distinct two points $[X], [X'] \in \Sigma$, there exists $0 < \Delta < 1$ such that $[X], [X'] \in \Sigma_\Delta$. Since $\mathcal{X}_\Delta$ and $\Sigma_\Delta \times \mathbb{R}_+$ are homeomorphic, Σ_Δ is metrizable. Thus, the two points $[X], [X']$ can be separated by neighbourhoods in Σ since Σ_Δ is open in Σ. The proof is completed. □

Before proving theorem 1.2 for the concentration topology, we study the scale-change action on Π and prove theorem 1.3 with the help of theorem 1.7, where the proof of theorem 1.7 is deferred to the next section. The concentration case of theorem 1.2 is obtained as a corollary of theorem 1.3.

PROPOSITION 3.11. *The quotient map $\pi\colon \Pi_* \to \Pi_*/\mathbb{R}_+$ is a principal $\mathbb{R}_+$-bundle.*

Proof. Assume that $\{\mathcal{P}_n\}_{n=1}^\infty \subset \Pi_*$ and $\{t_n\}_{n=1}^\infty \subset \mathbb{R}_+$ satisfy that $\mathcal{P}_n$ and $t_n\mathcal{P}_n$ converge weakly to $\mathcal{P} \in \Pi_*$ and $t\mathcal{P}$, respectively, as $n \to \infty$. Since $\mathcal{P} \in \Pi_*$, there exist real numbers $\kappa_0, \ldots, \kappa_N > 0$ with $\sum_{i=0}^N \kappa_i < 1$ such that

$$0 < \operatorname{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N) < +\infty$$

by theorem 1.7. By the same argument as in the proof of proposition 3.6, we have $t_n \to t$ as $n \to \infty$. The proof is completed. □

Proof of theorem 1.3 under assuming theorem 1.7. For any tuple $\kappa = (\kappa_0, \kappa_1, \ldots, \kappa_N)$ of positive real numbers with $\sum_{i=0}^N \kappa_i < 1$, we define

$$\Pi_\kappa := \left\{ \mathcal{P} \in \Pi \;\middle|\; \begin{array}{l} \operatorname{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N) < +\infty \text{and} \\ \operatorname{Sep}(\mathcal{P}; \kappa_0 + \delta, \ldots, \kappa_N + \delta) > 0 \text{for some } \delta > 0 \end{array} \right\}.$$

It is obvious that Π_κ is invariant under $\mathbb{R}_+$-action. We prove that Π_κ is open. Indeed, for any sequence $\{\mathcal{P}_n\}_{n=1}^\infty$ of pyramids convergent weakly to a pyramid $\mathcal{P}$, if $\operatorname{Sep}(\mathcal{P}_n; \kappa_0, \ldots, \kappa_N) = +\infty$, then

$$\operatorname{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N) \geq \limsup_{n\to\infty} \operatorname{Sep}(\mathcal{P}_n; \kappa_0, \ldots, \kappa_N) = +\infty$$

by theorem 2.18, and if $\operatorname{Sep}(\mathcal{P}_n; \kappa_0 + \delta, \ldots, \kappa_N + \delta) = 0$ for any $\delta > 0$, then

$$\operatorname{Sep}(\mathcal{P}; \kappa_0 + \delta, \ldots, \kappa_N + \delta) \leq \liminf_{n\to\infty} \operatorname{Sep}(\mathcal{P}_n; \kappa_0 + \frac{\delta}{2}, \ldots, \kappa_N + \frac{\delta}{2}) = 0$$

for any $\delta > 0$ by theorem 2.18. These imply that Π_κ is open.

We define a map $r_\kappa\colon \Pi_\kappa \to \mathbb{R}_+$ by

$$r_\kappa(\mathcal{P}) := \int_0^1 \operatorname{Sep}(\mathcal{P}; \kappa_0 + s, \ldots, \kappa_N + s)\, ds, \quad \mathcal{P} \in \Pi_\kappa.$$

Then, the map r_κ is continuous and 1-homogeneous on Π_κ, so that $\Pi_\kappa \to \Pi_\kappa/\mathbb{R}_+$ is trivial.

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The rest is to prove

$$\Pi_* = \bigcup_\kappa \Pi_\kappa.$$

By theorem 1.7, the inclusion $\bigcup_\kappa \Pi_\kappa \subset \Pi_*$ is obvious. To prove the reverse inclusion $\Pi_* \subset \bigcup_\kappa \Pi_\kappa$, we take any $\mathcal{P} \in \Pi_*$. By theorem 1.7, there exist real numbers $\kappa_0, \kappa_1, \ldots, \kappa_N > 0$ with $\sum_{i=0}^N \kappa_i < 1$ such that

$$0 < \mathrm{Sep}(\mathcal{P}; \kappa_0, \ldots, \kappa_N) < +\infty.$$

By the left-continuity of $\mathrm{Sep}(\mathcal{P}; s_0, \ldots, s_N)$ in s_i, there is $\varepsilon > 0$ such that

$$\mathrm{Sep}(\mathcal{P}; \kappa_0 - \varepsilon, \ldots, \kappa_N - \varepsilon) < +\infty,$$

which implies $\mathcal{P} \in \Pi_{(\kappa_0-\varepsilon,\ldots,\kappa_N-\varepsilon)}$ and then $\Pi_* \subset \bigcup_\kappa \Pi_\kappa$. Thus, we obtain the local triviality of $\pi\colon \Pi_* \to \Pi_*/\mathbb{R}_+$. The proof is completed. □

Proof of theorem 1.2 for the concentration topology. Assume that $\mathcal{X}$ is equipped with the concentration topology and consider the $\mathbb{R}_+$-action. Then, $\pi\colon \mathcal{X}_* \to \mathcal{X}_*/\mathbb{R}_+$ is the restriction of bundle $\pi\colon \Pi_* \to \Pi_*/\mathbb{R}_+$. Thus, it is also a principal $\mathbb{R}_+$-bundle and locally trivial (see [7]). This bundle is globally non-trivial because of theorem 1.1. We finish the proof. □

Corollary 3.12.

(i) There is no continuous 1-homogeneous map $r\colon \Pi_* \to \mathbb{R}_+$.
(ii) There is no continuous 1-homogeneous map $r\colon \mathcal{X}_* \to \mathbb{R}_+$ with respect to the concentration topology on $\mathcal{X}_*$.

We also obtain the following proposition.

Proposition 3.13. *Let $\mathcal{X}_{\mathrm{NA}}$ be the space of all non-atomic mm-spaces. Then, $\pi\colon \mathcal{X}_{\mathrm{NA}} \to \mathcal{X}_{\mathrm{NA}}/\mathbb{R}_+$ is a trivial bundle with respect to the box and concentration topologies.*

Proof. For the box topology, since $\mathcal{X}_{\mathrm{NA}}$ is an $\mathbb{R}_+$-invariant subspace of $\mathcal{X}_\Delta$ for any Δ, we see that $\pi\colon \mathcal{X}_{\mathrm{NA}} \to \mathcal{X}_{\mathrm{NA}}/\mathbb{R}_+$ is the restriction of trivial bundle $\pi\colon \mathcal{X}_\Delta \to \Sigma_\Delta$. Similarly, for the concentration topology, $\mathcal{X}_{\mathrm{NA}}$ is a subspace of Π_κ for any κ. This implies the triviality. □

4. Scale-invariant pyramids

In this section, we prove theorems 1.7 and 1.8. We recall that $\mathcal{A}$ is the set of all monotone non-increasing sequences of non-negative real numbers with total sum not greater than 1. We equip $\mathcal{A}$ with the weak topology as a closed convex subset of the space l^2. In particular, $\mathcal{A}$ is compact. For every $A = \{a_i\}_{i=1}^{\infty} \in \mathcal{A}$, we set

$$\mathcal{P}_A := \left\{ X \in \mathcal{X} \,\middle|\, \text{There exists a sequence } \{x_i\}_{n=1}^{\infty} \subset X \text{such that } \sum_{i=1}^{\infty} a_i \delta_{x_i} \le \mu_X \right\}.$$

We remark that $\mathcal{P}_A$ is a pyramid a priori, where the $\square$-closedness of $\mathcal{P}_A$ follows from the argument in the proof of claim 4.2 below.

4.1. Characterization of Fix(Π)

In order to prove theorem 1.7, we need a lemma.

LEMMA 4.1. *Let $a_1, \ldots, a_k$, $k < +\infty$, and ε be positive numbers with $\sum_{i=1}^{k} a_i < 1$ and $\min_{i=1,\ldots,k} a_i \ge \varepsilon > 0$. If an mm-space X admits distinct k points $\{x_i\}_{i=1}^{k}$ with*

$$\mu_X(\{x\}) \begin{cases} \ge a_i & \text{if } x = x_i, \\ < \varepsilon & \text{if } x \ne x_i, \end{cases} \quad \text{and} \quad \sum_{i=1}^{k} (\mu_X(\{x_i\}) - a_i) < \varepsilon,$$

then there exist positive numbers $\kappa_0, \ldots, \kappa_N > 0$ such that $\kappa_i \le \varepsilon$ for every i,

$$0 < 1 - \sum_{i=1}^{k} a_i - \sum_{i=0}^{N} \kappa_i \le \varepsilon, \quad \text{and} \quad \mathrm{Sep}(X; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) > 0.$$

Proof. Take any mm-space X having distinct k points $\{x_i\}_{i=1}^{k}$ with

$$\mu_X(\{x\}) \begin{cases} \ge a_i & \text{if } x = x_i, \\ < \varepsilon & \text{if } x \ne x_i. \end{cases}$$

Let $\{\xi_i\}_{i=1}^{\infty}$ be a countable dense subset of X and let $d_i := d_X(\xi_i, \cdot)$. Put

$$X_0 := X \setminus \bigcup_{i=1}^{k} U_{\varepsilon'}(x_i)$$

for some sufficiently small $\varepsilon' > 0$ with $\mu_X(X \setminus X_0) - \sum_{i=1}^{k} \mu_X(\{x_i\}) =: \eta_0 \ll \varepsilon$. We first find ε-atomic points $\alpha_1, \ldots, \alpha_m$ of $d_{1*}(\mu_X \lfloor_{X_0})$, i.e.,

$$d_{1*}(\mu_X \lfloor_{X_0})(\{t\}) \begin{cases} \ge \varepsilon & \text{if } t = \alpha_i, \\ < \varepsilon & \text{if } t \ne \alpha_i, \end{cases}$$

if these exist. We put

$$X_i := d_1^{-1}(\{\alpha_i\}) \cap X_0, \; i = 1, \ldots, m, \quad \text{and} \quad X^{(1)} := \bigcup_{i=1}^{m} X_i \subset X_0.$$

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There exist finitely many disjoint closed intervals $I_0, \dots, I_{l_1}$ of $\mathbb{R}$ such that

$$0 < d_{1*}(\mu_X\lfloor_{X_0})(I_i) =: \kappa_i \le \varepsilon,\ i = 0, \dots, l_1,$$
$$\text{and} \quad d_{1*}(\mu_X\lfloor_{X_0})(\mathbb{R} \setminus (\{\alpha_1, \dots, \alpha_m\} \sqcup \bigsqcup_{i=0}^{l_1} I_i)) =: \eta_1 \ll \varepsilon.$$

We put $A_i := d_1^{-1}(I_i) \cap X_0$. Note that $\mu_X(A_i) = \kappa_i$.

We next find ε-atomic points $\alpha_{i1}, \dots, \alpha_{im_i}$ of $d_{2*}(\mu_X\lfloor_{X_i})$ for $i = 1, \dots, m$ if these exist, and we set

$$X_{ij} := d_2^{-1}(\{\alpha_{ij}\}) \cap X_i,\ j = 1, \dots, m_i, \quad \text{and} \quad X^{(2)} := \bigcup_{i=1}^{m} \bigcup_{j=1}^{m_i} X_{ij} \subset X^{(1)}.$$

There exist finitely many disjoint closed intervals $I_{l_1+1}, \dots, I_{l_1+l_2}$ such that

$$0 < d_{2*}(\mu_X\lfloor_{X_{i(j)}})(I_j) =: \kappa_j \le \varepsilon,\ j = l_1+1, \dots, l_1+l_2, \text{for some } i(j) \in \{1, \dots, m\},$$
$$\text{and} \quad d_{2*}(\mu_X\lfloor_{X^{(1)}})(\mathbb{R} \setminus (\{\alpha_{ij}\}_{i,j} \sqcup \bigsqcup_{j=l_1+1}^{l_1+l_2} I_j)) =: \eta_2 \ll \varepsilon.$$

We put $A_j := d_2^{-1}(I_j) \cap X_{i(j)}$ for every $j = l_1+1, \dots, l_1+l_2$.

Repeating this construction, we obtain a monotone sequence $X^{(1)} \supset X^{(2)} \supset \cdots$ and a disjoint family $\{A_i\}$ on X.

We prove that $X^{(n)} = \emptyset$ for some n. It is sufficient to prove $\bigcap_{n=1}^{\infty} X^{(n)} = \emptyset$ since this implies $\mu_X(X^{(n)}) \to 0$ as $n \to \infty$, which contradicts the fact $\mu_X(X^{(n)}) \ge \varepsilon$ if $X^{(n)} \neq \emptyset$. Suppose that there exists a point $x_0 \in \bigcap_{n=1}^{\infty} X^{(n)} \neq \emptyset$. Then, there exists a sequence $\{i_k\}_k$ such that $x_0 \in X_{i_1 i_2 \cdots i_k}$ for any k, where $i_k \in \{1, \dots, m_{i_1 i_2 \cdots i_{k-1}}\}$. Then it holds that

$$\bigcap_{k=1}^{\infty} X_{i_1 i_2 \cdots i_k} = \{x_0\}.$$

Indeed, suppose that there exists another point $y \in \bigcap_{k=1}^{\infty} X_{i_1 i_2 \cdots i_k}$ and let $r_0 := d_X(x_0, y)/4$. There exists a sufficiently large k_0 such that $d_X(\xi_{k_0}, x_0) \le r_0$. Since $x_0, y \in X_{i_1 i_2 \cdots i_{k_0}}$, we have

$$d_X(\xi_{k_0}, x_0) = d_X(\xi_{k_0}, y) = \alpha_{i_1 i_2 \cdots i_{k_0}},$$

which implies the contradiction $4r_0 = d_X(x_0, y) \le d_X(\xi_{k_0}, x_0) + d_X(\xi_{k_0}, y) \le 2r_0$.

Thus, we obtain

$$\mu_X(\{x_0\}) = \mu_X(\bigcap_{k=1}^{\infty} X_{i_1 i_2 \cdots i_k}) = \lim_{k \to \infty} \mu_X(X_{i_1 i_2 \cdots i_k}) \ge \varepsilon,$$

but this contradicts the assumption of this lemma.

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Therefore, we have

$$\mathrm{Sep}(X; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) \geq \min\{\min_{i \neq j} d_X(x_i, x_j), \min_{i,j} d_X(x_i, A_j),$$
$$\min_{i \neq j} d_X(A_i, A_j)\} > 0,$$

where $N = l_1 + l_2 + \cdots + l_n$, and

$$\begin{aligned} 1 &= \mu_X(X_0) + \sum_{i=1}^{k} \mu_X(\{x_i\}) + \eta_0 \\ &= \mu_X(X^{(1)}) + \sum_{i=0}^{l_1} \kappa_i + \eta_1 + \sum_{i=1}^{k} \mu_X(\{x_i\}) + \eta_0 \\ &= \mu_X(X^{(2)}) + \sum_{i=0}^{l_1+l_2} \kappa_i + \sum_{i=0}^{2} \eta_i + \sum_{i=1}^{k} \mu_X(\{x_i\}) \\ &= \mu_X(X^{(n)}) + \sum_{i=0}^{N} \kappa_i + \sum_{i=0}^{n} \eta_i + \sum_{i=1}^{k} \mu_X(\{x_i\}). \end{aligned}$$

Therefore, taking η_i with $\sum_{i=0}^{n} \eta_i \leq \varepsilon - \sum_{i=1}^{k}(\mu_X(\{x_i\}) - a_i)$, we obtain the conclusion. □

Proof of theorem 1.7. Since (iv) ⇒ (i) ⇒ (ii) ⇒ (iii) is trivial, we prove (iii) ⇒ (iv). Assume that a pyramid $\mathcal{P}$ satisfies the condition (iii). Let $\{Y_n\}_{n=1}^{\infty}$ be an approximation of $\mathcal{P}$ and let

$$Y_\infty := \varprojlim Y_n$$

be the inverse limit of $\{Y_n\}_{n=1}^{\infty}$. There exists a probability measure μ_{Y_∞} such that

$$\pi_{n*}\mu_{Y_\infty} = \mu_{Y_n}$$

for any n, where $\pi_n \colon Y_\infty \to Y_n$ is the projection (see [2]). Note that Y_∞ admits an extended metric that can take values in $[0, +\infty]$ and π_n is 1-Lipschitz.

Let $\{y_i\}_{i=1}^{M} \subset Y_\infty$, $M \leq +\infty$, be the sequence of atomic points of μ_{Y_∞} and let

$$a_i := \mu_{Y_\infty}(\{y_i\})$$

for every i. By relabeling, we can assume that

$$a_1 \geq a_2 \geq \cdots \geq a_M \geq 0 =: a_{M+1} = a_{M+2} = \cdots$$

and then $A := \{a_i\}_{i=1}^{\infty} \in \mathcal{A}$. Then, for any n, we have

$$\sum_{i=1}^{M} a_i \delta_{\pi_n(y_i)} = \pi_{n*}(\sum_{i=1}^{M} a_i \delta_{y_i}) \leq \pi_{n*}\mu_{Y_\infty} = \mu_{Y_n}.$$

□

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Claim 4.2. *It holds that*

$$\mathcal{P} \subset \mathcal{P}_A.$$

Proof. Take any $X \in \mathcal{P}$. Then there exist Borel maps $f_n\colon Y_n \to X$ such that f_n is 1-Lipschitz up to ε_n with non-exceptional domain $\tilde{Y}_n$, that is, $\mu_{Y_n}(\tilde{Y}_n) \geq 1 - \varepsilon_n$ and

$$d_X(f_n(y), f_n(y')) \leq d_{Y_n}(y, y') + \varepsilon_n$$

for any $y, y' \in \tilde{Y}_n$, and $d_{\mathrm{P}}(f_{n*}\mu_{Y_n}, \mu_X) \leq \varepsilon_n$ for some $\varepsilon_n \to 0$ (see [10, lemma 4.6]).

We prove that, for each $i = 1, 2, \ldots, M$, the sequence $\{f_n \circ \pi_n(y_i)\}_{n=1}^{\infty} \subset X$ has a convergent subsequence. Suppose that $\{f_n \circ \pi_n(y_i)\}_{n=1}^{\infty}$ has no convergent subsequence. Then, there exist a real number $\eta > 0$ and a subsequence $\{n_k\}_{k=1}^{\infty}$ of $\{n\}$ such that $\{B_\eta(f_{n_k} \circ \pi_{n_k}(y_i))\}_{k=1}^{\infty}$ is a disjoint family. Since $\pi_{n_k}(y_i) \in \tilde{Y}_{n_k}$ for sufficiently large k and

$$B_{\frac{\eta}{2}}(\pi_{n_k}(y_i)) \cap \tilde{Y}_{n_k} \subset f_{n_k}^{-1}(B_{\frac{\eta}{2}+\varepsilon_{n_k}}(f_{n_k} \circ \pi_{n_k}(y_i))),$$

we have

$$\begin{aligned}0 < a_i &\leq \mu_{Y_{n_k}}(\{\pi_{n_k}(y_i)\}) \leq \mu_{Y_{n_k}}(B_{\frac{\eta}{2}}(\pi_{n_k}(y_i))) \\ &\leq f_{n_k*}\mu_{Y_{n_k}}(B_{\frac{\eta}{2}+\varepsilon_{n_k}}(f_{n_k} \circ \pi_{n_k}(y_i))) + \varepsilon_{n_k} \leq \mu_X(B_\eta(f_{n_k} \circ \pi_{n_k}(y_i))) + 2\varepsilon_{n_k}\end{aligned}$$

for sufficiently large k, which contradicts the disjointness of $\{B_\eta(f_{n_k} \circ \pi_{n_k}(y_i))\}_{n=1}^{\infty}$. Thus, $\{f_n \circ \pi_n(y_i)\}_{n=1}^{\infty}$ has a convergent subsequence.

Let

$$x_i := \lim_{k\to\infty} f_{n_k} \circ \pi_{n_k}(y_i),$$

where $\{n_k\}$ is a subsequence of $\{n\}$ such that $\{f_{n_k} \circ \pi_{n_k}(y_i)\}_{k=1}^{\infty}$ converges for any $i = 1, 2, \ldots, M$ (by the diagonal argument). We prove

$$\sum_{i=1}^{M} a_i \delta_{x_i} \leq \mu_X.$$

For any non-negative bounded continuous function $\varphi\colon X \to [0, +\infty)$, by Fatou's lemma, we have

$$\sum_{i=1}^{M} a_i\, \varphi(x_i) \leq \liminf_{k\to\infty} \sum_{i=1}^{M} a_i\, \varphi(f_{n_k} \circ \pi_{n_k}(y_i)) \leq \lim_{k\to\infty} \int_{Y_{n_k}} \varphi \circ f_{n_k}\, d\mu_{Y_{n_k}} = \int_X \varphi\, d\mu_X,$$

which implies $\sum_{i=1}^{M} a_i \delta_{x_i} \leq \mu_X$. Therefore, the proof of this claim is completed. □

We next prove the converse inclusion under the condition (iii). We take any $\varepsilon > 0$ and any positive integer k such that

$$k = \sup\{i \mid a_i \geq \varepsilon\}.$$

Then, for sufficiently large n, since $\pi_n\colon \{y_1, \ldots, y_k\} \to Y_n$ is injective, we have

$$\mu_{Y_n}(\{y\}) \begin{cases} \geq a_i & \text{if } y = \pi_n(y_i), \\ < \varepsilon & \text{if } y \neq \pi_n(y_i), \end{cases} \quad \text{and} \quad \sum_{i=1}^{k} (\mu_{Y_n}(\{\pi_n(y_i)\}) - a_i) < \varepsilon.$$

Indeed, if not, then the atomic part of μ_{Y_∞} on the inverse limit Y_∞ is not equal to $\sum_{i=1}^{M} a_i \delta_{y_i}$. Thus, by lemma 4.1, there exist $\kappa_0, \ldots, \kappa_N > 0$ such that $\kappa_i \leq \varepsilon$ for every i,

$$0 < 1 - \sum_{i=1}^{k} a_i - \sum_{i=0}^{N} \kappa_i \leq \varepsilon, \quad \text{and} \quad \operatorname{Sep}(Y_n; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) > 0$$

for some large n. Thus, we have

$$\operatorname{Sep}(\mathcal{P}; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) \geq \operatorname{Sep}(Y_n; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) > 0.$$

Combining this and the condition (iii) implies that

$$\operatorname{Sep}(\mathcal{P}; a_1, \ldots, a_k, \kappa_0, \ldots, \kappa_N) = \infty.$$

By the limit formula of the separation distance, we have

$$\lim_{\eta \to 0+} \lim_{n \to \infty} \operatorname{Sep}(Y_n; a_1 - \eta, \ldots, a_k - \eta, \kappa_0 - \eta, \ldots, \kappa_N - \eta) = \infty.$$

Now we prove the following claim.

Claim 4.3. *It holds that*

$$\mathcal{P} \supset \mathcal{P}_A.$$

Proof. We take any mm-space X admitting a sequence $\{x_i\}_{i=1}^{\infty} \subset X$ such that $\sum_{i=1}^{\infty} a_i \delta_{x_i} \leq \mu_X$. By the approximation, we can assume that X is finite. Indeed, there exist finite nets $\mathcal{N}_n$ of X and Borel maps $\xi_n\colon X \to \mathcal{N}_n$ such that $\lim_{n\to\infty} d_{\mathrm{P}}(\xi_{n*}\mu_X, \mu_X) = 0$. Then, we have

$$\sum_{i=1}^{\infty} a_i \delta_{\xi_n(x_i)} \leq \xi_{n*}\mu_X,$$

so that X can be approximated keeping our assumption. Let $\{z_1, \ldots, z_m\} := X$ and

$$\nu_X := \mu_X - \sum_{i=1}^{k} a_i \delta_{x_i}.$$

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Since $\lim_{\eta\to 0+}\lim_{n\to\infty}\mathrm{Sep}(Y_n; a_1-\eta,\dots,a_k-\eta,\kappa_0-\eta,\dots,\kappa_N-\eta)=\infty$, there exist Borel subsets $Y_{n,1},\dots,Y_{n,N+k+1}\subset Y_n$ for any sufficiently large n such that

$$\mu_{Y_n}(Y_{n,i})\ge a_i-\eta \text{for } i=1,\dots,k,\quad \mu_{Y_n}(Y_{n,j+k+1})\ge \kappa_j-\eta \text{for } j=0,\dots,N,$$
$$\text{and}\quad \min_{i\ne j} d_{Y_n}(Y_{n,i},Y_{n,j})\ge \operatorname{diam} X$$

for some $\eta<(N+k+1)^{-1}\varepsilon$. We define a Borel map $g_n\colon Y_n\to X$ satisfying

$$g_n(Y_{n,i})=x_i \text{for } i=1,\dots,k \quad\text{and}\quad g_n(Y_{n,j+k+1})=z_l \text{for } j_{l-1}\le j<j_l,$$

where

$$j_0:=0\quad\text{and}\quad j_l:=\max\left\{ j\ge j_{l-1} \,\middle|\, \nu_X(\{z_l\})\ge\sum_{i=j_{l-1}}^{j-1}\kappa_i\right\} \text{for } l=1,\dots,m.$$

Letting $\tilde{Y}_n:=\bigcup_{i=1}^{j_m+k}Y_{n,i}$, by the definition of g_n, we have

$$0\le\mu_X(\{z_l\})-g_{n*}(\mu_{Y_n}\lfloor_{\tilde{Y}_n})(\{z_l\})\le 2\varepsilon$$

for any $l=1,\dots,m$. In particular, $1-\mu_{Y_n}(\tilde{Y}_n)\le 2m\varepsilon$. Moreover, for any $B\subset X$, we have

$$g_{n*}\mu_{Y_n}(B)\le g_{n*}(\mu_{Y_n}\lfloor_{\tilde{Y}_n})(B)+2m\varepsilon\le\mu_X(B)+2m\varepsilon.$$

Thus, g_n is 1-Lipschitz up to $2m\varepsilon$ with non-exceptional domain $\tilde{Y}_n$ and $d_{\mathrm{P}}(g_{n*}\mu_{Y_n},\mu_X)\le 2m\varepsilon$. Since $Y_n\in\mathcal{P}$, taking $\varepsilon\to 0$, we obtain $X\in\mathcal{P}$ (see [10, corollary 4.7]). □

We finish the proof of this theorem.

4.2. Topological structure of Fix(Π)

The goal here is to prove theorem 1.8.

LEMMA 4.4. *The map $\mathcal{A}\ni A\mapsto\mathcal{P}_A\in\mathrm{Fix}(\Pi)$ is injective.*

Proof. Take any $A=\{a_i\}_{i=1}^\infty, A'=\{a'_i\}_{i=1}^\infty\in\mathcal{A}$ with $A\ne A'$. There exists a number k such that $a_i=a'_i$ for any $i<k$ and $a_k\ne a'_k$. We can assume that $a_k<a'_k$. An mm-space X is defined as the unit interval $([0,1],|\cdot|)$ with probability measure

$$\mu_X:=\sum_{i=1}^\infty a_i\delta_{2^{-i}}+\left(1-\sum_{i=1}^\infty a_i\right)\mathcal{L}^1,$$

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where $\mathcal{L}^1$ is the Lebesgue measure on $[0,1]$. Then, we have $X \in \mathcal{P}_A$ and $X \notin \mathcal{P}_{A'}$. Indeed, if $X \in \mathcal{P}_{A'}$, then there exists a sequence $\{x_i\}_{i=1}^\infty \subset X$ such that $\sum_{i=1}^\infty a'_i \delta_{x_i} \le \mu_X$. Since A is monotone non-increasing, we have

$$\sum_{i=1}^k a'_i \le \mu_X(\{x_1,\dots,x_k\}) \le \sum_{i=1}^k a_i < \sum_{i=1}^k a'_i,$$

which is a contradiction. Therefore, we obtain $\mathcal{P}_A \neq \mathcal{P}_{A'}$. This completes the proof. □

LEMMA 4.5. *The map $\mathcal{A} \ni A \mapsto \mathcal{P}_A \in \mathrm{Fix}(\Pi)$ is continuous.*

Proof. Assume that $A_n = \{a_{ni}\}_{i=1}^\infty \in \mathcal{A}$ converges weakly to $A = \{a_i\}_{i=1}^\infty \in \mathcal{A}$. Let us prove that $\mathcal{P}_{A_n}$ converges weakly to $\mathcal{P}_A$.

We first prove that $\lim_{n\to\infty} \square(X, \mathcal{P}_{A_n}) = 0$ for any $X \in \mathcal{P}_A$. By the standard approximation, X can be assumed to be a finite mm-space. Take any $\varepsilon > 0$ and find a number k such that $a_{k+1} < \varepsilon$. Then, for sufficiently large n, we have

$$|a_{ni} - a_i| < \frac{\varepsilon}{2^i} \text{for } i = 1,\dots,k \quad \text{and} \quad a_{n,k+1} < \varepsilon.$$

Since A_n is a monotone non-increasing sequence, $a_{n,k+1} < \varepsilon$ implies $\sup_{i>k} a_{ni} < \varepsilon$. Take such large n and fix it. Let $\{x_i\}_{i=1}^\infty$ be a sequence in X such that $\sum_{i=1}^\infty a_i\delta_{x_i} \le \mu_X$ and let $\{y_1,\dots,y_N\} := X$. We define

$$\tilde{X} := \{\xi_1,\dots,\xi_k,\eta_1,\dots,\eta_N\}$$

and define two maps $\varphi\colon \tilde{X} \to X$ and $\psi\colon X \to \tilde{X}$ by

$$\varphi(x) := \begin{cases} x_i & \text{if } x = \xi_i, \\ y_i & \text{if } x = \eta_i, \end{cases} \quad \text{and} \quad \psi(y_i) := \eta_i.$$

We now define two probability measures $\mu_{\tilde{X}}$ and $\mu_{\tilde{X}_n}$ on $\tilde{X}$ as

$$\mu_{\tilde{X}} := \sum_{i=1}^k a_i\delta_{\xi_i} + \psi_*(\mu_X - \sum_{i=1}^k a_i\delta_{x_i}), \quad \mu_{\tilde{X}_n}\lfloor_{\{\xi_1,\dots,\xi_k\}} := \sum_{i=1}^k a_{ni}\delta_{\xi_i}$$

and $\mu_{\tilde{X}_n}\lfloor_{\{\eta_1,\dots,\eta_N\}}$ is determined as follows. Find finitely many real numbers $b_{n1},\dots,b_{nM} \in [0,\varepsilon)$ with

$$\sum_{i=1}^M b_{ni} = 1 - \sum_{i=1}^\infty a_{ni}$$

and set

$$c_{nj} := \begin{cases} b_{nj} & \text{if } 1 \le j \le M, \\ a_{n,j-M+k} & \text{if } j > M. \end{cases}$$

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Note that $\sup_j c_{nj} < \varepsilon$. We define

$$\mu_{\tilde{X}_n}(\{\eta_i\}) := \sum_{j=j_{i-1}+1}^{j_i} c_{nj}$$

for $i = 1, \ldots, N$, where $j_0 := 0$, $j_N := +\infty$, and

$$j_i := \inf \left\{ j > j_{i-1} \;\middle|\; \sum_{l=j_{i-1}+1}^{j} c_{nl} \geq \mu_{\tilde{X}}(\{\eta_i\}) \right\}$$

for $i = 1, \ldots, N-1$. Under $\inf \emptyset = +\infty$, if there exists $i_0 < N$ such that $j_{i_0} = \cdots = j_N = +\infty$, then we understand

$$\mu_{\tilde{X}_n}(\{\eta_{i_0+1}, \ldots, \eta_N\}) = 0.$$

Letting $i_0 := \min \{1 \leq i \leq N \,|\, j_i = +\infty\}$, we have

$$\mu_{\tilde{X}}(\{\eta_i\}) \leq \mu_{\tilde{X}_n}(\{\eta_i\}) \leq \mu_{\tilde{X}}(\{\eta_i\}) + \varepsilon$$

for any $i < i_0$ by the definition. On the other hand, since

$$1 - \sum_{i=1}^{k} a_{ni} = \sum_{j=1}^{\infty} c_{nj} \leq \sum_{i=1}^{i_0} \mu_{\tilde{X}}(\{\eta_i\}) + (i_0 - 1)\varepsilon,$$

we have

$$\sum_{i=i_0+1}^{N} \mu_{\tilde{X}}(\{\eta_i\}) = 1 - \sum_{i=1}^{k} a_i - \sum_{i=1}^{i_0} \mu_{\tilde{X}}(\{\eta_i\}) \leq \sum_{i=1}^{k} |a_{ni} - a_i| + (i_0 - 1)\varepsilon \leq i_0\varepsilon.$$

These imply that

$$\begin{aligned} &|\mu_{\tilde{X}_n}(\{\eta_{i_0}\}) - \mu_{\tilde{X}}(\{\eta_{i_0}\})| \\ &\leq \sum_{i=1}^{i_0-1} |\mu_{\tilde{X}_n}(\{\eta_i\}) - \mu_{\tilde{X}}(\{\eta_i\})| + \sum_{i=i_0+1}^{N} \mu_{\tilde{X}}(\{\eta_i\}) + \sum_{i=1}^{k} |a_{ni} - a_i| \\ &\leq 2i_0\varepsilon \leq 2N\varepsilon. \end{aligned}$$

Hence, we have

$$d_{\mathrm{TV}}(\mu_{\tilde{X}_n}, \mu_{\tilde{X}}) = \frac{1}{2} \sum_{i=1}^{k} |a_{ni} - a_i| + \frac{1}{2} \sum_{i=1}^{N} |\mu_{\tilde{X}_n}(\{\eta_i\}) - \mu_{\tilde{X}}(\{\eta_i\})| \leq 2N\varepsilon.$$

Therefore, letting $X_n := (X, d_X, \varphi_*\mu_{\tilde{X}_n})$, we obtain $X_n \in \mathcal{P}_{A_n}$ and

$$\square(X, X_n) \leq 2d_{\mathrm{TV}}(\mu_X, \varphi_*\mu_{\tilde{X}_n}) \leq 2d_{\mathrm{TV}}(\mu_{\tilde{X}}, \mu_{\tilde{X}_n}) \leq 4N\varepsilon,$$

which imply $\lim_{n\to\infty} \square(X, \mathcal{P}_{A_n}) = 0$.

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We next prove that $\liminf_{n\to\infty} \square(X, \mathcal{P}_{A_n}) > 0$ for any $X \in \mathcal{X} \setminus \mathcal{P}_A$. It is sufficient to prove that if $X_n \in \mathcal{P}_{A_n}$ $\square$-converges to X, then we have $X \in \mathcal{P}_A$, due to considering the contraposition and extracting a subsequence. Assume that $X_n \in \mathcal{P}_{A_n}$ $\square$-converges to X. Let $\{x_{ni}\}_{i=1}^{\infty}$ be a sequence in X_n with

$$\sum_{i=1}^{\infty} a_{ni} \delta_{x_{ni}} \leq \mu_{X_n}.$$

There exist Borel maps $f_n \colon X_n \to X$ and a sequence $\varepsilon_n \to 0$ such that f_n is 1-Lipschitz up to ε_n and $d_{\mathrm{P}}(f_{n*}\mu_{X_n}, \mu_X) \leq \varepsilon_n$ (actually, f_n can be assumed to be an ε_n-mm-isomorphism but this is unnecessary here). The sequence $\{f_n(x_{ni})\}_{n=1}^{\infty}$ has a convergent subsequence by the same argument as in the proof of claim 4.2. Let

$$x_i := \lim_{k\to\infty} f_{n_k}(x_{n_k i}),$$

where $\{n_k\}$ is a subsequence of $\{n\}$ such that $\{f_{n_k}(x_{n_k i})\}_{k=1}^{\infty}$ converges for any i. Then, since A_n converges weakly to A and $f_{n*}\mu_{X_n}$ converges weakly to μ_X, we have

$$\sum_{i=1}^{\infty} a_i\, \varphi(x_i) \leq \liminf_{k\to\infty} \sum_{i=1}^{\infty} a_{n_k i}\, \varphi(f_{n_k}(x_{n_k i})) \leq \lim_{k\to\infty} \int_{X_{n_k}} \varphi \circ f_{n_k}\, d\mu_{X_{n_k}} = \int_X \varphi\, d\mu_X$$

for any non-negative bounded continuous function $\varphi \colon X \to [0, +\infty)$, where the first inequality follows from Fatou's lemma. This implies $\sum_{i=1}^{\infty} a_i \delta_{x_i} \leq \mu_X$, so that $X \in \mathcal{P}_A$. The proof of this lemma is now completed. $\square$

Proof of theorem 1.8. By lemmas 4.4 and 4.5, the map $\mathcal{A} \ni A \mapsto \mathcal{P}_A \in \mathrm{Fix}(\Pi)$ is a continuous bijection from the compact space $\mathcal{A}$ to the Hausdorff space $\mathrm{Fix}(\Pi)$. Thus, this is a homeomorphism. The proof is completed. $\square$

5. Contractibility of total and base spaces

In this section, we study the topology of the total space $\mathcal{X}_* = \mathcal{X} \setminus \{*\}$ and the base space $\Sigma = \mathcal{X}_*/\mathbb{R}_+$, and prove proposition 1.5. Since $\mathcal{X}_*$ and Σ (resp. Π_* and $\Pi_*/\mathbb{R}$) are weakly homotopy equivalent to each other, proposition 1.5 implies corollary 1.6 directly.

Proof of proposition 1.5(i). Let Z be an arbitrary mm-space with at least two different points and let $p \in [1, \infty]$. We define a map $H \colon [0,1] \times \mathcal{X}_* \to \mathcal{X}_*$ by

$$H(t, X) := (1-t)X \times_p tZ, \quad t \in [0,1],\ X \in \mathcal{X}_*,$$

where $\times_p$ means the l_p-product of two mm-spaces and we agree $0X := *$ for any $X \in \mathcal{X}$. We see that

$$H(0, X) = X \times_p * = X, \quad H(1, X) = * \times_p Z = Z,$$

and each mm-space $H(t, X)$ has at least two different points, that is, $H(t, X) \in \mathcal{X}_*$ for any $t \in [0,1]$ and $X \in \mathcal{X}_*$. Since the scaling and the l_p-product are continuous

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operations (see [9]), the map H is continuous with respect to both the box and concentration topologies. Thus, H is a deformation retraction of $\mathcal{X}_*$ onto $\{Z\}$. Therefore, $\mathcal{X}_*$ is contractible. $\square$

On another note, the following proposition can be obtained by the same proof as above.

PROPOSITION 5.1. *Let $\mathcal{F}$ be the family of all finite mm-spaces. Then, $\mathcal{X} \setminus \mathcal{F}$ is contractible.*

Proof. Just take Z as $([0,1], |\cdot|, \mathcal{L})$ in the proof of proposition 1.5(i). Then, $H(t,X)$ is a deformation retraction of $\mathcal{X} \setminus \mathcal{F}$ onto $\{Z\}$ since $H(t,X)$ is not finite. $\square$

We next prove proposition 1.5(ii) for pyramids. In order to prove this, we prepare the following lemma about the continuity of the l_p-product of pyramids. As for the l_p-product of pyramids, we refer to [4, 11].

LEMMA 5.2. *If a sequence $\{\mathcal{P}_n\}_{n=1}^\infty$ of pyramids converges weakly to a pyramid $\mathcal{P}$, then, for any mm-space Z and any $p \in [1,\infty]$, the sequence $\{\mathcal{P}_n \times_p \mathcal{P}_Z\}_{n=1}^\infty$ converges weakly to $\mathcal{P} \times_p \mathcal{P}_Z$, where the l_p-product of two pyramids $\mathcal{P}$ and $\mathcal{Q}$ is given by*

$$\mathcal{P} \times_p \mathcal{Q} := \overline{\bigcup_{X \in \mathcal{P}, Y \in \mathcal{Q}} \mathcal{P}_{X \times_p Y}}^{\square}.$$

Proof. The outline of this proof is similar to that of [10, theorem 1.2]. For simplicity of the proof, we assume that $p < +\infty$. The case of $p = +\infty$ can be proved in the same way.

We first prove that for any $Y \in \mathcal{P} \times_p \mathcal{P}_Z$,

$$\lim_{n\to\infty} \square(Y, \mathcal{P}_n \times_p \mathcal{P}_Z) = 0. \tag{5.1}$$

Note that (5.1) holds if and only if there exist mm-spaces $X_n \in \mathcal{P}_n$, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} \square(Y, \mathcal{P}_{X_n \times_p Z}) = 0.$$

It is sufficient to prove (5.1) for any mm-space Y with $Y \prec X \times_p Z$ for some $X \in \mathcal{P}$. Assume that $Y \prec X \times_p Z$ for some $X \in \mathcal{P}$. Since $\mathcal{P}_n \to \mathcal{P}$, there exists a sequence $\{X_n\}_{n=1}^\infty$ $\square$-converging to X such that $X_n \in \mathcal{P}_n$ for every n. Then, we have

$$\limsup_{n\to\infty} \square(X_n \times_p Z, X \times_p Z) \leq \lim_{n\to\infty} \square(X_n, X) = 0$$

by [9, proposition 4.1]. Since $Y \prec X \times_p Z$, we have

$$\limsup_{n\to\infty} \square(Y, \mathcal{P}_{X_n \times_p Z}) = 0$$

by [18, proposition 6.10]. Therefore, we obtain (5.1) for $Y \prec X \times_p Z$, and then for any $Y \in \mathcal{P} \times_p \mathcal{P}_Z$.

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We next prove that if an mm-space Y satisfies (5.1), then $Y \in \mathcal{P} \times_p \mathcal{P}_Z$ conversely, which implies that

$$\liminf_{n\to\infty} \square(Y, \mathcal{P}_n \times_p \mathcal{P}_Z) > 0$$

for any $Y \notin \mathcal{P} \times_p \mathcal{P}_Z$ by choosing an appropriate subsequence. Assume that an mm-space Y satisfies (5.1) and assume that Y has at least two different points since $* \in \mathcal{P} \times_p \mathcal{P}_Z$ is always true. Then there exist mm-spaces $X_n \in \mathcal{P}_n$, $n = 1, 2, \ldots$, such that

$$\lim_{n\to\infty} \square(Y, \mathcal{P}_{X_n \times_p Z}) = 0.$$

By [10, lemma 4.6], there exist a Borel measurable map $f_n \colon X_n \times_p Z \to Y$ and a Borel measurable subset $\tilde{Y}_n \subset X_n \times_p Z$ such that $d_{\mathrm{P}}(f_{n*}(\mu_{X_n} \otimes \mu_Z), \mu_Y) \le \varepsilon_n$, $\mu_n(\tilde{Y}_n) \ge 1 - \varepsilon_n$, and

$$\sup_{(x,z),(x',z')\in\tilde{Y}_n} (d_Y(f_n(x,z), f_n(x',z')) - d_{X_n\times_p Z}((x,z),(x',z'))) \le \varepsilon_n$$

for some $\varepsilon_n \to 0$.

Take any $\varepsilon > 0$ and fix it. There exist open subsets $Y_1, \ldots, Y_N \subset Y$ such that $\operatorname{diam} Y_j < \varepsilon$ and $\mu_Y(Y_j) > 0$ for any $j = 1, \ldots, N$,

$$\sum_{j=1}^{N} \mu_Y(Y_j) > 1 - \varepsilon, \quad \text{and} \quad \delta' := \min_{j\neq j'} d_Y(Y_j, Y_{j'}) > 0,$$

cf. [17, lemma 42]. Let

$$Y_0 := Y \setminus \bigsqcup_{j=1}^{N} Y_j$$

and take $y_j \in Y_j$ for each $j = 0, 1, \ldots, N$. In the case of $Y_0 = \emptyset$, it is rather easy and will be omitted. We define an mm-space $\dot{Y}$ by

$$\dot{Y} := (\{y_j\}_{j=0}^{N}, d_Y, \sum_{j=0}^{N} \mu_Y(Y_j)\delta_{y_j}).$$

Let $0 < \eta \ll \varepsilon$ be a small real number as

$$N\eta < \varepsilon \quad \text{and} \quad \delta := \frac{\delta'}{2} - \eta > 0.$$

Then, there exist pairwisely disjoint open subsets $Z_1, \ldots, Z_M \subset Z$ such that $\operatorname{diam} Z_i < \eta$ and $\mu_Z(Z_i) > 0$ for any $i = 1, \ldots, M$, and

$$\sum_{i=1}^{M} \mu_Z(Z_i) > 1 - \eta.$$

We define subsets $A_{ij}^n \subset X_n$ for every $i = 1, \ldots, M$, $j = 1, \ldots, N$ by

$$A_{ij}^n := \left\{ x \in X_n \,|\, \text{there exists } z \in Z_i \text{such that } (x,z) \in f_n^{-1}(Y_j) \cap \tilde{Y}_n \right\}$$

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and define a 1-Lipschitz map $\Phi_n \colon X_n \to (\mathbb{R}^{MN}, \|\cdot\|_\infty)$ by

$$\Phi_n(x) := (\min\{d_{X_n}(x, A^n_{ij}), \operatorname{diam} \dot{Y}\})_{i=1,\dots,M, j=1,\dots,N}$$

for $x \in X_n$. By letting

$$\nu_n := \Phi_{n*}\mu_{X_n},$$

since $\operatorname{supp} \nu_n \subset \{w \in \mathbb{R}^{MN} \mid \|w\|_\infty \leq \operatorname{diam} \dot{Y}\}$, the sequence $\{\nu_n\}_{n=1}^\infty$ is tight. Thus, we can assume that $\{\nu_n\}_{n=1}^\infty$ converges weakly to a Borel probability measure ν on $\mathbb{R}^{MN}$.

We now define an mm-space X by

$$X := (\mathbb{R}^{MN}, \|\cdot\|_\infty, \nu).$$

Note that $X \in \mathcal{P}$ because $(\mathbb{R}^{MN}, \|\cdot\|_\infty, \nu_n) \prec X_n \in \mathcal{P}_n$ for every n. Our goal is to construct a Borel map $\Psi \colon X \times_p Z \to \dot{Y}$ such that $d_{\mathrm{P}}(\Psi_*(\mu_X \otimes \mu_Z), \mu_{\dot{Y}}) < 2\varepsilon$ and

$$\sup_{(x,z),(x',z') \in \tilde{Y}} (d_{\dot{Y}}(\Psi(x,z), \Psi(x',z')) - d_{X \times_p Z}((x,z),(x',z'))) \leq 3\varepsilon$$

for some Borel subset $\tilde{Y} \subset X \times_p Z$ with $\mu_X \otimes \mu_Z(\tilde{Y}) > 1 - 2\varepsilon$. Indeed, if such a map Ψ exists, then we have $Y \in \mathcal{P}_{X \times_p Z}$ by applying [10, lemma 4.6] to the composition $\iota \circ \Psi$ of the inclusion map $\iota \colon \dot{Y} \to Y$ and the map $\Psi \colon X \times_p Z \to \dot{Y}$. Therefore, we obtain

$$Y \in \mathcal{P} \times_p \mathcal{P}_Z$$

since $\mathcal{P}_{X \times_p Z} \subset \mathcal{P} \times_p \mathcal{P}_Z$.

For each $i \in \{1, \dots, M\}$, we set $\operatorname{proj}_i \colon \mathbb{R}^{MN} \to \mathbb{R}^N$ the projection given by

$$\operatorname{proj}_i((w_{kl})_{k=1,\dots,M,l=1,\dots,N}) := (w_{il})_{l=1,\dots,N} = (w_{i1}, \dots, w_{iN})$$

for $(w_{kl})_{k,l} \in \mathbb{R}^{MN}$. Put

$$\nu_i := \operatorname{proj}_{i*}\nu \quad \text{and} \quad \Phi_{n,i} := \operatorname{proj}_i \circ \Phi_n.$$

We see that $\{\Phi_{n,i*}\mu_{X_n}\}_{n=1}^\infty$ converges weakly to ν_i on $\mathbb{R}^N$ as $n \to \infty$. For each $j \in \{1, \dots, N\}$, we define a closed subset $W_j \subset \mathbb{R}^N$ by

$$W_j := \left\{(w_1, \dots, w_N) \in \mathbb{R}^N \mid w_j = 0 \text{and } w_{j'} \geq \delta \text{for every } j' \neq j\right\}.$$

□

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We now prepare two claims.

Claim 5.3. *For any sufficient large n and for any $i = 1, \ldots, M$, $j = 1, \ldots, N$, we have*

$$\Phi_{n,i}(A^n_{ij}) \subset W_j.$$

Proof. Take any $x \in A^n_{ij}$. We have

$$(\Phi_{n,i}(x))_j = \min\{d_{X_n}(x, A^n_{ij}), \operatorname{diam} \dot{Y}\} = 0.$$

We prove that for any $j' \neq j$,

$$(\Phi_{n,i}(x))_{j'} = \min\{d_{X_n}(x, A^n_{ij'}), \operatorname{diam} \dot{Y}\} \geq \delta.$$

By the definition of δ, it is clear that $\delta \leq \delta' \leq \operatorname{diam} \dot{Y}$. We take any $x' \in A^n_{ij'}$. Then there exist $z, z' \in Z_i$ such that

$$(x, z) \in f_n^{-1}(Y_j) \cap \tilde{Y}_n \quad \text{and} \quad (x', z') \in f_n^{-1}(Y_{j'}) \cap \tilde{Y}_n.$$

Thus, we have

$$\begin{aligned}\delta' &\leq d_Y(Y_j, Y_{j'}) \leq d_Y(f_n(x, z), f_n(x', z')) \leq d_{X_n \times_p Z}((x, z), (x', z')) + \varepsilon_n \\ &= (d_{X_n}(x, x')^p + d_Z(z, z')^p)^{1/p} + \varepsilon_n \leq (d_{X_n}(x, x')^p + \eta^p)^{1/p} + \varepsilon_n\end{aligned}$$

which implies that $d_{X_n}(x, x') \geq \delta$ if n is large as $\varepsilon_n < \delta'/2$. Therefore, we have $d_{X_n}(x, A^n_{ij'}) \geq \delta$. This completes the proof. □

Claim 5.4. *For any $j = 1, \ldots, N$, we have*

$$\mu_Y(Y_j) \leq \sum_{i=1}^{M} \nu_i(W_j)\mu_Z(Z_i) + \eta.$$

Proof. A straight-forward calculation implies

$$\begin{aligned}\mu_Y(Y_j) &\leq \liminf_{n\to\infty} (\mu_{X_n} \otimes \mu_Z)(f_n^{-1}(Y_j) \cap \tilde{Y}_n) \\ &\leq \limsup_{n\to\infty} \sum_{i=1}^{M} (\mu_{X_n} \otimes \mu_Z)(f_n^{-1}(Y_j) \cap \tilde{Y}_n \cap (X_n \times Z_i)) + \eta \\ &\leq \limsup_{n\to\infty} \sum_{i=1}^{M} \mu_{X_n}(A^n_{ij})\mu_Z(Z_i) + \eta \\ &\leq \limsup_{n\to\infty} \sum_{i=1}^{M} \Phi_{n,i*}\mu_{X_n}(W_j)\mu_Z(Z_i) + \eta \\ &\leq \sum_{i=1}^{M} \nu_i(W_j)\mu_Z(Z_i) + \eta,\end{aligned}$$

where the fourth inequality follows from claim 5.3. □

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We define a Borel map $\Psi\colon X \times_p Z \to \dot{Y}$ by

$$\Psi(x,z) := \begin{cases} y_j & \text{if } \mathrm{proj}_i(x) \in W_j \text{and } z \in Z_i, \\ y_0 & \text{otherwise.} \end{cases}$$

First, we prove that

$$d_{\mathrm{P}}(\Psi_*(\mu_X \otimes \mu_Z), \mu_{\dot{Y}}) < 2\varepsilon. \tag{5.2}$$

For any $j = 1, \ldots, N$, we have

$$\mu_{\dot{Y}}(\{y_j\}) = \mu_Y(Y_j) \leq \sum_{i=1}^{M} \nu_i(W_j)\mu_Z(Z_i) + \eta$$

by claim 5.4. On the other hand, we have

$$\begin{aligned}\Psi_*(\mu_X \otimes \mu_Z)(\{y_j\}) &= \mu_X \otimes \mu_Z(\bigsqcup_{i=1}^{M} \{(x,z) \in X \times Z \mid \mathrm{proj}_i(x) \in W_j \text{and } z \in Z_i\}) \\ &= \sum_{i=1}^{M} \nu_i(W_j)\mu_Z(Z_i).\end{aligned}$$

Therefore, we have

$$\mu_{\dot{Y}}(\{y_j\}) \leq \Psi_*(\mu_X \otimes \mu_Z)(\{y_j\}) + \eta$$

for any $j = 1, \ldots, N$. Taking $\mu_{\dot{Y}}(\{y_0\}) < \varepsilon$ and $N\eta < \varepsilon$ into account, we obtain (5.2).

Let

$$\tilde{Y} := \bigsqcup_{i=1}^{M} (\mathrm{proj}_i^{-1}(\bigsqcup_{j=1}^{N} W_j) \times Z_i) \subset X \times Z.$$

We see that

$$\mu_X \otimes \mu_Z(\tilde{Y}) = \sum_{i=1}^{M}\sum_{j=1}^{N} \nu_i(W_j)\mu_Z(Z_i) \geq \sum_{j=1}^{N}(\mu_{\dot{Y}}(\{y_j\}) - \eta) \geq 1 - \varepsilon - N\eta > 1 - 2\varepsilon. \tag{5.3}$$

Finally, we prove that

$$\sup_{(x,z),(x',z') \in \tilde{Y}} (d_{\dot{Y}}(\Psi(x,z), \Psi(x',z')) - d_{X \times_p Z}((x,z),(x',z'))) \leq 3\varepsilon. \tag{5.4}$$

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Take any $(x,z),(x',z')\in\tilde{Y}$ and find $i,i'\in\{1,\ldots,M\}$ and $j,j'\in\{1,\ldots,N\}$ such that $\mathrm{proj}_i(x)\in W_j$, $z\in Z_i$, $\mathrm{proj}_{i'}(x')\in W_{j'}$, and $z'\in Z_{i'}$. There exist $x_n\in A^n_{ij}$ and $x'_n\in A^n_{i'j'}$, $n=1,2,\ldots$, such that

$$\|\Phi_n(x_n)-x\|_\infty,\|\Phi_n(x'_n)-x'\|_\infty\to 0$$

as $n\to\infty$. There exists $z_n\in Z_i$ such that $(x_n,z_n)\in f_n^{-1}(Y_j)\cap\tilde{Y}_n$. For any $\tilde{x}_n\in A^n_{i'j'}$ and $\tilde{z}_n\in Z_{i'}$ with $(\tilde{x}_n,\tilde{z}_n)\in f_n^{-1}(Y_{j'})\cap\tilde{Y}_n$, we have

$$\begin{aligned} d_{\dot{Y}}(y_j,y_{j'}) &\le d_Y(Y_j,Y_{j'})+2\varepsilon\le d_Y(f_n(x_n,z_n),f_n(\tilde{x}_n,\tilde{z}_n))+2\varepsilon\\ &\le d_{X_n\times_p Z}((x_n,z_n),(\tilde{x}_n,\tilde{z}_n))+\varepsilon_n+2\varepsilon\\ &\le (d_{X_n}(x_n,\tilde{x}_n)^p+(d_Z(z,z')+2\eta)^p)^{1/p}+\varepsilon_n+2\varepsilon\\ &\le (d_{X_n}(x_n,\tilde{x}_n)^p+d_Z(z,z')^p)^{1/p}+\varepsilon_n+2\varepsilon+2\eta, \end{aligned}$$

which implies that

$$d_{\dot{Y}}(y_j,y_{j'})\le(((\Phi_n(x_n))_{i'j'})^p+d_Z(z,z')^p)^{1/p}+\varepsilon_n+2\varepsilon+2\eta.$$

Thus, we have

$$\begin{aligned} d_{\dot{Y}}(y_j,y_{j'}) &\le \limsup_{n\to\infty}(((\Phi_n(x_n))_{i'j'})^p+d_Z(z,z')^p)^{1/p}+3\varepsilon\\ &=\limsup_{n\to\infty}(((\Phi_n(x_n)-\Phi_n(x'_n))_{i'j'})^p+d_Z(z,z')^p)^{1/p}+3\varepsilon\\ &\le\limsup_{n\to\infty}(\|\Phi_n(x_n)-\Phi_n(x'_n)\|_\infty^p+d_Z(z,z')^p)^{1/p}+3\varepsilon\\ &=(\|x-x'\|_\infty^p+d_Z(z,z')^p)^{1/p}+3\varepsilon\\ &=d_{X\times_p Z}((x,z),(x',z'))+3\varepsilon. \end{aligned}$$

This implies (5.4). Thus, the map $\Psi\colon X\times_p Z\to\dot{Y}$ satisfies (5.2), (5.3), and (5.4), so that we obtain $Y\in\mathcal{P}_{X\times_p Z}\subset\mathcal{P}\times_p\mathcal{P}_Z$. Therefore, the sequence $\{\mathcal{P}_n\times_p\mathcal{P}_Z\}_{n=1}^\infty$ converges weakly to the pyramid $\mathcal{P}\times_p\mathcal{P}_Z$.

Remark 5.5. We conjecture that if two sequences $\{\mathcal{P}_n\}_{n=1}^\infty$ and $\{\mathcal{Q}_n\}_{n=1}^\infty$ of pyramids converge weakly to pyramids $\mathcal{P}$ and $\mathcal{Q}$, respectively, then $\{\mathcal{P}_n\times_p\mathcal{Q}_n\}_{n=1}^\infty$ converges weakly to $\mathcal{P}\times_p\mathcal{Q}$ as $n\to\infty$ for all $p\in[1,\infty]$. Furthermore, this may be true for the setting that extends [9]. However, we need some new idea/proof that does not rely on the partition of the fixed mm-space Z.

Proof of proposition 1.5(ii). Let Z be an arbitrary mm-space with at least two different points and let $p\in[1,\infty]$. We define a map $H\colon[0,1]\times\Pi_*\to\Pi_*$ by

$$H(t,\mathcal{P}):=F_t(\mathcal{P})\times_p\mathcal{P}_{tZ},\quad t\in[0,1],\ \mathcal{P}\in\Pi_*,$$

where $F_t(s):=\min\{s,\frac{1}{t}-1\}$ for $s\ge 0$, which is a monotone metric preserving function, and $F_t(\mathcal{P})$ for a pyramid $\mathcal{P}$ is given by $F_1(\mathcal{P}):=\mathcal{P}_*$ and

$$F_t(\mathcal{P}):=\overline{\bigcup_{X\in\mathcal{P}}\mathcal{P}_{(X,F_t\circ d_X,\mu_X)}}^{\square}\in\Pi$$

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for $t \in [0,1)$. It is a known fact that $F_0(\mathcal{P}) = \mathcal{P}$ and the map $(t,\mathcal{P}) \mapsto F_t(\mathcal{P})$ is continuous on $[0,1] \times \Pi_*$. We remark that the map $(t,\mathcal{P}) \mapsto (1-t)\mathcal{P}$ is not continuous at $t=1$. See [11, §5] for more details. We see that

$$H(0,\mathcal{P}) = F_0(\mathcal{P}) \times_p \mathcal{P}_* = \mathcal{P}, \quad H(1,\mathcal{P}) = F_1(\mathcal{P}) \times_p \mathcal{P}_Z = \mathcal{P}_Z.$$

Moreover, $H(t,\mathcal{P}) \in \Pi_*$ since $H(t,\mathcal{P}) \neq \mathcal{P}_*$ and $H(t,\mathcal{P})$ has finite observable diameter if $t>0$. We prove the continuity of H. Assume that $t_n \to t$ on $[0,1]$ and $\mathcal{P}_n \to \mathcal{P}$ on Π_*. If $t>0$, then we have

$$H(t_n,\mathcal{P}_n) = \frac{t_n}{t}\left(\frac{t}{t_n} F_{t_n}(\mathcal{P}_n) \times_p \mathcal{P}_{tZ}\right) \to F_t(\mathcal{P}) \times_p \mathcal{P}_{tZ} = H(t,\mathcal{P})$$

as $n \to \infty$ by lemma 5.2 and [10, corollary 1.5]. If $t=0$, then we have

$$\begin{aligned}\rho(H(t_n,\mathcal{P}_n), H(0,\mathcal{P})) &\le \rho(H(t_n,\mathcal{P}_n), F_{t_n}(\mathcal{P}_n)) + \rho(F_{t_n}(\mathcal{P}_n),\mathcal{P}) \\ &\le \mathrm{ObsDiam}(t_n Z) + \rho(F_{t_n}(\mathcal{P}_n),\mathcal{P})\end{aligned}$$

by [11, proposition 3.4], which implies that

$$\lim_{n\to\infty} \rho(H(t_n,\mathcal{P}_n), H(0,\mathcal{P})) = 0.$$

Thus, the map H is continuous, so that it is a deformation retraction of Π_* onto $\{Z\}$. Therefore, Π_* is contractible. □

6. Further questions

It is asked in [1, Question 9.1] if the Gromov–Hausdorff space is homeomorphic to l^2. In our previous article [11], we have proved that $\mathcal{X}$ is not homeomorphic to l^2 with respect to the concentration topology. The following question remains.

QUESTIONA 6.1. Is $\mathcal{X}$ homeomorphic to l^2 with respect to the box topology?

We also ask the following.

QUESTIONA 6.2. If two sequences $\{\mathcal{P}_n\}_{n=1}^\infty$ and $\{\mathcal{Q}_n\}_{n=1}^\infty$ of pyramids converge weakly to pyramids $\mathcal{P}$ and $\mathcal{Q}$, respectively, then does $\{\mathcal{P}_n \times_p \mathcal{Q}_n\}_{n=1}^\infty$ converge weakly to $\mathcal{P} \times_p \mathcal{Q}$ as $n \to \infty$ for all $p \in [1,\infty]$?

QUESTIONA 6.3. Are Σ and $\Pi_*/\mathbb{R}_+$ contractible?

QUESTIONA 6.4. Are the spaces $\mathcal{X}$ and $\mathcal{X}_*$ locally contractible with respect to the box and/or concentration topologies?

QUESTIONA 6.5. Are Π and Π_* locally contractible?

The following is already stated in our previous article [11].

QUESTIONA 6.6. Is the observable metric on $\mathcal{X}$ geodesic? Is the pyramidal compactification Π of $\mathcal{X}$ geodesic?

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