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Fibers of Polynomial Mappings at Infinity and a Generalized Malgrange Condition

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Abstract. Let f be a complex polynomial mapping. We relate the behaviour of f 'at infinity' to the characteristic cycle associated to the projective closures of fibres of f. We obtain a condition on the characteristic cycle which is equivalent to a condition on the asymptotic behaviour of some of the minors of the Jacobian matrix of f. This condition generalizes the condition in the hypersurface case known as Malgrange's condition. The relation between this condition and the behavior of the characteristic cycle is a partial generalization of Parusinski's result in the hypersurface case. We show that the new condition implies the C^∞-triviality of f.

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Let $f\colon \mathbf{C}^n \to \mathbf{C}^p$ be a polynomial mapping with $n > p$. A value $t_0 \in \mathbf{C}^p$ of f is called *typical* if f is a C^∞ trivial fibration over a neighborhood of t_0 and *atypical* otherwise. The set of atypical values, consists of the critical values of f and, maybe, some other values coming from the 'singularities of f at infinity'. In the case where $p = 1$, the atypical values have been studied by many authors ([Hà–Lê], [Hà], [Pa1], [Pa2], [S–T], or [Z].) In this paper we consider the extent to which some recent results by Parusinski can be extended to the case where $p > 1$.

As in the hypersurface case, we consider the family $\overline{f}\colon X \to \mathbf{C}^p$ of projective closures of fibres of f, X being the closure of the graph of f in $\mathbf{P}^n \times \mathbf{C}^p$. In the case where $p = 1$, Parusinski was able to relate the vanishing cycles of $\overline{f}$, the characteristic cycle of X, and a condition of Malgrange to give a sufficient condition ensuring that a value t_0 is not atypical.

In this paper, using the theory of the integral closure of modules, we are able to relate the condition on the characteristic cycle of X that appears in Parusinski's paper with the asymptotic behavior of some of the minors of the Jacobian matrix of f. This condition generalizes the condition in the hypersurface case known as Malgrange's condition. We use this condition to give a sufficient condition for a value t_0 in $\mathbf{C}^p$ not to be atypical.

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For the special case when the points of X at infinity form a complete intersection with isolated singularities, we give a numerical condition for a value t_0 in $\mathbf{C}^p$ not to be atypical.

Currently, the notion of vanishing cycles is defined only for functions, so the extension of this part of Parusinski's paper remains to be done.

Let $f(x_1,\dots,x_n)$ be a complex polynomial mapping with components f_i, $1 \leqslant i \leqslant p$ of degree d_i and let $\tilde{f}_i(x_0,x_1,\dots,x_n)$ be the homogenization of f_i. Consider the family of the projective closures of the fibres of f given by $\overline{f}\colon X \to \mathbf{C}^p$, where X is the closure of the graph of f in $\mathbf{P}^n \times \mathbf{C}^p$, and $\overline{f}$ is induced by the projection on the second factor. Let $H_\infty = \{x_0 = 0\} \subset \mathbf{P}^n$ be the hyperplane at infinity and let $X_\infty = X \cap (H_\infty \times \mathbf{C}^p)$. Denote by $I_T(f)$ the ideal generated by the terms of highest degree of the elements of the ideal generated by the $\{f_i\}$. We denote the terms of highest degree of f_i by f_{d_i}. The cone at infinity, C_∞, of a fiber $f^{-1}(t)$ is obtained by forming the closure of the fiber in $\mathbf{P}^n$ and intersecting with the hyperplane at infinity. If all of the fibers of f have dimension $n-p$, then the cone at infinity does not depend on t, is equal to the vanishing of $I_T(f)$ and hence $X_\infty = C_\infty \times \mathbf{C}^p$.

Let $\mathbf{C}_X$ denote the constant sheaf on X extended by zero onto $\mathbf{P}^n \times \mathbf{C}^p$. Let $\mathrm{Car}(X) \subset T^*(\mathbf{P}^n \times \mathbf{C}^p)$ denote the characteristic cycle of $\mathbf{C}_X$ (cf. [Br] or [Sa]). We denote the projection from $\mathbf{P}^n \times \mathbf{C}^p$ to $\mathbf{C}^p$ by π. The relative cotangent bundle of π in $T^*(\mathbf{P}^n \times \mathbf{C}^p)$ is denoted $T^*{}_\pi(\mathbf{P}^n \times \mathbf{C}^p)$.

DEFINITION 1. We say that $\overline{f}$ is noncharacteristic at $p \in X$ if the fiber of $\mathrm{Car}(X) \cap T^*{}_\pi(\mathbf{P}^n \times \mathbf{C}^p)$ over p is empty.

Similarly we say that $\overline{f}$ is noncharacteristic over t_0 (or over t_0 at ∞) if $\overline{f}$ is noncharacteristic at every $p \in \overline{f}^{-1}(t_0)$ (resp. at every $p \in \overline{f}^{-1}(t_0) \cap X_\infty$). If $\overline{f}$ is not non-characteristic at p, then we call p a characterisitc point.

To state our generalization of Malgrange's condition, we need some notation. Let $M_I(f)$ with $I = (i_1 < i_2 < \cdots < i_p)$ denote the maximal minor of the Jacobian matrix of f formed from the columns indexed by I. Let $M_J(f, j)$ denote the minor of the Jacobian matrix of size $(p-1) \times (p-1)$ using the columns indexed by J, and all the rows of the Jacobian matrix, except for the jth row. If $p = 1$, then by convention we set $M_J(f, j) = 1$. Then the generalized Malgrange condition holds for $t_0 \in \mathbf{C}^p$ if, for $|x|$ large enough and for $f(x)$ close to t_0,

$$\exists_{\delta>0}\ ||x|| \frac{(\Sigma_I ||M_I(f)||^2)^{1/2}}{(\Sigma_{J,j} ||M_J(f,j)||^2)^{1/2}} \geqslant \delta. \tag{GM}$$

We now begin to develop the connection between condition GM and the notion of characteristic points. Fix $p_0 \in X_\infty$. We assume that $p_0 = ((0\colon 0\colon \dots\colon 0\colon 1), 0, \dots, 0) \in \mathbf{P}^n \times \mathbf{C}^p$, so that $y_i = x_i/x_n$ for $i = 0,\dots,n-1$, and $t_1,\dots,t_p$, form a local system of coordinates at p. We say that f *is fair at* p_0 if in this new coordinate system X is defined by

$$F_i(y_0, y_1, \dots, y_{n-1}, t_1, \dots, t_p) = \tilde{f}_i(y_0, y_1, \dots, y_{n-1}, 1) - t_i y_0^{d_i} = 0.$$

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This amounts to assuming that the graph of f is dense in the zero set of F near p_0 which is equivalent to assuming that the terms f_{d_i} define a complete intersection of codimension p in a neighborhood of p_0 in $(H_\infty \times \mathbf{C}^p)$. Let g denote the restriction to X of y_0; if Ω is a neighborhood of p_0, then $T^*{}_g(\Omega)$ denotes the relative conormal of g in Ω. The next result links the notion of a characteristic point and the behavior of g.

PROPOSITION 2. *Suppose X, f, Ω, p_0 as above. Then p_0 is a characteristic point iff the fiber of $T^*{}_g(\Omega) \cap T^*{}_\pi(\mathbf{P}^n \times \mathbf{C}^p)$ over p_0 is nonempty.*

Proof. The proof is essentially the same as Corollary 1.5 of [Pa2], which in turn depends on one of the main results of [BMM]. □

The next step is to use the theory of the integral closure of modules to interpret the condition that the fiber of $T^*{}_g(\Omega) \cap T^*{}_\pi(\mathbf{P}^n \times \mathbf{C}^p)$ over p_0 is empty. We begin with a definition.

DEFINITION 3. Suppose X, x is a complex analytic germ, M a submodule of $\mathcal{O}^p_{X,x}$. Then $h \in \mathcal{O}^p_{X,x}$ is in the integral closure of M, denoted $\overline{M}$, iff for all $\phi \colon \mathbf{C}, 0 \to X, x$, $h \circ \phi \in (\phi^* M)\mathcal{O}_{\mathbf{C}}$.

A submodule N of M is a reduction of M if $\overline{N} = \overline{M}$. If G is a complex analytic mapping, $G \colon X \to \mathbf{C}^p$, then the submodule of $\mathcal{O}_X{}^p$ generated by the partial derivatives of G is called the Jacobian module of G and is denoted $JM(G)$. If H is a linear space, let $JM(G)_H$ denote the submodule of $JM(G)$ generated by $\partial G/\partial v$ for all $v \in H$. As the next proposition shows, the theory of integral closure allows us to control limiting tangent hyperplanes to analytic sets, and to fibers of analytic maps.

PROPOSITION 4. *Suppose X^d is an equidimensional complex analytic germ in $\mathbf{C}^n$, 0, defined by a map germ F. Suppose $g \colon \mathbf{C}^n \to \mathbf{C}^p$, and let G be the map germ with components (g, F). Suppose V is a linear subspace of $\mathbf{C}^n$. Then no hyperplane H containing V is a limiting tangent hyperplane to the fibers of $g|X$ at* 0 *iff $JM(G)_V$ is a reduction of $JM(G)$.*

Proof. The proof is essentially the same as Theorem 2.2 of [G1] □

We can describe the condition on the conormal of g in Proposition 2 in integral closure terms. We denote by Y the linear space spanned by the $\{\partial/\partial y_i\}$, where $0 \leqslant i \leqslant n-1$, Y_+ the linear space spanned by the $\{\partial/\partial y_i\}$, where $1 \leqslant i \leqslant n-1$.

PROPOSITION 5. *Suppose X, f, Ω, p_0 as Proposition* 2. *Then p_0 is not a characteristic point iff*

$$\partial F/\partial t_i \in \overline{JM(F)_{Y_+}} 1 \leqslant i \leqslant p \ \text{ at the point } p_0. \tag{6}$$

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Proof. We apply Proposition 4, with g and F playing the same parts as in Proposition 4. Because f is fair, we know that X is equidimensional, and that X is defined by F. Propositions 2 and 4 then imply that p_0 is not a characteristic point iff $\partial G/\partial t_i \in \overline{JM(G)_Y}$ for all i at p_0. However, an examination of the Jacobian matrix of G and of $\partial G/\partial t_i$, reveals that this is equivalent to $\partial F/\partial t_i \in \overline{JM(F)_{Y_+}}$. □

At this point it is helpful to note some of the relations between the partials of f and F.

$$\frac{\partial F_j}{\partial y_i}\Big/ {y_0}^{d_j-1} = \frac{\partial f_j}{\partial x_i}, \quad 1 \leqslant i \leqslant n-1, 1 \leqslant j \leqslant p, \tag{7}$$

$$\frac{\partial F_j}{\partial t_i} = {y_0}^{d_j}\delta_{i,j}, \quad 1 \leqslant j \leqslant p, \tag{8}$$

$$\frac{\partial F_j}{\partial y_0}\Big/ {y_0}^{d_j-1} = x_1\frac{\partial f_j}{\partial x_1} + \cdots + x_n\frac{\partial f_j}{\partial x_n}. \tag{9}$$

In (8), $\delta_{i,j}$ is the Kronecker delta. Now we need to reformulate (6) in a way that takes into account $\partial F_j/\partial y_0$. To do this and to make the transition from X to $\mathbf{C}^n$, we need to introduce a few more ideas from the theory of integral closure.

DEFINITION 10. Suppose M is submodule of $\mathcal{O}^p_{X,x}$, $h \in \mathcal{O}^p_{X,x}$. Then h is *strictly dependent* on M if for all $\phi\colon \mathbf{C}, 0 \to X, x$ we have $h \circ \phi \in m_1\phi^*M$, where m_1 is the maximal ideal in $\mathcal{O}_{\mathbf{C}}$. We denote the set of elements strictly dependent on M by $\overline{M}^+$.

The connection between the integral closure of ideals and modules is given by the next Proposition. We denote the $(p-k)$th fitting ideal of $\mathcal{O}^p_{X,x}/M$ by $J_k(M)$; if h is an element of $\mathcal{O}^p_{X,x}$, we denote by (h, M) the module generated by M and h.

PROPOSITION 11. *Suppose M is a submodule of $\mathcal{O}^p_{X,x}$, $h \in \mathcal{O}^p_{X,x}$ and the rank of (h, M) is k on each component of X, x. Then $h \in \bar{M}$ iff $J_k(h, M) \subset \overline{J_k(M)}$.*

Proof. Cf. [1.7] and [1.8] of [G2]. □

We are now ready to reformulate condition (6).

PROPOSITION 12. *Suppose X, f, Ω as in Proposition* 2, *then condition* (6) *holds iff*

$$\partial F/\partial t_i \in \overline{\{y_0\partial F/\partial y_0, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1}\}} \quad \forall i \tag{13}$$

for y in X and close to p_0. Moreover, (6) *implies*

$$y_0\partial F/\partial y_0 \in \overline{\{\partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1}\}}^+ \tag{14}$$

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for y in X_∞ close to p_0.

Proof. Clearly, (6) implies (13). Assume (13) holds. Suppose $x\colon \mathbf{C}, 0 \to X, p$. Since $F(x(s)) \equiv 0$

$$0 = \frac{\mathrm{d}}{\mathrm{d}s} F(y(s), \qquad t(s)) = \sum_{i=1}^{p} \frac{\mathrm{d}t_i}{\mathrm{d}s}\frac{\partial F}{\partial t_i} + \frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial F}{\partial y_0} + \sum_{i=1}^{n-1} \frac{\mathrm{d}y_i}{\mathrm{d}s}\frac{\partial F}{\partial y_i},$$

which gives

$$\frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial F}{\partial y_0} \in x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right).$$

Suppose that the order of $\mathrm{d}y_0/\mathrm{d}s$ in s is k. Then the order of $y_0(s)$ must be $k+1$. We have then that

$$\frac{\mathrm{d}y_0}{\mathrm{d}s}\frac{\partial F}{\partial y_0} \in x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right)$$

holds iff

$$s^k \frac{\partial F}{\partial y_0} \in x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right)$$

iff

$$y_0 \frac{\partial F}{\partial y_0} \in m_1 x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right). \tag{15}$$

Now we can apply Nakayama's lemma and (12) to deduce (6); for combining (12) and (15) we get

$$x^*(JM(F)_{Y_+}) + m_1 x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right)$$
$$= x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right).$$

Applying Nakayama's lemma gives

$$x^*(JM(F)_{Y_+}) = x^* \left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right).$$

Since x is arbitrary, we get

$$\overline{JM(F)_{Y_+}} = \overline{\left(\frac{\partial F}{\partial t_1}, \dots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \dots, \frac{\partial F}{\partial y_{n-1}}\right)}$$

which gives the first part of the Proposition.

From (15) it follows that

$$y_0 \frac{\partial F}{\partial y_0} \in \overline{\left(\frac{\partial F}{\partial t_1}, \ldots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \ldots, \frac{\partial F}{\partial y_{n-1}}\right)}^{+}.$$

The desired result now follows since $JM(F)_{Y_+}$ is a reduction of

$$\left(\frac{\partial F}{\partial t_1}, \ldots, \frac{\partial F}{\partial t_p}, \frac{\partial F}{\partial y_1} \ldots, \frac{\partial F}{\partial y_{n-1}}\right). \qquad \square$$

The following corollary will play an important role in showing that condition (GM) implies the C^∞ triviality of the fibers of f.

COROLLARY 16. *If p_0 is not a characteristic point, then the maximal minors of the matrix with columns $\partial F/\partial y_i$, $1 \leqslant i \leqslant n-1$ do not simultaneously vanish on $\Omega - X_\infty$.*

Proof. We have just shown that the hypothesis implies that $JM(F)_{Y_+}$ is a reduction of the module generated by $(y_0(\partial F/\partial y_0), JM(F)_{Y_+})$, and the partials of F with respect to the t_i. This implies that the corresponding quotient sheaves have the same support. Off of X_∞ the support of the second quotient sheaf is just the singular set of X which is empty off of X_∞. This implies that the ideal of maximal minors in the corollary cannot vanish off of X_∞. $\square$

We are now ready to show the equivalence of condition GM and p_0 being a non-characteristic point.

THEOREM 17. *Suppose f and X as in Proposition* 2. *Then $\overline{f}$ is noncharacteristic over t_0 at infinity if and only if condition GM holds for t_0.*

Proof. For the first part of the proof, we work in a neighborhood of a point p_0 of X_∞ in the fiber of $\overline{f}$. We use the same set-up as in the previous proofs. Then we know that $\overline{f}$ is noncharacteristic at p_0 if and only if

$$\partial F/\partial t_i \in \overline{\{y_0 \partial F/\partial y_0, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1}\}} \quad \forall i$$

for all y in X such that $f(y)$ is sufficiently close to t_0, and x is sufficiently large. By Proposition 11, we know that this is equivalent to $y_0{}_i^{d} M(J,i,F) \in \overline{(M(I,F))}$, where $M(I,F)$ is a maximal minor with multi index I, of the matrix whose columns are $\{y_0 \partial F/\partial y_0, \partial F/\partial y_1, \ldots, \partial F/\partial y_{n-1}\}$, and $M(J,i,F)$ is a maximal minor of the same matrix, with the ith row deleted.

If necessary, shrink the neighborhood, so that $||x_n|| \geqslant ||x_i|| \forall i$ off X_∞. A property of the integral closure of ideals ([L-T]) implies that there exists C such that

$$C \sup_I ||(M(I,F)(z)|| \geqslant \sup_{J,i} ||y_0||^{d_i} ||M(J,i,F)(z)||.$$

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Because the set of points where $y_0 \neq 0$ is dense in X, we can divide both sides of the above inequalities by $||y_0||^k$ where $k = \sum(d_i - 1)$, and get an equivalent inequality. Dividing the ith row of each minor by $(y_0)^{d_i-1}$ and using the formulae 7 and 9 we get

$$C \sup_I ||(M'(I, f)(z)|| \geqslant \sup_{J,i} ||1/x_n||||M'(J, i, f)(z)||.$$

Here, if $i_n \neq n$ we have $M'(I, f) = M(I, f)$ otherwise, replace the column vector with the $\partial f/\partial x_n$ terms by $\sum(x_i/x_n)\partial f/\partial x_i$. A similar substitution should be made to define the $M'(J, i, f)(z)$ terms. Multiplying both sets of terms by $||x_n||$ and using the fact that $||x_n|| \geqslant ||x_i||\forall i$, it is easy to see that this inequality is equivalent to GM.

Because the fibers of $\overline{f}$ are compact, and the GM condition independent of the point at infinity, it follows that there exists a neighborhood of the fiber of $\overline{f}$ over t_0 with x sufficiently large, such that GM holds on the neighborhood. □

THEOREM 18. *Suppose X, f as in Proposition* 2.

(i) *If $\overline{f}$ is noncharacteristic over t_0 then f is C^∞ trivial over a neighborhood of t_0, that is t_0 is typical.*
(ii) *Similarly if $\overline{f}$ is noncharacteristic over t_0 at infinity then f is C^∞-trivial over a neighborhood of t_0 and near infinity* (*i.e. in the complement of a sufficiently big ball in* $\mathbf{C}^n$).

Proof. We will construct p smooth vectorfields V_i such that

(i) $V_i(f_j) = \delta_{i,j}$.
(ii) $\langle x, V_i\rangle = 0$.
(iii) V_i is well defined for all x sufficiently large, for $f(x)$ sufficiently close to t_0.

Integrating these vector fields will produce p C^∞ flows, which will enable us to flow from the fiber over t_0 to any nearby fiber; each flow will adjust one of the component functions of f. Each vectorfield will be a normalized sum of basic vectorfields. We construct these as follows. Augment the Jacobian matrix of f with the row vector $\overline{x}$; replace the ith row of the augmented matrix by $(\partial/\partial x_1, \ldots, \partial/\partial x_n)$. If I is a multindex of length $p+1$, then the submatrix with columns indexed by I is a square matrix of size $p+1$. Expanding along the i-th row produces a vectorfield which we denote by $V_{i,I}$. It is clear that $V_{i,I}(f_j) = 0$ for $i \neq j$ and $\langle x, V_{i,I}\rangle = 0$, for both expressions are just the determinant of a matrix with a row repeated. Denote the minor of the original augmented matrix by $m(I, f, x)$; then we define V_i to be

$$V_i = \frac{\sum_I \overline{m(I, f, x)} V_{i,I}}{\sum_I ||m(I, f, x)||^2}.$$

Properties (i) and (ii) are clear; it remains to show that (iii) holds. Once we show (iii), then it will be obvious from (i) that V_i will never be zero.

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To prove this we will work in a neighborhood of $p = ((0\colon 0\colon \cdots \colon 0\colon 1), 0, \dots, t_0)$. From the form of V_i, it is clear that V_i fails to be defined iff all of the $m(I, f, x)$ vanish. So it suffices to consider those minors for which $i_{p+1} = n$. Expanding a minor of this type along the top row, we get

$$m(I, f, x) = (-1)^p \overline{x_n} m(I', f) + \sum (-1)^{i-1} \overline{x_i} m(J, f), \tag{19}$$

where I' is the multisubindex of I of length p_0, $i_p \neq n$ and J is a multisubindex of I of length p_0 with $i_p = n$. If we change into the y coordinate system, and multiply through by ${y_0}^k \overline{y_0}$ where $k = \sum (d_i - 1)$, we get

$${y_0}^k \overline{y_0} m(I, f, x) = (-1)^p m(I', F) + \sum (-1)^{i-1} \overline{y_i} m(J', F).$$

Here the last column of the matrix which gives $m(J', F)$ is

$$y_0 \frac{\partial F}{\partial y_0} - \left(y_1 \frac{\partial F}{\partial y_1} + \cdots + y_{n-1} \frac{\partial F}{\partial y_{n-1}} \right).$$

Expanding this minor out, gives a sum of minors (with sign). If we collect terms, one term is $(-1)^p (1 + \sum (-1)^{j-1} y_{i_j} \overline{y_{i_j}}) m(I', F)$. Some of the other terms are sums of minors of form $y_i \overline{y_j} m(K, F)$, where $m(K, F)$ is a maximal minor of the matrix of partials of F with respect to the y_i, $i > 0$. The rest of the terms are of the form $\overline{y_i} m(L, F)$, where the $m(L, F)$ are maximal minors of the matrix with columns $\{y_0 \partial F/\partial y_0, \partial F/\partial y_1, \dots, \partial F/\partial y_{n-1}\}$. Because $y_0 \partial F/\partial y_0$ is strictly dependent on the module generated by the partials with respect to the other y_i, these minors are much smaller than the minors $m(I', F)$. By Corollary 16, we know that some minor of type $m(I', F)$ is nonzero. So, consider the norm of the right hand side of (19) for an I' which is maximal in norm. This expression will be bounded below, for y_i small enough, by a constant multiple of $||m(I', F)||$, where the constant depends only on the y_i, not on the minor. It follows, that $m(I, f, x)$ is nonzero at this point, hence V_i is well-defined. □

We now wish to restrict to the case where the cone at infinity is a local complete intersection with isolated singularities. We assume $I_T(f)$ defines the cone with reduced structure. We also want to work at points (p_0, t) at infinity such that in a neighborhood of (p_0, t) the fiber over t is smooth at points not in X_∞. If $t \in \mathbf{C}^p$, let X_t be the fiber over t, $JM(F_t)$ obtained by restricting $JM(F)$ to X_t. We want to work with the multiplicity of the module $JM(F_t)_{Y^+}$, but for the multiplicity to be defined, we need that $JM(F_t)_{Y^+}$ has finite colength. It turns our that our geometric hypotheses are exactly what's needed to ensure this.

PROPOSITION 19. *Suppose at $(p, t) \in X_\infty$ we have that C_∞ is a local complete intersection at p_0 with an isolated singularity. Suppose further that in a neighborhood of (p_0, t) the fiber over t is smooth at points not in X_∞. Then $JM(F_t)_{Y^+}$ has finite colength.*

Proof. The module $JM(F_t)_{Y^+}$ is generated by the partial derivatives of F with respect to $y_1, \ldots, y_{n-1}$; denote the matrix with these partial derivatives as columns by M_{Y^+}. We need to show that this matrix has maximal rank except possibly at (p_0, t). There are two cases to consider, that with $y_0 = 0$ and $y_0 \neq 0$. If $y_0 = 0$ then M_{Y^+} specializes to the corresponding matrix gotten by using the terms of f of highest degree. By the condition on the cone at infinity, this matrix has maximal rank except at (p_0, t). Suppose $y_0 \neq 0$. Then the smoothness condition ensures that M_Y has maximal rank at finite points. Suppose that there exists a curve on the fiber over t such that $y_0|X_t$ is not a submersion along this curve. Then the y_0 coordinate of this curve must be constant, hence the curve cannot pass through (p_0, t). Since $y_0|X_t$ is a submersion close to (p_0, t), it follows that the rank of M_{Y^+} must be maximal.

We can now give a numerical criterion for a point (p_0, t) to not be atypical.

THEOREM 20. *Suppose (p_0, t) as above, p_0 a singular point of the cone at infinity. Suppose the multiplicity of $JM(F_t)_{Y^+}$ is constant in a neighborhood of (p_0, t) in $p \times \mathbf{C}^p$, then (p_0, t) is not atypical.*

Proof. We are going to show that the constancy of the multiplicity implies that the condition (6) of Proposition 5 holds. First we note that f is fair by our setup and condition (6) holds generically in our neighborhood. This follows because the inclusion is implied by the the a_g condition applied to the pair $(X_0, p \times \mathbf{C}^p)$. Here X_0 denotes the set of points of X where the function g is a submersion. (Recall that g is just the restiction of y_0 to X.) We know that the a_g condition holds generically because g is a function. Now the constancy of the multiplicity, coupled with the fact that the inclusion holds generically, allows us to apply the principle of specialization of integral dependence for modules. [G–K]. This implies that the inclusion holds at all points close to (p_0, t) including (p_0, t). □

Note that if p_0 is a smooth point of the cone at infinity, then there is nothing to prove; for then the module $JM(F)_{Y^+}$ has colength 0, so the inclusion of Proposition 5 is trivial.

There is an interesting interpretation of the multiplicity of $JM(F_t)_{Y^+}$ which we now describe.

PROPOSITION 21. *The multiplicity of $JM(F_t)_{Y^+}$ at (p_0, t) is the sum of the Milnor number of X_t and the Milnor number of C_∞.*

Proof. The number of generators of $JM(F_t)_{Y^+}$ is $n - 1$. Meanwhile, it is a submodule of $\mathcal{O}^p_{X_t}$ and the dimension of X_t is $n - p$. Now the number of generators of a minimal reduction of $JM(F_t)_{Y^+}$ is $\dim(X_t) + p - 1 = n - 1$ so $JM(F_t)_{Y^+}$ is already a minimal reduction. Hence the multiplicity of $JM(F_t)_{Y^+}$ is just its colength ([B–R]), since the fiber is a complete intersection hence $\mathcal{O}_{X_t}$ is Cohen-Macaulay. By a theorem of Buchsbaum and Rim ([B–R]), this is the colength of the ideal of maximal minors. Now by a theorem of Lê and Greuel ([Gr], [Le]) the

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colength of the ideal of maximal minors is the sum of the Milnor number of X_t and the Milnor number of the slice by y_0, which is just the cone at infinity. □

COROLLARY 22. *Suppose* (p_0, t) *as above,* p_0 *a singular point of the cone at infinity. Suppose the Milnor number of* X_t *is constant in a neighborhood of* (p_0, t) *in* $p \times \mathbf{C}^p$, *then* (p_0, t) *is not atypical.*

Proof. Since the cone at infinity is independent of t, its Milnor number remains constant, so the multiplicity of $JM(F_t)_{Y^+}$ is constant iff the Milnor number of X_t is constant. Now apply Theorem 20. □

We can also show that if the Milnor number differs from that of the generic fiber, then the point is a characteristic point.

THEOREM 23. *Suppose at* $(p, t) \in X_\infty$ *we have that* C_∞ *is a local complete intersection at* p_0 *with an isolated singularity. Suppose further that in a neighborhood of* (p_0, t) *the fiber over* t *is smooth at finite points. Then if the Milnor number of* X_t *is greater than the generic value,* (p_0, t) *is a characteristic point.*

Proof. Suppose (p_0, t) is not a characteristic point. Then by Proposition 5, we have that $\partial F/\partial t_i \in \overline{JM(F)_{Y_+}} 1 \leqslant i \leqslant p$. This implies that the cosupport of $JM(F)_{Y^+}$ lies in X_∞; however we already know that the cosupport of $JM(F)_{Y^+}$ in X_∞ consists of $S(C_\infty) \times \mathbf{C}^p$. This implies that the cosupport of $JM(F)_{Y^+}$ does not split. We know that $JM(F)_{Y^+}$ restricted to the fibers over $\mathbf{C}^p$ has the minimal number of generators for a module of finite colength, hence we can apply Proposition 1.5 of [G-K] to deduce that the multiplicity, and hence the Milnor number is constant. □

To complete the extension of Parusinski's work, it remains to show that the characteristic points are atypical. As mentioned earlier, the problem is that the notion of vanishing cycles is only well defined for functions.

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