AN INEQUALITY BETWEEN NUMERICAL HOMOTOPY INVARIANTS

M. J. M. PRIDDIS

1. Introduction. In (**1**), Berstein and Ganea defined the nilpotency class of a based topological space. For a based topological space X we write nil X for the nilpotency class of the group ΩX in the category of based topological spaces and based homotopy classes. Hilton, in (**3**), defined the nilpotency class, nil class K of a based semi-simplicial (s.-s.) complex; actually, the restriction of connectedness can be removed. Hence, by using the total singular complex functor S, an invariant (nil class SX) can be defined for a based topological space X. In this note, it is our purpose to prove that nil class $SX \geqq$ nil X for X of the based homotopy type of a CW-complex.

2. Preliminaries. By a space we shall always mean a based topological space. We further assume that all maps and homotopies keep base points fixed.

If X and Y are two spaces, then the wedge $X \vee Y$ is considered as a subset of the Cartesian product $X \times Y$ and $\overset{n}{V}X$ will denote the n-fold wedge of X. Let ∇_n: $\overset{n}{V}X \to X$ be the folding map for $n \geqq 2$ and let ∇_1: $X \to X$ be the identity map of X.

The fibre of the inclusion map $X \vee Y \to X \times Y$ is denoted by* $X \flat Y$. By definition, $X \flat Y$ is the space of paths in $X \times Y$ that begin at the base point and end in $X \vee Y$. The map, i: $X \flat Y \to X \vee Y$, which projects a path onto its end point, is essentially the inclusion of the fibre into the total space.

For the s.-s. case by a complex we shall mean a Kan complex with base point and all s.-s. maps and homotopies are assumed to preserve base points.

All the above concepts apply to the s.-s. case and we use the same symbol to denote the s.-s. analogue, except that we indicate the construction of the flat product. First, we digress to give the construction and main properties of the s.-s. analogue of the mapping tract functor.

Proposition 1. *Let f: $K \to L$ be an s.-s. map between an s.-s. K with base point and an s.-s. Kan complex L with base point. Then there exists an s.-s. fibre space (M, p, L) with the following properties:*

(i) *M is a Kan complex with base point, p is base-point preserving and the fibre is a Kan complex;*

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*Called the flat product of X and Y.

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(ii) *There exist base-point preserving s.-s. maps* $f_1: K \to M$ *and* $f_2: M \to K$ *such that* $f_2 f_1 = \text{id}$ *and* $f_1 f_2 \simeq \text{id}$, *relative to the base point.*

Proof. M is obtained as the total space of the fibre map over $K \times L$ induced by $f \times \text{id}$ from the fibre space $(\text{Hom}(\Delta[1], L), \text{Hom}(\text{inj}, \text{id}), \text{Hom}(\Delta[1], L))$; see (**5**) for details. The first assertion of (i) follows from (**5**) and the others can be easily verified, as can (ii).

To simplify notation, we often refer to the fibre map (K, f, L) rather than the actual fibre map (M, p, L).

For two complexes K and L let $k: K \vee L \to K \times L$ denote the inclusion of the union of K and L, united at their base points into their Cartesian product. We define $K \flat L$ to be the fibre of the fibre map associated with k according to Proposition 1. The composition of the inclusion of $K \flat L$ into the total space followed by the projection onto $K \vee L$ is denoted by j.

For spaces X, Y let SX, SY denote their total singular complexes. It can be readily verified that the inclusion $q: SX \vee SY \to S(X \vee Y)$ is a weak homotopy equivalence, where, for definiteness, we take $\pi_n(K)$ to be defined as $\pi_n(S|K|)$ for a not necessarily Kan complex K.

3. The $\flat$-invariant. For a positive integer n and a space X we define the nth flat product $\flat^n X$ and maps $i_n: \flat^n X \to \overset{n}{V} X$ as follows:

$$n = 1, \quad \flat^n X = X_1, \qquad i_n = \text{id},$$
$$n > 1, \quad \flat^n X = (\flat^{n-1} X) \flat X, \quad i_n = (i_{n-1} \vee \text{id}) i,$$

where

$$i: (\flat^{n-1} X) \flat X \to \flat^{n-1} X \vee X.$$

3.1. *Definition.* The flat invariant, $\flat - X$, of a space X is the least integer $n \geqq 0$ for which the map $\nabla_{n+1} i_{n+1}: \flat^{n+1} X \to X$ is null-homotopic; if no such integer exists, we write $\flat - X = \infty$.

3.2. PROPOSITION. *If X is dominated by Y, then $\flat - X \leqq \flat - Y$.*

Proof. We may assume that $\flat - Y = n < \infty$. Let $f: X \to Y$ and $g: Y \to X$ be maps such that $gf \simeq 1$. Then f induces $\flat^{n+1} f: \flat^{n+1} X \to \flat^{n+1} Y$ and $\overset{n+1}{V} f: \overset{n+1}{V} X \to \overset{n+1}{V} Y$, giving commutativity in the diagram

$$\begin{array}{ccccc} \flat^{n+1} X & \xrightarrow{i_{n+1}} & \overset{n+1}{V} X & \xrightarrow{\nabla_{n+1}} & X \\ \flat^{n+1} f \downarrow & & \overset{n+1}{V} f \downarrow & & f \downarrow \\ \flat^{n+1} Y & \xrightarrow{i_{n+1}} & \overset{n+1}{V} Y & \xrightarrow{\nabla_{n+1}} & Y \end{array}$$

Then $\nabla_{n+1} i_{n+1} \simeq gf\, \nabla_{n+1} i_{n+1} = g \nabla_{n+1} i_{n+1} \flat^{n+1} f \simeq 0$.

3.3. COROLLARY. *$\flat - X$ depends only on the homotopy type of X.*

In exactly the same way, a definition of the flat invariant can be given for a complex K, which can also be shown to be a homotopy-type invariant. The same notation is used except that i_n and i are replaced by j_n and j, respectively. We remark that in both cases $X \flat X = X_0 \flat X_0$, where X_0 is the path component of a space or complex containing the base point. Thus, in dealing with the flat invariant, there is no loss in generality in assuming that the spaces are path connected and the complexes connected.

Reverting to the topological category, define $G' - \text{nil}\, X = \sup \text{nil}\, \pi(A, X)$, where A ranges over all cogroup-like spaces with non-degenerate base point and π denotes the collection of homotopy classes of maps from A to X.

The proof of Theorem 6.11 of (**1**) readily dualizes to give the following results.

3.4. PROPOSITION. *Let X have a non-degenerate base point. Then*

$$G' - \text{nil}\, X \geqq \text{nil}\, X.$$

3.5. THEOREM. *Let X be a space, then $\flat - X \geqq G' - \text{nil}\, X$.*

Proof. Suppose that $\flat - X < k$ and let A be a cogroup-like space with non-degenerate base point. We consider maps $f_1, \ldots, f_k$: $A \to X$ and let

$$k_f = f_1 \vee \ldots \vee f_k$$

and $\flat^k f$ the map induced from k_f. Consider the diagram

$$\begin{array}{ccccccc} A & \xrightarrow{c'k} & \overset{k}{V}A & \xrightarrow{k_f} & \overset{k}{V}X & \xrightarrow{\nabla_k} & X \\ & & i_k \uparrow & & i_n \uparrow & & \\ & & \flat^k A & \xrightarrow{\flat^k f} & \flat^k X & & \end{array}$$

where $c'k$ is the k-fold co-commutator map of A. Now $c'k$ can be factored through $\flat^k A$, say $i_k g \simeq c'k$. Then

$$\nabla_k{}^k f c'n \simeq \nabla_k{}^k f i_k g = \nabla_k i_k \flat^k f \simeq 0.$$

3.6. COROLLARY. *If X has a non-degenerate base point, then $\flat - X \geqq \text{nil}\, X$.*

This can be regarded as a partial generalization of Theorem 3.1 of (**3**).

4. The main inequality. Now we connect the two flat invariants.

4.1. LEMMA. *Let X be a space. Then for all positive integers n there is an s.-s. map ψ_n: $\flat^n(SX) \to S(\flat^n X)$ which is a homotopy equivalence and*

$$S(\nabla_n i_n)\psi_n \simeq \nabla_n j_n.$$

Proof. The assertion for $n = 1$ is trivial. We proceed by induction. Consider the diagram

$$\begin{array}{ccccc} S((\flat^{n-1}X)\flat X) & \xrightarrow{S(i)} & S(\flat^{n-1}X \vee X) & \xrightarrow{S(k)} & S(\flat^{n-1}X \times X) \\ & & q_n \uparrow & & \uparrow p_n \\ (\flat^{n-1}SX)\flat SX & \xrightarrow{j} & \flat^{n-1}SX \vee SX & \xrightarrow{k} & \flat^{n-1}SX \times SX; \end{array}$$

here $p_n = \psi_{n-1} \times 1$, $q_n = q(\psi_{n-1} \vee 1)$ (on identifying $S(\flat^{n-1}X \times X)$ with $S^{n-1}X \times SX$) and k: $\flat^{n-1}(SX) \vee SX \to \flat^{n-1}SX \times SX$ are the natural inclusions.

The square commutes up to homotopy and since S preserves fibrations, there is an s.-s. map ψ_n: $\flat^n SX \to S(\flat^n X)$ such that $S(i)\psi_n \simeq q_n j$.

An application of π_* to the diagram then shows that ψ_n is a weak homotopy equivalence; and since both $S(\flat^n X)$ and $\flat^n(SX)$ are Kan complexes, ψ_n is a homotopy equivalence.

The second half of the lemma then follows easily from the inductive hypothesis.

4.2. THEOREM. $\flat - X \geqq \flat - SX$ *and if* X *is of the homotopy type of a* CW-*complex, then* $\flat - X = \flat - SX$.

Proof. The first part follows immediately from the lemma and the second part follows from (**6**).

We now connect the simplicial flat invariant with the group complex nilpotency class.

Let A be a group complex, $A_{(n)}$ the nth free derived group complex of A, l_n the inclusion homomorphism $A_{(n)} \to \overset{n}{*}A$, and ∇_n the folding map: $\overset{n}{*}A \to A$; see (**3**) for the explicit definitions and for the definition of the $\bar{W}$ functor of Kan.

4.3. LEMMA. *For each positive integer* n, *there is a complex s.-s. homotopy equivalence* ϕ_n: $\flat^n(\bar{W}A) \to \bar{W}(A_{(n)})$ *and* $\bar{W}(\nabla_n l_n)\phi_n \simeq \nabla_n j_n$.

The proof is similar to that of 4.1, using the following straightforward proposition.

4.4. PROPOSITION. *Let* p: $A \to B$ *be an epimorphism of group complexes, then* $\bar{W}p$: $\bar{W}A \to \bar{W}B$ *is an s.-s. fibre map.*

4.5. THEOREM. nil class $A \geqq \flat - \bar{W}A$ *and if* A *is a free group complex, then* nil class $A = \flat - \bar{W}A$.

Proof. The first part follows immediately from 4.3 and the second follows by using the properties of the function α of (**4**).

4.6. COROLLARY. *If* K *is a complex, then* nil class $K = \flat - K$.

Proof. By the obvious modification of the argument of § 13 in (**4**), it can be shown that K and $\bar{W}GK$ are of the same homotopy type.

Finally, using 3.6, 4.2, and 4.6 we deduce the following theorem.

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4.7. THEOREM. *Let X be of the homotopy type of a* CW-*complex, then* $\flat - X = \text{nil class } SX$ *and so* $\text{nil class } S(X) \geqq \text{nil } X$.

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Birmingham University,
Birmingham, England;
Liverpool University,
Liverpool, England

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