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OPTIMAL STOPPING UNDER GENERAL DEPENDENCE CONDITIONS

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ABSTRACT. Let $\{X_n\}$ be a sequence of random variables, not necessarily independent or identically distributed, put $S_n=\sum_1^n X_i$ $(S_0=0)$ and $M_n=\max_{0\le k\le n}|S_k|$. Effective bounds on $E(M_n^\nu)$ in terms of assumed bounds on $E|\sum_i^j X_k|^\nu$, for $\nu>0$, are used to identify conditions under which an extended-valued stopping time τ exists. That is these inequalities are used to guarantee the existence of the stopping time τ such that $E(|S_\tau|/a_\tau)=\sup_{t\in T_\infty} E(|S_t|/a_t)$, where T_∞ denotes the class of randomized extended-valued stopping times based on $S_1, S_2, \ldots$ and $\{a_n\}$ is a sequence of constants. Specific applications to stochastic processes of the time series type are considered.

1. **Introduction.** Let $\{X_n\}$ be a sequence of random variables, not necessarily independent or identically distributed, put $S_n=\sum_1^n X_i$ $(S_0=0)$ and $M_n=\max_{0\le k\le n}|S_k|$. Effective bounds on $E(M_n^\nu)$ in terms of assumed bounds on $E(|\sum_i^j X_k|^\nu)$, for $\nu>0$, are used to identify conditions under which an extended-valued stopping time τ exists. That is, these inequalities are used to guarantee the existence of the stopping time τ such that $E(|S_\tau|/a_\tau)=\sup_{t\in T_\infty} E(|S_t|/a_t)$, where T_∞ denotes the class of randomized extended-valued stopping times based on $S_1, S_2, \ldots$ and $\{a_n\}$ is a sequence of constants. Examples of random sequences which are neither independent random variables nor martingales are provided to illustrate the generality of the conditions guaranteeing the existence of the optimal stopping rules.

2. **General problem.** A game is conducted in which a player observes sequentially the random variables $X_1, X_2, \ldots$. The player may stop observing the random variables at any time n, basing the decision on whether or not to stop only on the observations $X_1, \ldots, X_n$. A player stopping at the nth stage receives a reward $Y_n=Y_n(X_1,\ldots,X_n)$, a known function of $X_1,\ldots,X_n$. It is of interest to find stopping times τ which maximize the expected reward $E\{Y_\tau\}$.

More formally, the problem consists of a probability space (Ω, B, P), a sequence of σ-algebras $B_1\subseteq B_2\subseteq\cdots\subseteq B_\infty=B$, and a sequence of random variables $Y_1, Y_2, \ldots, Y_\infty$ such that Y_n is B_n-measurable for $1\le n\le\infty$. Let T_∞

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denote the collection of all extended-value stopping rules with respect to $B_1, \ldots, B_\infty$. Specifically, T_∞ consists of those random variables t taking values in $\{1, 2, \ldots, \infty\}$ such that $\{t = n\} \in B_n$ for all n. The problem is to determine under what conditions does there exist $\tau \in T_\infty$ such that

$$E\{Y_\tau\} = \sup_{t \in T_\infty} E\{Y_t\} \quad \text{where} \quad Y_t = \sum_{n=1}^{\infty} Y_n I_{(t=n)}.$$

In seeking solutions to the above problem, the following theorem from [Theorem 4 of Siegmund (1967)] is useful.

THEOREM A. *Assume that*

(1) $$E\left\{\sup_{n \le \infty} Y_n\right\} < \infty$$

(2) $$Y_\infty \ge \limsup_n Y_n \quad a.s.$$

Then there exists $\tau \in T_\infty$ *such that*

$$E\{Y_\tau\} = \sup_{t \in T_\infty} E\{Y_t\}.$$

3. **The reward sequence:** $|S_n|/a_n$. Now suppose we examine the particular reward sequence

$$Y_n = \frac{1}{a_n}\left|\sum_1^n X_k\right|,$$

where $\{a_n\}$ is an increasing sequence of positive constants and $\{X_n\}$ is a sequence of random variables. In order to apply the results of Theorem A to this particular reward sequence, it will be necessary to consider the following maximal inequality from Longnecker and Serfling (1977).

THEOREM B. *Let* $W_1, W_2, \ldots, W_n$ *be arbitrary random variables. Suppose that for constants* $\nu > 0$ *and* $\gamma > 1$,

(3) $$E\left\{\left|\sum_{k=i}^{j} W_k\right|^\nu\right\} \le (g(i, j))^\gamma \quad (all\ 1 \le i \le j \le n),$$

where g satisfies either

(4) $$g(i, j) + g(j+1, k) \le g(i, k) \quad (all\ 1 \le i \le j \le k \le n)$$

or

(5) $$g(i, j)/g(1, n) \le (j - i + 1)/n \quad (all\ 1 \le i \le j \le n).$$

Then

(6) $$E\left\{\max_{1 \le i \le n}\left|\sum_{k=1}^{i} W_k\right|^\nu\right\} \le A_{\nu,\gamma}(g(1, n))^\gamma,$$

where $A_{\nu,\gamma}$ *is a function of just* ν *and* γ.

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The following theorem will establish the existence of an optimal stopping time.

THEOREM 1. *Let* $\{a_i\}$ *be an increasing sequence of positive constants and* $\{X_i\}$ *be a sequence of random variables such that*

$$E\left\{\left|\sum_{k=i}^{j} \frac{1}{a_k} X_k\right|^{\nu}\right\} \le \left(\sum_{k=i}^{j} u_k\right)^{\gamma}, \quad (all\ 1 \le i \le j \le n), \tag{7}$$

where $\nu > 1$, $\gamma > 1$, *and* $\{u_n\}$ *is a nonnegative sequence of constants such that* $\sum_1^\infty u_n < \infty$.

Then there exists $\tau \in T_\infty$ *such that*

$$E\left\{\frac{1}{a_\tau}\left|\sum_{k=1}^{\tau} X_k\right|\right\} = \sup_{t \in T_\infty} E\left\{\frac{1}{a_t}\left|\sum_{k=1}^{t} X_k\right|\right\}. \tag{8}$$

Proof. It will be shown that conditions (1) and (2) of Theorem A hold with

$$Y_n = \frac{1}{a_n}|S_n| \quad (S_0 = 0),$$

where $S_n = \sum_1^n X_k$. Let $B_n = \sum_1^n \frac{1}{a_k} X_k$. Then it follows that

$$\max_{1 \le i \le n} Y_i \le 2 \max_{1 \le i \le n} |B_i|.$$

By Theorem B,

$$E\left\{\max_{1 \le i \le n} |B_i|^{\nu}\right\} \le A_{\nu,\gamma}\left(\sum_1^n u_k\right)^{\gamma}.$$

Thus,

$$\begin{aligned} E\left\{\sup_{n \le \infty} Y_n\right\} &= E\left\{\lim_{n\to\infty} \max_{1 \le i \le n} Y_i\right\} \\ &= \lim_{n\to\infty} E\left\{\max_{1 \le i \le n} Y_i\right\} \\ &\le 2 \lim_{n\to\infty} E\left\{\max_{1 \le i \le n} |B_i|\right\} \\ &\le 2 \lim_{n\to\infty} E^{1/\nu}\left\{\max_{1 \le i \le n} |B_i|^{\nu}\right\} \\ &\le 2A_{\nu,\gamma}^{1/\nu}\left(\sum_1^\infty u_k\right)^{\gamma/\nu} < \infty. \end{aligned}$$

Hence condition (1) of Theorem A holds.

By applying Theorem B a second time, it can be shown that $B_n = \sum_1^n (1/a_k)X_k$ converges a.s. Hence $Y_n = 1/a_n |S_n|$ converges a.s. to 0 by the Kronecker

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Lemma. Hence $Y_\infty = 0 = \lim_n Y_n$ a.s., and condition (2) of Theorem A holds.

4. **Applications.** The derivation of optimal stopping times for the reward sequence $|S_n|/a_n$ has been dealt with extensively in the literature. Some particular results are the following.

Case 1: If $\{X_i\}$ are i.i.d. with $E\{X_i\} \equiv 0$ and $a_n = n$, then Davis (1973) proves that condition (A) holds iff $E\{|X_1| \log|X_1|\} < \infty$.

Case 2: Klass (1974) considers the case $\{X_i\}$ i.i.d with $E\{X_1\} \equiv 0$. He proves that condition (A) holds with $a_n = n^{1/\alpha}$, $1 < \alpha < 2$, iff $E\{|X_1|^\alpha\} < \infty$.

Case 3: Basu and Chow (1977) obtain optimal rules for the reward sequence $X_n = n^{-\alpha} |S_n|^\beta$ for constants $2\alpha > \beta > 0$, where $(S_n, \mathcal{F}_n, n \geq 1)$ is a martingale with $E(|S_n - S_{n-1}|^{\max(2,\beta)} | \mathcal{F}_{n-1}) \leq C < \infty$ a.s.

To apply Theorem 1 to specific sequences $\{X_n\}$ the results of Longnecker and Serfling (1978) are useful. In this paper they introduce several dependence restrictions of the weak multiplicative type. The dependence restrictions involve product moments $E\{X_{i_1} X_{i_2} \cdots X_{i_\nu}\}$ rather than conditional expectations or joint distribution functions. Hence, the conditions are relatively easy to check in practice.

One of these types of restrictions is

Definition 2. A sequence $\{X_i\}$ is said to satisfy condition (B) with respect to an even integer ν, a sequence of constants $\{a_i\}$ and a function g of $\nu/2$ arguments if for all $1 \leq i_1 < i_2 < \cdots < i_\nu$,

$$|E\{X_{i_1} X_{i_2} \cdots X_{i_\nu}\}| \leq g(i_2 - i_1, i_4 - i_3, \ldots, i_\nu - i_{\nu-1}) a_{i_1} \cdots a_{i_\nu} \tag{9}$$

and

$$\sum_{k=1}^{\infty} \sum_{j_1=1}^{k} \cdots \sum_{j_{\nu/2-1}=1}^{k} g(j_1, \ldots, j_{\nu/2-1}, k) < \infty. \tag{10}$$

Under this restriction, the following moment inequality is derived.

Theorem C. *Let $\{X_i\}$ satisfy Condition (B) and let $b_i = (E\{X_i^\nu\})^{1/\nu} < \infty$, for ν an even integer. Let β be given by*

$$\beta = \sum_{k=1}^{\infty} \sum_{j_1=1}^{k} \cdots \sum_{j_{\nu/2-1}=1}^{k} g(j_1, \ldots, j_{\nu/2-1}, k) < \infty$$

Then

$$E\left\{\left(\sum_{k=1}^{n} c_k X_k\right)^\nu\right\} \leq (\nu!\beta + D_\nu)^{\nu/2} \left(\sum_{k=1}^{n} b_k^2 c_k^2\right)^{\nu/2},$$

where D_ν is a function of just ν.

Thus condition (7) of Theorem 1 holds $\nu > 2$, $\gamma = \nu/2$ and

$$u_k = (\nu!\beta + D_\nu) b_k^2 a_k^{-2}.$$

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Similar results for several other weak multiplicative type sequences are derived. Also a version of the above theorem for strictly stationary, strongly mixing sequences can be derived.

The following stochastic processes from Longnecker and Serfling (1975) demonstrate the extent to which Theorem 1 extends previous results concerning optimal stopping rules.

1. *The random telegraph signal.* Let $X(t)$ be a stochastic process taking values +1 or −1 in continuous time t. Let the changes of sign occur according to a Poisson process. The process was examined analytically by Kenrick (1929), as a model for treating a telegraph message of random lengths. It was studied by Magness (1954) as an example of non-Gausian noise. Further, Wonham and Fuller (1958), noting that $X(t)$ may be generated by standard electronic methods, have shown that random signals having specified probability density functions may be generated as the output of a smoothing network with $X(t)$ as input.

The relevant moments of $X(t)$ are contained in Wonham and Fuller (1958). They established, for any even integer ν,

$$(11)\quad E\{X(t)X(t+\tau_1)\cdots X(t+\tau_{\nu-1})\}=\exp\{-2\mu(|\tau_1|-|\tau_2|+|\tau_3|-\cdots+|\tau_{\nu-1}|)\}$$

for $-\infty<\tau_1<\cdots<\tau_{\nu-1}<\infty$ and μ is the average number of sign changes per unit time. (The odd product moments are 0.) Consider the discrete-time sequence $\{X_k\}$ given by

$$(12)\quad X_1=X(0),\ X_2=X(-1),\ldots.$$

Applying (11) with $t=-i_1+1$, $\tau_1=i_1-i_2$, $\tau_2=i_1-i_3,\ldots,\tau_{\upsilon-1}=i_1-i_\nu$, where $1\le i_1<\cdots<i_\nu$, it can be seen that

$$(13)\quad \begin{aligned} E\{X_{i_1}X_{i_2}\cdots X_{i_\nu}\}&=\exp\{-2\mu(|i_1-i_2|-|i_1-i_3|+\cdots+|i_1-i_\nu|)\}\\ &=\exp\{-2\mu[(i_2-i_1)+(i_4-i_3)+\cdots+(i_\nu-i_{\nu-1})]\}.\end{aligned}$$

Thus, $\{X_i\}$ satisfies Condition B for any even integer $\nu\ge 4$ with

$$g(j_1,j_2,\ldots,j_{\nu/2-1},k)=\exp\{-2\mu[j_1+j_2+\cdots+j_{\nu/2-1}+k]\}.$$

Also, $\{X_i\}$ is seen to be neither independent nor a martingale by an examination of (13) and by noting that $E[X(t)]\equiv 0$ and $E[X^2(t)]\equiv 1$.

2. *The square of a Gaussian time series.* Another process considered by Magness (1954) for quantitative illustration of non-Gaussianity is

$$(14)\quad X(t)=2^{-\frac{1}{2}}[Z^2(t)-1],$$

where $Z(t)$ is a Gaussian process with $E[Z(t)]\equiv 0$, $E[Z^2(t)]\equiv 1$, and $E[Z(t)Z(t+\tau)]=r(\tau)$. Consider the associated discrete-time sequence $\{X_i\}$, where

$$(15)\quad X_k=X(k),\quad k=1,2,\ldots.$$

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It is readily seen that $E\{X_k\} \equiv 0$ and $E\{X_k^2\} \equiv 1$ and that, for any even integer ν, $E\{X_k^\nu\} \equiv C_\nu < \infty$. It is found that

$$E\{X_iX_j\} = r^2(j-i) \tag{16}$$

and

$$\begin{aligned} E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\} = & \; 4r(i_2-i_1)r(i_3-i_1)r(i_4-i_3)r(i_4-i_1) \\ & + 4r(i_3-i_1)r(i_4-i_3)r(i_4-i_2)r(i_2-i_1) \\ & + 4r(i_4-i_1)r(i_4-i_2)r(i_3-i_2)r(i_3-i_1) \\ & + r^2(i_4-i_1)r^2(i_3-i_2) + r^2(i_3-i_1)r^2(i_4-i_2) \\ & + r^2(i_2-i_1)r^2(i_4-i_3). \end{aligned} \tag{17}$$

For those processes $\{Z(t)\}$ having $|r(\tau)|$ nonincreasing as $|\tau|$ increases, (17) yields, for $i_1 < i_2 < i_3 < i_4$,

$$|E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\}| \le 15\,|r(i_2-i_1)r(i_4-i_3)|. \tag{18}$$

Thus, $\{X_i\}$ satisfies Condition B with $\nu = 4$ and

$$g(j_1, k) = 15\,|r(j_1)r(k)|$$

provided that $\sum_1^\infty j\,|r(j)| < \infty$. However, an examination of (16) indicates that $\{X_i\}$ is neither independent nor a martingale unless $r(j) \equiv 0$ or $r^2(j) \equiv 1$, respectively.

3. *Moving Averages.* Given a time series $\{Y_n\}$, a related series $\{X_n\}$ generated by

$$X_n = \sum_{k=0}^{\infty} c_k W_{n-k} \tag{19}$$

is called a "moving average". Such series arise in connection with various types of sequences $\{W_n\}$ of dependent random variables, in problems of representation of time series with absolutely continuous special distribution and in related prediction problems. In other types of application, the sequence $\{W_n\}$ represents some non-ideal input process and the constants $\{c_i\}$ are "design" constants selected to make the output process $\{X_n\}$ have desired statistical properties in terms of performance characteristics.

A particular case of broad applicability is the familiar moving average model

$$X_n = \sum_{k=0}^{\infty} a^k W_{n-k}. \tag{20}$$

where $|a| < 1$ and $\{W_n\}$ is a sequence of independent random variables. Again, $\{X_n\}$ satisfies Condition B but is neither independent nor a martingale. This can be seen by computing

$$\begin{aligned} E\{X_{i_1}X_{i_2}X_{i_3}X_{i_4}\} &= \sum_{k=-\infty}^{i_1} a^{i_1+i_2+i_2+i_4-4k} \\ &= (1-a^4)^{-1}a^{i_2+i_3+i_4-3i_1} \end{aligned}$$

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Thus, $\{X_n\}$ satisfies Condition B with

$$g(j_1, k) = (1 - a^4)^2 a^{j_1+k}.$$

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