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On generation of the coefficient field of a primitive Hilbert modular form by a single Fourier coefficient

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Abstract. Let f be a primitive Hilbert modular form over F of weight k with coefficient field E_f, generated by the Fourier coefficients $C(\mathfrak{p}, f)$ for $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F)$. Under certain assumptions on the image of the residual Galois representations attached to f, we calculate the Dirichlet density of $\{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F) | E_f = \mathbb{Q}(C(\mathfrak{p}, f))\}$. For $k = 2$, we show that those assumptions are satisfied when $[E_f : \mathbb{Q}] = [F : \mathbb{Q}]$ is an odd prime. We also study analogous results for F_f, the fixed field of E_f by the set of all inner twists of f. Then, we provide some examples of f to support our results. Finally, we compute the density of $\{\mathfrak{p} \in \mathrm{Spec}(\mathcal{O}_F) | C(\mathfrak{p}, f) \in K\}$ for fields K with $F_f \subseteq K \subseteq E_f$.

1 Introduction

The study of the Fourier coefficients of modular forms is an active area of research in number theory. It is well known that, for any primitive form f over $\mathbb{Q}$, the Fourier coefficients of f generate a number field E_f. In [KSW08], Koo, Stein, and Wiese proved that the set of primes p for which the pth Fourier coefficient of f generates E_f has density 1, if f does not have any nontrivial inner twists. To the best of the authors' knowledge, the analogous question is still open for Hilbert modular forms, which is the objective of our study in this article.

For a primitive form f over a totally real number field F, let E_f denote the number field generated by the Fourier coefficients $C(\mathfrak{p}, f)(\mathfrak{p} \in P)$ of f, where $P = \mathrm{Spec}(\mathcal{O}_F)$, the set of all prime ideals of $\mathcal{O}_F$ (cf. [Shi78]). We first state a result, for primitive forms f over F of weight 2, the set of $\mathfrak{p} \in P$ with $\mathbb{Q}(C(\mathfrak{p}, f)) = E_f$ has Dirichlet density 1, if $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime (cf. Theorem 3.1). We then state and prove a general result for primitive forms f of weight k (cf. Theorem 3.6), under some assumptions on the image of the residual Galois representations $\bar{\rho}_{f,\lambda}$ attached to f and $\lambda \in \mathrm{Spec}(\mathcal{O}_{E_f})$ (cf. equation (2.2)). We then show that these assumptions on the image of $\bar{\rho}_{f,\lambda}$ are satisfied for primitive forms f over F of weight 2, if $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime (cf. Theorem 3.1). The proof of Theorem 3.1 depends on the works of Dimitrov (cf. [Dim05]), Dimitrov and Dieulefait (cf. [DD06]). We continue a similar study for F_f, the fixed field of E_f by the set of all inner twists of f, and show that the set of $\mathfrak{p} \in P$ with $\mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f$ has density 1 (cf. Section 2.1 for the definition of $C^*(\mathfrak{p}, f)$).

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This article builds on the ideas of Koo et al. in [KSW08] for primitive forms over $\mathbb{Q}$. One of the vital ingredients in the proof of [KSW08, Theorem 1.1] is a theorem of Ribet (cf. [Rib85, Theorem 3.1]), where he explicitly described the image of l-adic residual Galois representation $\bar{\rho}_{f,l}$ attached to primitive form f and a prime l. This result played a crucial role in obtaining certain sharp bounds for the images of $\bar{\rho}_{f,l}$, which was helpful in their proof. Unfortunately, in our context, an analog of Ribet's result does not seem to exist in the literature. In order to get a similar sharp bound for the images of $\bar{\rho}_{f,\lambda}$, we have to work with some assumptions (cf. equation (3.1)). This explains the reason for our assumptions in Theorems 3.1 and 3.6. Using L-functions and modular forms database (LMFDB), we produce examples of primitive forms f of parallel weight 2 in support of Theorem 3.1 (cf. Examples 4–6).

We also calculate the density of $\mathfrak{p} \in P$ for which $C(\mathfrak{p}, f) \in K$, where $K \subseteq E_f$ is a subfield. This density depends on whether $F_f \subseteq K$ or not. If $F_f \nsubseteq K$, then it is zero (cf. Lemma 4.1); otherwise, it is nonzero and completely determined by the inner twists of f associated with K (cf. Proposition 4.3).

1.1 Structure of the article

The article is organized as follows. In Section 2, we collate all the preliminaries that are required to prove our main theorems (cf. Theorems 3.1 and 3.6). We also introduce the notion of inner twists and study their properties quite elaborately. In Section 3, we state and prove Theorem 3.1 and its generalization (cf. Theorem 3.6) for primitive forms f over F of parallel weight 2 and weight k, respectively. We also prove analogous results for F_f and study their consequences. In Section 4, we calculate the Dirichlet density of $\mathfrak{p} \in P$ with $C(\mathfrak{p}, f) \in K$ for any field K with $K \subseteq E_f$.

2 Preliminaries

Let F be a totally real number field of degree n. Let $\mathcal{O}_F$, $\mathfrak{n}$, and $\mathfrak{D}$ denote the ring of integers, an ideal, and the absolute different of F, respectively.

2.1 Notations

Throughout this article, we fix to use the following notations.

- Let $\mathbb{P}$ denote the set of all primes in $\mathbb{Z}$, $P = \mathrm{Spec}(\mathcal{O}_F)$.
- Let $k = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$ such that $k_i \geq 2$ and $k_1 \equiv k_2 \equiv \cdots \equiv k_n \pmod 2$. Let $k_0 := \max\{k_1, k_2, \ldots, k_n\}$, $n_0 = k_0 - 2$.
- For any number field K, denote $G_K := \mathrm{Gal}(\bar{K}/K)$. Let L be a subfield of K. For a prime ideal $\mathfrak{q}$ in K lying above $\mathfrak{p} = \mathfrak{q} \cap L$ in L, let $\mathrm{e}(\mathfrak{q}/\mathfrak{p})$ and $\mathrm{f}(\mathfrak{q}/\mathfrak{p})$ denote the ramification degree and inertia degree of $\mathfrak{q}$ over $\mathfrak{p}$, respectively.

For any Hecke character Ψ of F with conductor dividing $\mathfrak{n}$ and infinity-type $2 - k_0$, let $S_k(\mathfrak{n}, \Psi)$ denote the space of all Hilbert modular newforms over F of weight k, level $\mathfrak{n}$, and character Ψ. A primitive form is a normalized Hecke eigenform in the space of newforms. The ideal character corresponding to Ψ of F is denoted by Ψ^*.

For a primitive form $f \in S_k(\mathfrak{n}, \Psi)$, let $C(\mathfrak{b}, f)$ denote the Fourier coefficient of f corresponding to an integral ideal $\mathfrak{b}$ of $\mathcal{O}_F$ and $C^*(\mathfrak{b}, f) := \frac{C(\mathfrak{b}, f)^2}{\Psi^*(\mathfrak{b})}$ for all ideal $\mathfrak{b}$

with $(\mathfrak{b},\mathfrak{n}) = 1$. Write $E_f = \mathbb{Q}(C(\mathfrak{b}, f))$, $F_f = \mathbb{Q}(C^*(\mathfrak{b}, f))$, where $\mathfrak{b}$ runs over all the integral ideals of $\mathcal{O}_F$ with $(\mathfrak{b},\mathfrak{n}) = 1$. Let $\mathcal{P}_f := \mathrm{Spec}(\mathcal{O}_{E_f})$, the set of all prime ideals of $\mathcal{O}_{E_f}$. For any two subfields F_1, F_2 such that $\mathbb{Q} \subseteq F_2 \subseteq F_1 \subseteq E_f$, we let

$$\mathfrak{f}_{\lambda,F_1,F_2} := \mathfrak{f}(\lambda \cap F_1/\lambda \cap F_2)$$

for $\lambda \in \mathcal{P}_f$. The following proposition describes some properties of E_f.

Proposition 2.1 [Shi78] *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form of weight k, level $\mathfrak{n}$, and character Ψ with coefficient field E_f. Then:*

(1) *E_f is a finite Galois extension of $\mathbb{Q}$.*
(2) *$\Psi^*(\mathfrak{m}) \in E_f$, for all ideals $\mathfrak{m} \subseteq \mathcal{O}_F$.*
(3) *E_f is either a totally real or a complex multiplication (CM) field.*
(4) *$E_f = \mathbb{Q}(\{C(\mathfrak{p}, f)\}_{\mathfrak{p}\in S})$, where $S \subseteq P$ with S^c is finite.*
(5) *$\overline{C(\mathfrak{p}, f)} = \Psi^*(\mathfrak{p})^{-1}C(\mathfrak{p}, f)$ for all $\mathfrak{p} \in P$ with $(\mathfrak{p},\mathfrak{n}) = 1$.*

2.2 Galois representations attached to f

Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form of weight k, level $\mathfrak{n}$, and character Ψ with coefficient field E_f. For $\lambda \in \mathcal{P}_f$, by the works of Ohta, Carayol, Blasius-Rogawski, and Taylor (cf. [Tay89] for more details), there exists a continuous Galois representation

$$\rho_{f,\lambda} : G_F \to \mathrm{GL}_2(E_{f,\lambda}),$$

which is absolutely irreducible, totally odd, and unramified outside $\mathfrak{n}q$, where $q \in \mathbb{P}$ is the rational prime lying below λ. Here, $E_{f,\lambda}$ denote the completion of E_f at λ. For all primes $\mathfrak{p}$ of $\mathcal{O}_F$ with $(\mathfrak{p}, \mathfrak{n}q) = 1$, we have

$$\text{(2.1)} \quad \mathrm{tr}(\rho_{f,\lambda}(\mathrm{Frob}_\mathfrak{p})) = C(\mathfrak{p}, f) \text{ and } \det(\rho_{f,\lambda}(\mathrm{Frob}_\mathfrak{p})) = \Psi^*(\mathfrak{p})N(\mathfrak{p})^{k_0-1}$$

(cf. [Car86]). By taking a Galois stable lattice, we define

$$\text{(2.2)} \qquad \bar{\rho}_{f,\lambda} := \rho_{f,\lambda} (\mathrm{mod}\ \lambda) : G_F \to \mathrm{GL}_2(\mathbb{F}_\lambda)$$

whose semi-simplification is independent of the choice of a lattice. We conclude this section by recalling the Chebotarev density theorem (cf. [Ser81]).

Theorem 2.2 *Let C be a conjugacy class of $G := \bar{\rho}_{f,\lambda}(G_F)$. The natural density of $\{\mathfrak{p} \in P : [\bar{\rho}_{f,\lambda}(\mathrm{Frob}_\mathfrak{p})]_G = C\}$ is $\frac{|C|}{|G|}$.*

2.3 Inner twists and its properties

We now define inner twists associated with a primitive form and describe some of its properties. This notion is quite useful in Section 4.

Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F. For any Hecke character Φ of F, let f_Φ denote the twist of f by Φ (cf. [SW93, Section 5]). The Fourier coefficients of f and f_Φ are related as follows.

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Proposition 2.3 [SW93, Proposition 5.1] *Let f, f_Φ be as above. If $\mathfrak{n}_0$ and $\mathfrak{m}_0$ are the conductors of Ψ and Φ, respectively, then $f_\Phi \in S_k\big(\mathrm{lcm}(\mathfrak{n}, \mathfrak{m}_0\mathfrak{n}_0, \mathfrak{m}_0^2), \Psi\Phi^2\big)$ and $C(\mathfrak{m}, f_\Phi) = \Phi^*(\mathfrak{m})C(\mathfrak{m}, f)$ for all ideals $\mathfrak{m}$ of $\mathcal{O}_F$.*

Definition 2.4 We say a primitive form f is said to be of CM type, if there exists a nontrivial Hecke character Φ of F such that $C(\mathfrak{p}, f) = \Phi^*(\mathfrak{p})C(\mathfrak{p}, f)$ for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$. We say that f is non-CM if f is not of CM type.

We are now ready to define inner twists.

Definition 2.5 (Inner twists) Let $f \in S_k(\mathfrak{n}, \Psi)$ be a non-CM primitive form over F. For any Hecke character Φ of F, we say that the twist f_Φ of f is inner if there exists a field automorphism $\gamma : E_f \to E_f$ such that $\gamma(C(\mathfrak{p}, f)) = C(\mathfrak{p}, f_\Phi)$ for almost all prime ideals $\mathfrak{p}$ of $\mathcal{O}_F$.

Remark 2.6 For any primitive form f, the identity map $\mathrm{id} : E_f \to E_f$ induces an inner twist of f and we refer to it as the trivial inner twist of f.

Let $\Gamma \leq \mathrm{Aut}(E_f)$ denote the subgroup of γ associated with all the inner twists of f. Let $F_f := E_f^\Gamma$, the fixed field of E_f by Γ. By Galois theory, E_f is a finite Galois extension of F_f. Some properties of F_f are given below.

Lemma 2.7 *The field F_f is totally real and $C^*(\mathfrak{p}, f) \in \mathbb{Q}(C(\mathfrak{p}, f))$.*

Proof By Proposition 2.1, we have $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)\overline{C(\mathfrak{p}, f)}$. This shows that F_f is totally real. By Proposition 2.1, if E_f is totally real, then $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)^2 \in \mathbb{Q}(C(\mathfrak{p}, f))$. If E_f is a CM field, then $\mathbb{Q}(C(\mathfrak{p}, f))$ is preserved under complex conjugation. Hence, $C^*(\mathfrak{p}, f) = C(\mathfrak{p}, f)\overline{C(\mathfrak{p}, f)} \in \mathbb{Q}(C(\mathfrak{p}, f))$. ∎

We now examine the existence of trivial, nontrivial inner twists for any primitive form f.

Lemma 2.8 *If $f \in S_k(\mathfrak{n}, \Psi)$ is a non-CM primitive form over F with a nontrivial Hecke character Ψ, then f has a nontrivial inner twist.*

Proof Let $\sigma : E_f \to E_f$ be an automorphism defined by $\sigma(x) = \overline{x}$, for all $x \in E_f$. By Proposition 2.1, we have $\sigma(C(\mathfrak{p}, f)) = \Psi^*(\mathfrak{p})^{-1}C(\mathfrak{p}, f)$ for all $\mathfrak{p}$ with $(\mathfrak{p}, \mathfrak{n}) = 1$. By Proposition 2.3, f has a nontrivial inner twist given by $(\Psi^*)^{-1}$. ∎

We now give some examples of primitive forms with a nontrivial inner twist.

Example 1 Consider a non-CM primitive form f labeled as `2.2.8.1-41.1-a` in [LMFDB], defined over $F = \mathbb{Q}(\sqrt{2})$ of weight $(2, 2)$, level $[41, 41, 2\sqrt{2} - 7]$, and with trivial character. The coefficient field $E_f = \mathbb{Q}(\sqrt{2})$ and $F_f = \mathbb{Q}$.

Example 2 Consider a non-CM primitive form f labeled as `2.2.12.1-13.1-a` in [LMFDB], defined over $F = \mathbb{Q}(\sqrt{3})$ of weight $(2, 2)$, level $[13, 13, \sqrt{3} + 4]$, and with trivial character. The coefficient field $E_f = \mathbb{Q}(\sqrt{2})$ and $F_f = \mathbb{Q}$.

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Example 3 Consider a non-CM primitive form f labeled as `2.2.24.1-9.1-a` in [LMFDB], defined over $F = \mathbb{Q}(\sqrt{6})$ of weight $(2,2)$, level $[9, 3, 3]$, and with trivial character. The coefficient field $E_f = \mathbb{Q}(\sqrt{6})$ and $F_f = \mathbb{Q}$.

In Examples 1–3, the coefficient field $E_f \neq F_f$. Hence, these primitive forms f have a nontrivial inner twist.

Lemma 2.9 *Suppose $f \in S_k(\mathfrak{n}, \Psi)$ is a non-CM primitive form over F with $[E_f : \mathbb{Q}]$ is an odd prime. If E_f is totally real, then f does not have any nontrivial inner twists. If $\Psi = \Psi_0$ is a trivial character, then E_f is totally real.*

Proof Let $\mathfrak{p} \in P$ be a prime with $(\mathfrak{p}, \mathfrak{n}) = 1$. Since E_f is totally real, $C(\mathfrak{p}, f)^2 \in F_f$. Since $[E_f : \mathbb{Q}]$ is prime, the field F_f is either $\mathbb{Q}$ or E_f. If $F_f = \mathbb{Q}$, then $[\mathbb{Q}(C(\mathfrak{p}, f)) : \mathbb{Q}]$ is either 1 or 2. This contradicts to that $[E_f : \mathbb{Q}]$ is an odd prime. Therefore, $F_f = E_f$. Hence, f does not have any nontrivial inner twists. ∎

3 Statement and proof of the main theorem

In this section, we shall state and prove the main theorem of this article.

Theorem 3.1 (Main Theorem) *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of parallel weight 2, level $\mathfrak{n}$, and character Ψ, which is not a theta series. Let E_f denote the coefficient field of f. Suppose $[F : \mathbb{Q}] = [E_f : \mathbb{Q}]$ is an odd prime. Then,*

$$\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = E_f\}\right) = 1,$$

where $\delta_D(S)$ denotes the Dirichlet density of $S \subseteq P$.

3.1 Images of the residual Galois representations

We now determine the images of the residual Galois representations attached to primitive forms of parallel weight 2. The work of Dimitrov in [Dim05] is quite influential in this section.

Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight $k = (k_1, k_2, \ldots, k_n)$, level $\mathfrak{n}$, and character Ψ. Recall that $k_0 = \max\{k_1, \ldots, k_n\}$ and ω_q is the mod q cyclotomic character. Then, $\bar{\Psi}{\omega_q}^{k_0-2}$ is a character on G_F. Let $\hat{F}$ be the compositum of the Galois closure of F in $\bar{\mathbb{Q}}$ and the subfield of $\bar{\mathbb{Q}}$ given by $(\bar{F})^{\mathrm{Ker}(\bar{\Psi}{\omega_q}^{k_0-2})}$. Then, $\hat{F}$ is a Galois extension of F and $G_{\hat{F}} \trianglelefteq G_F$. A combination of Propositions 3.8 and 3.9 in [Dim05] would imply the following proposition.

Proposition 3.2 *Let f be a primitive form which is not a theta series. For almost all $q \in \mathbb{P}$, there exists a power $\hat{q}$ of q such that either*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \simeq \left\{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in (\mathbb{F}_q^{\times})^{k_0-1}\right\}$$

or

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \simeq \left\{g \in \mathbb{F}_{\hat{q}^2}^{\times}\mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in (\mathbb{F}_q^{\times})^{k_0-1}\right\}$$

holds.

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3.2 Key proposition in the proof of Theorem 3.1

We will now determine the image of $\bar{\rho}_{f,\lambda}$ for primitive forms f in Theorem 3.1. More precisely, we have

Proposition 3.3 *Let $f \in S_k(\mathfrak{n}, \Psi)$ be as in Theorem 3.1. For any $\lambda \in \mathrm{Spec}(\mathcal{O}_{E_f})$ lying above q, we have*

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^d}) : \det(\gamma) \in \mathbb{F}_q^\times\},$$

for infinitely many $q \in \mathbb{P}$ with $d = \mathrm{f}(\lambda/q)$.

Before we start the proof of Proposition 3.3, we recall some necessary results.

Proposition 3.4 [Mar77] *Let $K/\mathbb{Q}$ be a cyclic Galois extension of degree n. For $1 \le r \mid n$, let $S_r := \{q \in \mathbb{P} : e(\lambda|q) = 1$ & $\mathrm{f}(\lambda/q) = r$ for some prime ideal $\lambda|q\}$. Then, $\delta_D(S_r) = \frac{\varphi(r)}{n}$.*

Corollary 3.5 *Let f be as in Theorem 3.1. Then, there exists infinitely many primes $q \in \mathbb{P}$ which are inert in both F and E_f.*

For $q \in \mathbb{P}$, let λ be a prime ideal of $\mathcal{O}_{E_f}$ lying above q.

Proof of Proposition 3.3 We adopt the technique in [DD06, Theorem 3.1] to prove this proposition. In our case, $k_0 = 2$, $\Psi = \Psi_{\mathrm{triv}}$, and hence $G_{\hat{F}} = G_F$. By Proposition 3.2, for all primes $q \gg 1$, there exists a power $\hat{q}$ of q, and we have either $\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$, or $\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathbb{F}_{\hat{q}^2}^\times \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$. We now show that the latter case will not occur.

Suppose that $\bar{\rho}_{f,\lambda}(G_F) \simeq \{\gamma \in \mathbb{F}_{\hat{q}^2}^\times \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(\gamma) \in \mathbb{F}_q^\times\}$ for some prime power $\hat{q}$ of q with $q \gg 1$. By the argument in the proof of [Dim05, Proposition 3.9], we get that $\mathbb{F}_q \subseteq \mathbb{F}_{\hat{q}^2} \subseteq \mathbb{F}_\lambda$. However, this cannot happen because d is odd and $2|[\mathbb{F}_\lambda : \mathbb{F}_q]$. Therefore,

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{\hat{q}}) : \det(g) \in \mathbb{F}_q^\times\}$$

for $q \in \mathbb{P}$ with $q \gg 1$. Now, choose a prime $q \in \mathbb{P}$ which is inert in both F and E_f. Let $v \in P$ be the unique prime in F that lying above q, and let I_v be the inertia group at v. By [Dim05, Corollary 2.13] or by the discussion before [DD06, Proposition 1], the possible tame characters for $\bar{\rho}_{f,\lambda}|I_v$ are of level d or $2d$, since $d = \mathrm{f}(v|q)$. Hence, we have $\mathbb{F}_{q^d} \subseteq \mathbb{F}_{\hat{q}} \subseteq \mathbb{F}_\lambda$. By Corollary 3.5, there exists infinitely many such primes. Since $\mathrm{f}(\lambda|q) = d$, the tame characters of level $2d$ cannot occur in $\bar{\rho}_{f,\lambda}|I_v$, and therefore $\mathbb{F}_{q^d} = \mathbb{F}_{\hat{q}} = \mathbb{F}_\lambda$. Therefore, we have

$$\bar{\rho}_{f,\lambda}(G_F) \simeq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^d}) : \det(g) \in \mathbb{F}_q^\times\}$$

for infinitely many $q \in \mathbb{P}$ with $\mathrm{f}(\lambda/q) = \mathrm{f}(v/q) = d$. We are done with the proof. ∎

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3.3 A result for primitive forms of weight k

In this section, we prove Theorem 3.1 for primitive forms of weight k. If k is of parallel weight 2 and $[F:\mathbb{Q}] = [E_f:\mathbb{Q}]$ is an odd prime, then we show that the assumptions in Theorem 3.6 are satisfied. Hence, Theorem 3.1 is a consequence of Theorem 3.6.

Theorem 3.6 *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight k, level $\mathfrak{n}$, and character Ψ. For any subfield $\mathbb{Q} \subseteq L \subsetneq E_f$, assume that*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \supseteq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^{\mathrm{f}}}) : \det(\gamma) \in (\mathbb{F}_q^{\times})^{k_0-1}\} \text{ with } \mathrm{f} = \mathrm{f}(\lambda|q), \tag{3.1}$$

for infinitely many $\lambda \in \mathcal{P}_f$ with $\mathrm{f}_{\lambda,E_f,L} > 1$, where $q \in \mathbb{P}$ lying below λ. Then,

$$\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\}\right) = 0$$

for all proper subfields L of E_f.

The following proposition (cf. [KSW08, Proposition 2.1(c)]) is helpful in the proof of Theorem 3.6.

Proposition 3.7 *Let $R \subseteq \tilde{R}$ be two subgroups of $\mathbb{F}_{q^r}^{\times}$ for some $q \in \mathbb{P}$ and $r \in \mathbb{N}$. Let $G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^r}) : \det(g) \in \tilde{R}\} \leq \mathrm{GL}_2(\mathbb{F}_{q^r})$. Let $P(x) = x^2 - ax + b \in \mathbb{F}_{q^r}[x]$. Then, $\sum_C |C| \leq 2|\tilde{R}/R|(q^2+q)$, where the sum carries over all the conjugacy classes C of G with characteristic polynomial equals to $P(x)$.*

Proof of Theorem 3.6 Let $\mathcal{O}_{E_f}, \mathcal{O}_L$ denote the ring of integers of E_f, L, respectively. Let T be the set of all $\lambda \in \mathrm{Spec}(\mathcal{O}_{E_f})$ such that equation (3.1) holds. By assumption, T is an infinite set. For any $Q \in T$, let Q_L, q be the prime ideals of $\mathcal{O}_L, \mathbb{Z}$ lying below Q, respectively. Let $\mathbb{F}_{q^r} = \mathcal{O}_L/Q_L$, $\mathbb{F}_{q^{rm}} = \mathcal{O}_{E_f}/Q$ for some $r \geq 1, m \geq 2$.

Let $R := (\mathbb{F}_q^{\times})^{k_0-1}$, $W \leq \mathbb{F}_{q^{rm}}^{\times}$ denote the image of Ψ^* mod Q and $\tilde{R} := \langle R, W\rangle$, the subgroup of $\mathbb{F}_{q^{rm}}^{\times}$ generated by R and W. Then, $|R| \leq q-1$, $|W| \leq |(\mathcal{O}_E/\mathfrak{n})^{\times}|$, and hence $|\tilde{R}| \leq |R||W|$. Let $G := \bar{\rho}_{f,Q}(G_F)$ be the image of residual Q-adic Galois representation $\bar{\rho}_{f,Q}$. By equations (2.1) and (2.2), $G \leq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rm}}) : \det(g) \in \tilde{R}\}$ is a subgroup.

Let $M_Q := \bigsqcup_C \{\mathfrak{p} \in P : [\bar{\rho}_{f,Q}(\mathrm{Frob}_{\mathfrak{p}})]_G = C\}$, where C carries over all the conjugacy classes of G with characteristic polynomial $x^2 - ax + b \in \mathbb{F}_{q^{rm}}[x]$ such that $a \in \mathbb{F}_{q^r}$ and $b \in \tilde{R}$. There are at most $q^r|R||W|$ such polynomials. By equation (2.1), we have $a \equiv C(\mathfrak{p}, f) \pmod{Q}$. Since $a \in \mathbb{F}_{q^r}$, we get $C(\mathfrak{p}, f) \pmod{Q} \in \mathbb{F}_{q^r}$. Hence,

$$M_Q \supseteq \{\mathfrak{p} \in P : C(\mathfrak{p}, f) \pmod{Q} \in \mathbb{F}_{q^r}\}. \tag{3.2}$$

By Theorem 2.2, we have $\delta_D(M_Q) = \sum_C \frac{|C|}{|G|}$. Now, by Proposition 3.7, we get

$$\delta_D(M_Q) \leq \frac{q^r|R||W| \times 2|\tilde{R}/R|(q^2+q)}{|G|} = \frac{2|R||W|^2 q^r(q^2+q)}{|G|}. \tag{3.3}$$

Since $G_F \supseteq G_{\hat{F}}$, by equation (3.1), we get a lower bound to $|G|$ as

$$|G| \geq |R| \times |\mathrm{SL}_2(\mathbb{F}_{q^{rm}})|. \tag{3.4}$$

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Combine equations (3.4) and (3.3) to get

$$\delta_D(M_Q) \le \frac{2|W|^2|R|q^r(q^2+q)}{|R| \times |\mathrm{SL}_2(\mathbb{F}_{q^{rm}})|} = \frac{2|W|^2 q^{r+3}}{q^{3rm}(q-1)}.$$

Since $m \ge 2, r \ge 1$, we get $\delta_D(M_Q) \le O\left(\frac{1}{q^2}\right)$. Since T is an infinite set, q is unbounded. The inclusion of the sets in equation (3.2) implies $\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\right\} \subseteq \bigcap_{Q \in T} M_Q$. Therefore, we have

$$\delta_D\left(\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in L\right\}\right) = 0.$$

This completes the proof of Theorem 3.6. ∎

The above theorem holds even if the inclusion in equation (3.1) holds up to conjugation.

Corollary 3.8 *Let f be as in Theorem 3.6, which satisfies equation (3.1) for any subfield $\mathbb{Q} \subseteq L \subsetneq E_f$. Then, $\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = E_f\right\}\right) = 1$.*

Proof Let $\mathfrak{p} \in P$ be a prime ideal with $\mathbb{Q}\big(C(\mathfrak{p}, f)\big) \subsetneq E_f$. Then, $C(\mathfrak{p}, f) \in L$ for some proper subfield L of E_f. Since $[E_f : \mathbb{Q}]$ is a finite separable extension, there are only finitely many subfields between $\mathbb{Q}$ and E_f. By Theorem 3.6, we have $\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) \subsetneq E_f\right\}\right) = 0$. This completes the proof of the corollary. ∎

We have some remarks to make.

- By the definition of a CM primitive form f, the density of $\mathfrak{p} \in P$ for which $C(\mathfrak{p}, f) = 0$ is at least $\frac{1}{2}$.
- For a non-CM primitive form f, the density of primes $\mathfrak{p}$ for which $C(\mathfrak{p}, f) = 0$ is 0. This is a special case of the famous Sato–Tate equidistribution theorem of Barnet-Lamb, Gee, and Geraghty [BGG11, Corollary 7.17] (cf. [DK20, Theorem 4.4] for more details).

The assumption (3.1) of Theorem 3.6 implies $E_f \neq \mathbb{Q}$ and the set of $\mathfrak{p} \in P$ with $\mathbb{Q}\big(C(\mathfrak{p}, f)\big) = E_f$ has density 1 (cf. Corollary 3.8) implies that the form f has to be non-CM.

3.4 The proof of Theorem 3.1 with supporting examples

In this section, we give a proof of Theorem 3.1 and provide some examples of f in support of it.

Proof of Theorem 3.1 Since $[E_f : \mathbb{Q}]$ is an odd prime, the only proper subfield L of E_f is $\mathbb{Q}$. By Proposition 3.3, f satisfies the assumption (3.1) of Theorem 3.6. Hence, by Corollary 3.8, the proof of Theorem 3.1 follows. ∎

We now give some examples of primitive forms f in support of Theorem 3.1.

Example 4 Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_7)^+$ with generator ω having minimal polynomial $x^3 - x^2 - 2x + 1$, with weight $(2, 2, 2)$, level $[167, 167, \omega^2 + \omega - 8]$, and with trivial character. This Hilbert modular form f is

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labeled as `3.3.49.1-167.1-a` in [LMFDB]. The coefficient field E_f of f is $\mathbb{Q}(\alpha)$, where α is a root of the irreducible polynomial $x^3 - x^2 - 4x - 1 \in \mathbb{Q}[x]$.

Example 5 Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_9)^+$ with generator ω having minimal polynomial $x^3 - 3x - 1$, with weight $(2, 2, 2)$, level $[71,\ 71,\ \omega^2 + \omega - 7]$, and with trivial character. This Hilbert modular form f is labeled as `3.3.81.1-71.1-a` in [LMFDB]. The coefficient field E_f of f is $\mathbb{Q}(\beta)$, where β is a root of the irreducible polynomial $x^3 - x^2 - 4x + 3 \in \mathbb{Q}[x]$.

Example 6 Consider a non-CM primitive form f defined over $F = \mathbb{Q}(\zeta_7)^+$ with generator ω having minimal polynomial $x^3 - x^2 - 2x + 1$, with weight $(2, 2, 2)$, level $[239,\ 239,\ 6\omega^2 - 5\omega - 7]$, and with trivial character. This Hilbert modular form f is labeled as `3.3.49.1-239.1-a` in [LMFDB]. The coefficient field E_f of f is $\mathbb{Q}(\theta)$, where θ is a root of the irreducible polynomial $x^3 - 12x - 8 \in \mathbb{Q}[x]$.

The primitive forms f in Examples 4–6 are of parallel weight 2 with $[F : \mathbb{Q}] = [E_f : \mathbb{Q}] = 3$, and hence they satisfy the hypothesis of Theorem 3.1. Moreover, E_f is totally real, so by Lemma 2.9, these primitive forms f do not have any nontrivial inner twists.

3.5 Computation of some Dirichlet density for F_f

In this section, we shall state and prove a variant of Theorem 3.6 and Corollary 3.8 for F_f. In fact, we compute the Dirichlet density of the set $\{\mathfrak{p} \in P : \mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f\}$.

Theorem 3.9 *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a primitive form defined over F of weight k, level $\mathfrak{n}$, and character Ψ. For any subfield $\mathbb{Q} \subseteq L \subsetneq F_f$, assume that*

$$\bar{\rho}_{f,\lambda}(G_{\hat{F}}) \supseteq \{\gamma \in \mathrm{GL}_2(\mathbb{F}_{q^{\mathrm{f}}}) : \det(\gamma) \in (\mathbb{F}_q^\times)^{k_0 - 1}\} \text{ with } \mathrm{f} = \mathrm{f}_{\lambda, F_f, \mathbb{Q}}, \tag{3.5}$$

for infinitely many $\lambda \in \mathcal{P}_f$ with $\mathrm{f}_{\lambda, F_f, L} > 1$, where $q \in \mathbb{P}$ lying below λ. Then,

$$\delta_D\left(\{\mathfrak{p} \in P : C^*(\mathfrak{p}, f) \in L\}\right) = 0.$$

The above theorem holds even if the inclusion in equation (3.5) holds up to conjugation.

Proof In this proof, we follow the notations as in Theorem 3.6. Let $\mathcal{O}_{F_f}$ be the ring of integers of F_f. For any $Q \in T$, let Q_F be the prime ideal of $\mathcal{O}_{F_f}$ lying below Q. Let $\mathcal{O}_L/Q_L = \mathbb{F}_{q^r}, \mathcal{O}_{F_f}/Q_F = \mathbb{F}_{q^{rm}}$, and $\mathcal{O}_{E_f}/Q = \mathbb{F}_{q^{rms}}$ for some $r \geq 1$, $m \geq 2$, and $s \geq 1$. Then, $G \subseteq \{g \in \mathrm{GL}_2(\mathbb{F}_{q^{rms}}) : \det(g) \in \tilde{R}\}$. Now, arguing as in the proof of Theorem 3.6, we get $\delta_D(M_Q) \leq \frac{4|W|^3 q^{r+3}}{q^{3rm}(q-1)}$. Since $m \geq 2, r \geq 1$, we get $\delta_D(M_Q) \leq O\left(\frac{1}{q^2}\right)$. Therefore, we have

$$\delta_D\left(\{\mathfrak{p} \in P : C^*(\mathfrak{p}, f) \in L\}\right) = 0.$$

This completes the proof of Theorem 3.9. ∎

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Corollary 3.10 *Let f be as in Theorem 3.9, which satisfies equation (3.5) for any subfield $\mathbb{Q} \subseteq L \subsetneq F_f$. Then,*

$$\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) = F_f\right\}\right) = 1.$$

Proof Suppose $\mathfrak{p} \in P$ is a prime such that $L = \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) \subsetneq F_f$ is a proper subfield of F_f. Since $[F_f : \mathbb{Q}]$ is a finite separable extension, there are only finitely many subfields between $\mathbb{Q}$ and F_f, and by Theorem 3.9, we get

$$\delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C^*(\mathfrak{p}, f)\big) \subsetneq F_f\right\}\right) = 0.$$

This completes the proof of the corollary. ∎

Corollary 3.11 *Let f and $\bar{\rho}_{f,\lambda}$ be as in Theorem 3.9. Then, we have*

$$\delta_D\left(\left\{\mathfrak{p} \in P : F_f \subseteq \mathbb{Q}(C(\mathfrak{p}, f))\right\}\right) = 1.$$

Proof Suppose $\mathfrak{p} \in P$ with $\mathbb{Q}(C^*(\mathfrak{p}, f)) = F_f$. From Lemma 2.7, we have $F_f = \mathbb{Q}(C^*(\mathfrak{p}, f)) \subseteq \mathbb{Q}(C(\mathfrak{p}, f))$. Corollary 3.10 implies the result. ∎

In Examples 4–6, we have that E_f and F are of degree 3 over $\mathbb{Q}$ and $E_f = F_f$. Since there are no proper subfields of F_f, by Proposition 3.3, we conclude that these examples satisfy the hypothesis (3.5) of Theorem 3.9.

4 Computation of the Dirichlet density for subfields of E_f

In Section 3, we computed the Dirichlet density of $\mathfrak{p} \in P$ such that $C(\mathfrak{p}, f)$, $C^*(\mathfrak{p}, f)$ generate E_f, F_f, respectively. In this section, for any subfield K of E_f, we compute the Dirichlet density of the set $\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = K\right\}$. It is quite surprising that this density depends on whether $F_f \subseteq K$ or not.

We now calculate the density of $\mathfrak{p} \in P$ such that $C(\mathfrak{p}, f) \in K$ when $F_f \nsubseteq K$. The following lemma is an analog of [KSW08, Corollary 1.3(a)].

Lemma 4.1 *Let f be as in Theorem 3.9. Let $K \subseteq E_f$ be a subfield such that $F_f \nsubseteq K$. Then, $\delta_D\left(\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\right\}\right) = \delta_D\left(\left\{\mathfrak{p} \in P : \mathbb{Q}\big(C(\mathfrak{p}, f)\big) = K\right\}\right) = 0.$*

Proof Since $F_f \nsubseteq K$, we get $\left\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\right\} \subseteq \left\{\mathfrak{p} \in P : F_f \nsubseteq \mathbb{Q}(C(\mathfrak{p}, f))\right\}$. The proof now follows from Corollary 3.11. ∎

Let $\Gamma' = \{\gamma_1, \ldots, \gamma_r\}$ be a subgroup of the inner twists Γ associated with f. Let $\Psi_{\gamma_1}, \ldots, \Psi_{\gamma_r}$ be the corresponding Hecke characters, and their ideal Hecke characters $\Psi^*_{\gamma_1}, \ldots, \Psi^*_{\gamma_r}$ can be thought of as characters on G_F. For each $i \in \{1, 2, \ldots, r\}$, define $H_{\gamma_i} := \mathrm{Ker}(\Psi^*_{\gamma_i})$ and set $H^{\Gamma'} := \bigcap_{i=1}^r H_{\gamma_i} \leq G_F$. Let $K_{H^{\Gamma'}}$ denote the fixed field of $H^{\Gamma'}$. In particular, $F \subseteq K_{H^{\Gamma'}} \subseteq \bar{F}$.

Lemma 4.2 *Let Γ', $H^{\Gamma'}$, and $K_{H^{\Gamma'}}$ be as above. Then,*

$$\{\mathfrak{p} \in P : \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\} = \left\{\mathfrak{p} \in P : \Psi^*_\gamma(\mathfrak{p}) = 1,\ \forall \gamma \in \Gamma'\right\}.$$

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Proof Let $m = [K_{H^{\Gamma'}} : F]$.

$$\begin{aligned}&\{\mathfrak{p} \in P :\ \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\}\\&= \{\mathfrak{p} \in P :\ \mathfrak{p} = \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_m \text{ for prime ideals } \mathfrak{p}_j \text{ in } K_{H^{\Gamma'}} \text{ with } 1 \le j \le m\}\\&= \{\mathfrak{p} \in P :\ \Psi^*_{\gamma_i}(\mathfrak{p}_j) = 1\ \forall i \in \{1, 2, \ldots, r\},\ \forall j \in \{1, 2, \ldots, m\}\}\\&= \{\mathfrak{p} \in P :\ \Psi^*_{\gamma_i}(\mathfrak{p}) = 1\ \forall i \in \{1, 2, \ldots, r\}\}.\end{aligned}$$

This completes the proof of the lemma. ∎

We are now in a position to compute the density of the set $\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\}$ if $F_f \subseteq K$. The following proposition generalizes [KSW08, Corollary 1.3(b)] to primitive forms.

Proposition 4.3 *Let $f \in S_k(\mathfrak{n}, \Psi)$ be a non-CM primitive form defined over F. For any subfield K with $F_f \subseteq K \subseteq E_f$, there exists a subgroup Γ' of Γ such that $K = E_f^{\Gamma'}$ and $\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\}\right) = \frac{1}{[K_{H^{\Gamma'}}:F]}$.*

Proof Since E_f/F_f is Galois, there exists $\Gamma' \subseteq \Gamma$ such that $K = E_f^{\Gamma'}$. Hence,

$$\begin{aligned}\{\mathfrak{p} \in P : C(\mathfrak{p}, f) \in K\} &= \{\mathfrak{p} \in P : \gamma(C(\mathfrak{p}, f)) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\}\\&= \{\mathfrak{p} \in P : \Psi^*_\gamma(\mathfrak{p})C(\mathfrak{p}, f) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\}.\end{aligned}$$

Since $\delta_D\left(\{\mathfrak{p} \in P : C(\mathfrak{p}, f) = 0\}\right) = 0$ (cf. [DK20, Theorem 4.4(1)]) and by Chebotarev density theorem, we have

$$\begin{aligned}&\delta_D\left(\{\mathfrak{p} \in P : \Psi^*_\gamma(\mathfrak{p})C(\mathfrak{p}, f) = C(\mathfrak{p}, f) \text{ for all } \gamma \in \Gamma'\}\right)\\&\quad= \delta_D\left(\{\mathfrak{p} \in P : \Psi^*_\gamma(\mathfrak{p}) = 1 \text{ for all } \gamma \in \Gamma'\}\right)\\&\quad\underset{\text{Lemma (4.2)}}{=} \delta_D\left(\{\mathfrak{p} \in P : \mathfrak{p} \text{ splits completely in } K_{H^{\Gamma'}}\}\right) = \frac{1}{[K_{H^{\Gamma'}} : F]}.\end{aligned}$$

This completes the proof of the proposition. ∎

The following corollary is an application of Proposition 4.3 and an analog of [KSW08, Corollary 1.4].

Corollary 4.4 *Let f, K be as in Proposition 4.3 and $K = E_f^{\Gamma'}$ for $\Gamma' \le \Gamma$. Then,*

$$\begin{aligned}&\delta_D\left(\{\mathfrak{p} \in P : \mathbb{Q}(C(\mathfrak{p}, f)) = K\}\right)\\&\quad= \delta_D\left(\{\mathfrak{p} \in P : \Psi^*_\gamma(\mathfrak{p}) = 1,\ \forall \gamma \in \Gamma' \text{ and } \Psi^*_\omega(\mathfrak{p}) \ne 1,\ \forall \omega \in \Gamma - \Gamma'\}\right).\end{aligned}$$

These results illustrate that the Dirichlet density of $\mathfrak{p} \in P$ such that $\mathbb{Q}(C(\mathfrak{p}, f)) = K$, with $F_f \subseteq K \subseteq E_f$, is determined by the inner twists of f associated with K.

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