A Generalisation of a Theorem of Mercer.

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§ 1. It is well known that, if $nt_n = s_1 + s_2 + \ldots + s_n$, the convergence of s_n to a limit implies the convergence of t_n to the same limit. The converse theorem, that the convergence of t_n implies the convergence of s_n, is false. Mercer[1] proved, however, that if $s_n + at_n \to (1 + a)\, l$ when $a > -1$, then both s_n and t_n tend to l. This theorem has recently been extended in various directions.[2] In the present note the case of Abel limits is considered.

We say that s_n tends to l in Abel's sense,[3] or symbolically, that

$$A\text{-lim } s_n = l$$

if the series $\sum_1^\infty s_n x^n$ is convergent for $0 < x < 1$ and

$$\lim_{x \to 1-0} (1 - x) \sum_1^\infty s_n x^n = l.$$

It can easily be shown that, if

$$A\text{-lim } s_n = l,$$

then

$$A\text{-lim } t_n = l.$$

That the converse of this is untrue may be seen from the example

$$\sum_1^\infty t_n x^n = \sin \frac{1}{1 - x}$$

for which A-lim $t_n = 0$. Here

$$\sum_1^\infty s_n x^n = \frac{x}{1 - x} \cos \frac{1}{1 - x}$$

so that A-lim s_n does not exist. We shall prove, for limits in Abel's sense, the following analogue of Mercer's Theorem.

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[1] *Proc. London Math. Soc.*, (2), 5, (1907), 206-224

[2] *Cf.* Vijayaraghavan, *Journal London Math. Soc.*, 3, (1928), 130-134, (who gives references to previous work on the subject); Copson and Ferrar, *ibid.*, 4, (1929), 258-264 ; 5 (1930), 21-27.

[3] See, for example, Knopp, *Infinite Series*, (1928), 498 *et seq.*

THEOREM I. *If A-lim* $(s_n + at_n) = l\,(1+a)$ *when* $a > -1$, *then*

$$A\text{-}lim\ s_n = A\text{-}lim\ t_n = l.$$

By considering the sequence $s_n - l$, we see that it is sufficient to prove the theorem in the case $l = 0$.

§ 2. *Proof of Theorem I.*

We are given that $\sum_1^\infty (s_n + at_n)\,x^n$ is convergent for $0 < x < 1$, and that

$$\lim_{x \to 1-0} (1-x) \sum_1^\infty (s_n + at_n)\,x^n = 0;$$

we have to prove that $\sum_1^\infty s_n x^n$ and $\sum_1^\infty t_n x^n$ are convergent for $0 < x < 1$, and that

$$\lim_{x \to 1-0} (1-x) \sum_1^\infty s_n x^n = 0$$

$$\lim_{x \to 1-0} (1-x) \sum_1^\infty t_n x^n = 0$$

provided that $a > -1$.

Now the convergence of $\sum_1^\infty (s_n + at_n)\,x^n$ for $0 < x < 1$ implies that, for every positive value of ϵ,

$$y_n = s_n + at_n = O\,(1+\epsilon)^n.$$

But the equation

$$s_n + at_n = y_n$$

may be written

$$(n+a)\,t_n - (n-1)\,t_{n-1} = y_n,$$

a difference equation whose solution is, for $n > N$,

$$t_n = \frac{\Gamma(n)}{\Gamma(n+a+1)} \left[C + \sum_N^n \frac{y_p\,\Gamma(p+a)}{\Gamma(p)} \right].$$

By the use of Stirling's asymptotic formula for the Gamma-function, we obtain, since $a > -1$,

$$|\,t_n\,| < K_1\,n^{-a-1} + K_2\,n^{-a-1} \sum_N^n |\,y_p\,|\,p^a$$

$$< K_1\,n^{-a-1} + K_3\,(1+\epsilon)^n n^{-a-1} \sum_N^n p^a$$

$$< K\,(1+\epsilon)^n,$$

where K, K_1, K_2, K_3 denote positive constants. Since this holds for every positive value of ϵ, it follows that $\Sigma t_n x^n$ converges for $0 < x < 1$. Further since $s_n = nt_n - (n-1) t_{n-1}$, $\Sigma s_n x^n$ converges for $0 < x < 1$.

Write now

$$s(x) = \sum_1^\infty s_n x^n, \quad t(x) = \sum_1^\infty t_n x^n.$$

It can easily be shown that

$$t(x) = \int_0^x \frac{s(u)}{u(1-u)}\, du = C + \int_\alpha^x \frac{s(u)}{u(1-u)}\, du$$

where $\alpha > 0$. We are given that, as $x \to 1 - 0$,

$$(1-x)\, s(x) + a(1-x)\, t(x) \to 0,$$

a being greater than -1; this may be written

$$(1-x)\, s(x) + a(1-x) \int_\alpha^x \frac{s(u)}{u(1-u)}\, du \to 0.$$

Put now $x = \exp(-1/t)$, $(1-x)\, s(x) = g(t)$; then we have

$$g(t) + a(1 - e^{1/t}) \int_\beta^t \frac{g(\theta)\, d\theta}{(1 - e^{-1/\theta})^2 . \theta^2} \to 0$$

as $t \to +\infty$. Lastly, substitute $g(t) = h(t) . t^2 . (1 - e^{-1/t})^2$; then

$$h(t) + \frac{a}{t(1 - e^{-1/t})} \cdot \frac{1}{t} \int_\beta^t h(\theta)\, d\theta \to 0$$

as $t \to +\infty$.

Now $a / t(1 - e^{-1/t}) \to a > -1$, so that, applying a recently proved theorem,[1] we have $h(t) \to 0$, and consequently $g(t) \to 0$. We have thus shown that

$$\lim_{x \to 1-0} (1-x)\, s(x) = 0,$$

which proves the theorem.

§3. The following theorem involving the Cesàro limit of integral order k is of a type similar to that just discussed, but is very easy to prove.

THEOREM II. *If C_k-lim $(s_n + at_n) = l(1+a)$ when $a > -1$, then*

$$C_k\text{-}lim\ s_n = C_k\text{-}lim\ t_n = l.$$

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[1] *Journal London Math. Soc.*, **4** (1929), 258-264; Theorem IV.

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