A DIFFERENTIAL GEOMETRY ASSOCIATED WITH DISSIPATIVE SYSTEMS

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Introduction. Consider the following problem of Lagrange in the calculus of variations: relative to differentiable curves $x^i(t)$ satisfying $x^i(t_0) = x^i_0$ and $x^i(t_1) = x^i_1$ find a curve minimizing

$$(1) \quad \begin{cases} \int_{t_0}^{t_1} F\{x^\alpha, \dot{x}^\alpha, \lambda\}\, dt \\ \text{subject to the restraint}^* \\ \dot{\lambda} - F\{x^\alpha, \dot{x}^\alpha, \lambda\} = 0 \text{ and } \lambda(t_0) = 0 . \end{cases}$$

By integrating the equation of restraint in (1) it follows that the problem of Lagrange can be re-formulated: minimize $\lambda(t_1)$ given by the integral equation

$$(1') \qquad \lambda(t_1) = \int_{t_0}^{t_1} F\{x^\alpha(t), \dot{x}^\alpha(t), \lambda(t)\}\, dt$$

relative to the same curves as before. Assume that F is

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* If the restraint were of the form $\dot{\lambda} - F(x^\alpha, \dot{x}^\alpha) = 0$, this would be a special case of A. Lichnerowicz, Les espaces variationnels généralisés, Ann. Sci. École norm. sup. (3) 62, 339-384 (1945).

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positive homogeneous of degree one (briefly plus-one) in the $\dot{x}^\alpha$ so that F^2 is plus-two while

$$(2) \qquad g_{ij}(x^\alpha, \dot{x}^\alpha, \lambda) = \frac{1}{2} \frac{\partial^2 F^2}{\partial \dot{x}^j \partial \dot{x}^i} = \frac{1}{2} F^2_{\dot{x}^i \dot{x}^j}$$

is plus-zero in $\dot{x}^\alpha$. The standard properties of homogeneous functions (see [1]) imply that (1') may be written in the form

$$(3) \qquad \lambda(t) = \int_{t_0}^{t} \{ g_{ij}(x^\alpha, \dot{x}^\alpha, \lambda)\dot{x}^i \dot{x}^j\}^{1/2} \, d\tau \, .$$

Given a curve, (3) defines its λ-length and the extremals of (3) define geodesics and distances in a geometry which will be called <u>symmetric Finsler fatigue</u>. If the g_{ij} are independent of $\dot{x}^\alpha$, the geometry becomes <u>symmetric Riemann fatigue</u>. The word symmetric is used here to stress the fact that as given by (2) the g_{ij} are symmetric. The present paper is primarily concerned with the differential geometry of symmetric Riemann fatigue. Symmetric Finsler fatigue is studied in [2], while non-symmetric Riemann fatigue (except for the brief comments concluding this paper) will be studied in a latter paper. Its motivation will be seen in (iv) below.

It will be seen that the tensors g_{ij} and $\partial_\lambda g_{ij} = \frac{\partial g_{ij}}{\partial \lambda}$ play a fundamental role. Tensors formed from only the g_{ij} and its derivatives with respect to x^α are called <u>conservative</u>; if $\partial_\lambda g_{ij}$ or its derivatives appear, the tensor is called <u>dissipative</u>. The principle results of the present paper are:

(i) a distinction between conservative and dissipative covariant differentiation denoted respectively by $A^{\cdot\cdot}_{\cdot\cdot|i}$ and $A^{\cdot\cdot}_{\cdot\cdot,i}$;

(ii) the extremals of (3), or geodesics, depend on $\partial_\lambda g_{ij}$

and are dissipative. Coordinate transformations leave λ invariant, being the length of a curve, and hence lead to conservative tensors. The equation

$$\frac{\delta \dot{x}^i}{\delta t} = \epsilon \{ -g^{i\alpha} \partial_\lambda g_{\alpha\beta} \dot{x}^\beta + \frac{1}{2} \partial_\lambda g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \dot{x}^i \}$$

defines geodesics for $\epsilon = 1$ and auto-parallel curves for $\epsilon = 0$;

(iii) a conservative curvature tensor $R^i_{ojk\ell}$ having all the properties usually found in Riemannian geometry relative to conservative differentiation; a dissipative curvature tensor $R^i_{jk\ell}$ with the properties

$$R_{ij} = R^\alpha_{ij\alpha}$$

$$F_{ij} = R^\alpha_{\alpha ij} = R_{ji} - R_{ij} = -F_{ji}$$

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0$$

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0 \quad ;$$

(iv) if in addition it is assumed that for fixed x^α_0 the $g_{ij}(x^\alpha_0, \lambda)$ vary proportionately with λ (conformal tangent spaces) then the geodesics and auto-parallel curves equation may be written

$$\frac{\delta \dot{x}^i}{\delta t} = \frac{-\epsilon}{2} g^{i\alpha} (\partial_\lambda g_{\alpha\beta}) \dot{x}^\beta$$

while the geodesics become

$$g^{i\alpha}_{\ \ ,\alpha} = \rho \dot{x}^i \quad \text{where} \quad \rho = \partial_\lambda g^{\alpha\beta} \partial_\alpha \lambda \partial_\beta \lambda$$

if λ is an appropriate solution of the Hamilton-Jacobi equation.

In view of (iv) and the analogous equations of electro-magnetic theory, the significance of a non-symmetric Riemann fatigue geometry is clear.

Preliminary Theorems.

LEMMA 1. The Euler-Lagrange equations associated with (1) or (1') are

$$\frac{d}{dt} F_{\dot{x}^i} - F_{x^i} = F_{\dot{x}^i} F_\lambda . \tag{4}$$

Proof. Consider μ as Lagrange multiplier and set $G = F + \mu(\dot{\lambda} - F)$. Then $\frac{d}{dt}(G_{\dot{x}^i}) - G_{x^i} = 0$ and $\frac{d}{dt}(G_{\dot{\lambda}}) - G_\lambda = 0$ become

$$\dot{\mu} F_{\dot{x}^i} = (1-\mu)(\frac{d}{dt} F_{\dot{x}^i} - F_{x^i}) \text{ and } \dot{\mu} = (1-\mu) F_\lambda .$$

Eliminating $\dot{\mu}$ and $(1-\mu)$ yields the lemma.

Since the $g_{ij}(x^\alpha, \lambda)$ in the expression

$$\lambda(t) = \int_{t_0}^{t} \{ g_{ij}(x^\alpha, \lambda) \dot{x}^i \dot{x}^j \}^{1/2} d\tau$$

depend on x^α and on λ which in turn may also be a function of x^α, for clarity let $\partial_k = \frac{\partial}{\partial x^k}$ always imply λ fixed and let $D_k = D_{x^k}$ denote

$$D_k = \partial_k + \partial_k \lambda . \partial_\lambda ,$$

the total partial derivative with respect to x^α. The operation D_k implies of course a specified function $\lambda(x^\alpha)$. With this convention we may state the following

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THEOREM 1. In symmetric Riemann fatigue the Euler-Lagrange equations (geodesic equations) may be written in the form

$$\frac{\delta \dot{x}^i}{\delta t} = -g^{i\alpha}\partial_\lambda g_{\alpha\beta}\dot{x}^\beta + \frac{1}{2}\partial_\lambda g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta \dot{x}^i ,$$

where

$$\frac{\delta \dot{x}^i}{\delta t} = \ddot{x}^i + \gamma^i_{jk}\dot{x}^j \dot{x}^k$$

$$\gamma^i_{jk} = \frac{1}{2} g^{i\alpha}\{\partial_j g_{\alpha k} + \partial_k g_{\alpha j} - \partial_\alpha g_{jk}\}$$

and where the parameter is chosen such that

$$\dot{\lambda} = \{g_{ij}\dot{x}^i \dot{x}^j\}^{1/2} = 1 .$$

Proof. An immediate result of substituting $F = \{g_{ij}\dot{x}^i\dot{x}^j\}^{1/2}$ into (4).

In [3] it is shown that given a function $H\{x^\alpha, p_\alpha, \lambda\}$ which is plus-one in p_α and such that $\det(H^2_{p_i p_j}) \neq 0$, one can always associate a Lagrangian $F\{x^\alpha, \dot{x}^\alpha, \lambda\}$, plus-one in $\dot{x}^\alpha$, such that the characteristic equations of the partial differential equation

(5) $$H\{x^\alpha, p_\alpha, \lambda\} = 1 , \quad \text{where } p_\alpha = \partial_\alpha \lambda ,$$

coincide with the Euler-Lagrange equations associated with (1) or (1'). Since the present paper is concerned with Riemann fatigue it suffices to show that the function

(6) $$H\{x^\alpha, p_\alpha, \lambda\} = \{g^{ij}(x^\alpha, \lambda)p_i p_j\}^{1/2}$$

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is a Hamiltonian for $F\{x^\alpha, \dot{x}^\alpha, \lambda\} = \{g_{ij}(x^\alpha, \lambda)\dot{x}^i\dot{x}^j\}^{1/2}$. Here g^{ij} denotes the inverse matrix of g_{ij}, assuming as always that $\det(g_{ij}) \neq 0$.

LEMMA 2. The characteristic equations of the partial differential equation

$$(7.1) \qquad \{g^{ij}(x^\alpha, \lambda)p_i p_j\}^{1/2} = 1, \quad \text{where } p_i = \partial_i \lambda,$$

can be written in the form

$$(7.2) \qquad \dot{x}^i = g^{ij}p_j, \quad \text{implying } p_i = g_{ij}\dot{x}^j,$$

$$(7.3) \qquad \dot{p}_i = -\frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial x^i}p_\alpha p_\beta - \frac{1}{2}\frac{\partial g^{\alpha\beta}}{\partial \lambda}p_\alpha p_\beta p_i,$$

$$(7.4) \qquad \dot{\lambda} = \{g^{ij}p_i p_j\}^{1/2} = 1.$$

Proof. The characteristic equations for the general equation (5) are given by [4]

$$\dot{\lambda} = \sum_{\alpha=1}^{n} p_\alpha H_{p_\alpha} = H \text{ by homogeneity,}$$

$$\dot{x}^i = H_{p_i},$$

$$\dot{p}_i = -H_{x^i} - p_i H_\lambda,$$

so that (7.2), (7.3) and (7.4) follow immediately given (7.1).

If (7.1) is to be the Hamilton-Jacobi equation associated with (1) or (1'), one has merely to prove the following

THEOREM 2. The characteristic equations (7.2) and (7.3) coincide with the Euler-Lagrange equations as given in Theorem 1, so that (6) defines the Hamiltonian and (7.1) the Hamilton-Jacobi equation in symmetric Riemann fatigue.

Proof. By (7.2) $p_i = g_{ij}\dot{x}^j$, so that

$$\dot{p}_i = \frac{\partial g_{ij}}{\partial x^k}\dot{x}^k\dot{x}^j + \frac{\partial g_{ij}}{\partial \lambda}\dot{\lambda}\dot{x}^j + g_{ij}\ddot{x}^j .$$

But by (7.4), $\dot{\lambda} = 1$. Finally, since $g_{i\alpha}g^{\alpha j} = \delta_i^j$, it follows that

$$\partial_i g^{\alpha\beta} p_\alpha p_\beta = g_{\alpha k} g_{\beta j} \partial_i g^{\alpha\beta}\dot{x}^j\dot{x}^k = -\partial_i g_{jk}\dot{x}^j\dot{x}^k$$

and, similarly,

$$\partial_\lambda g^{\alpha\beta} p_\alpha p_\beta p_i = -\partial_\lambda g_{jk} g_{i\alpha}\dot{x}^j\dot{x}^k\dot{x}^\alpha .$$

Substituting in (7.3) yields

$$g_{ij}\ddot{x}^j + \partial_k g_{ij}\dot{x}^j\dot{x}^k - \frac{1}{2}\partial_i g_{jk}\dot{x}^j\dot{x}^k = -\partial_\lambda g_{ij}\dot{x}^j + \frac{1}{2}\partial_\lambda g_{jk} g_{i\alpha}\dot{x}^j\dot{x}^k\dot{x}^\alpha ,$$

proving the theorem.

A solution of (7.1) which is zero on a set P_0 will be called a distance function from P_0, and the geodesics which coincide with the corresponding characteristics (7.2) and (7.3) will be called λ-geodesics from P_0.

Tensors in Riemann Fatigue. To recapitulate the main formulas

$$F\{x^\alpha, \dot{x}^\alpha, \lambda\} = \{g_{ij}(x^\alpha, \lambda)\dot{x}^i\dot{x}^j\}^{1/2} ,$$

$$H\{x^\alpha, p_\alpha, \lambda\} = \{g^{ij}(x^\alpha, \lambda)p_i p_j\}^{1/2} ,$$

where g^{ij} and g_{ij} are symmetric inverse matrices. If λ is a distance function and $x^i(t)$ the corresponding λ-geodesic then

(8) $$\dot{x}^i = g^{ij} p_j = g^{ij} \frac{\partial \lambda}{\partial x^j} .$$

The geodesics are given by

(9) $$\frac{\delta \dot{x}^i}{\delta t} = \frac{1}{2} \partial_\lambda g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta \dot{x}^i - g^{i\alpha} \partial_\lambda g_{\alpha\beta} \dot{x}^\beta ,$$

where

$$\frac{\delta \dot{x}^i}{\delta t} = \ddot{x}^i + \gamma^i_{jk} \dot{x}^j \dot{x}^k$$

provided the parameter is chosen such that $\dot{\lambda} = 1$. Finally

(10) $$\partial_\lambda g_{ij} = -g_{i\alpha} g_{j\beta} \partial_\lambda g^{\alpha\beta}$$

and

$$\partial_k g_{ij} = -g_{i\alpha} g_{j\beta} \partial_k g^{\alpha\beta} .$$

Since the γ^i_{jk} are defined in terms of ∂_i implying λ fixed, many of the standard identities from Riemannian geometry carry over. In particular [5]

(11) $$\begin{cases} \partial_k g_{ij} = g_{\alpha j} \gamma^\alpha_{ik} + g_{\alpha i} \gamma^\alpha_{jk} \\ \gamma^\alpha_{\alpha i} = \partial_i \log \sqrt{g} \quad \text{where} \quad g = \det g_{ij} , \end{cases}$$

(assuming g positive, otherwise $\sqrt{-g}$ and

(12) $$\partial_i \gamma^{\alpha}_{\alpha j} = \partial_j \gamma^{\alpha}_{\alpha i} .$$

Since (3) or (3') defines a functional on curves, λ is invariant under coordinate transformations. Denoting the new coordinate system by primed indices it follows that the g_{ij} transform according to

(13) $$g_{i'j'}(x^{\alpha'}, \lambda) = A^i_{i'} A^j_{j'} g_{ij}(x^{\alpha}, \lambda) .$$

(This assumes that the coordinate transformation does not depend on λ, an assumption made throughout the remainder.) Differentiating with respect to λ implies

(14) $$\frac{\partial g_{i'j'}}{\partial \lambda} = A^i_{i'} A^j_{j'} \frac{\partial g_{ij}}{\partial \lambda}$$

so that g_{ij} and $\frac{\partial g_{ij}}{\partial \lambda}$ are tensors.

Perform the operation $D_{k'}$ on both sides of (13). It will be seen below that the result is the same whether $D_{k'}$ or $\partial_{k'}$ is used. Since $D_{k'} A^i_{j'} = \partial_{k'} A^i_{j'}$, the transformation not depending on λ, one obtains

$$D_{k'} g_{i'j'} = A^i_{i'} A^j_{j'} A^k_{k'} (D_k g_{ij}) + A^i_{i'k'} A^j_{j'} g_{ij} + A^i_{i'} A^j_{j'k'} g_{ij} .$$

By cyclic permutation this yields

(15) $$(D_{j'} g_{k'i'} + D_{i'} g_{j'k'} - D_{k'} g_{i'j'})$$

$$= A^i_{i'} A^j_{j'} A^k_{k'} (D_j g_{ki} + D_i g_{jk} - D_k g_{ij}) + 2A^i_{i'j'} A^j_{k'} g_{ij} .$$

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However, expanding $D_{i'} = \partial_{i'} + \frac{\partial}{\partial\lambda} \cdot \partial_{i'}\lambda$, it follows that the left side of (15) can be written

$$(\partial_{j'} g_{k'i'} + \partial_{i'} g_{j'k'} - \partial_{k'} g_{i'j'}) + (\partial_\lambda g_{k'i'} \partial_{j'} \lambda + \partial_\lambda g_{j'k'} \partial_{i'} \lambda - \partial_\lambda g_{i'j'} \partial_{k'} \lambda) .$$

In view of (14), the second term becomes

$$A^k_{k'} A^i_{i'} (\partial_\lambda g_{ki})(\partial_j \lambda) A^j_{j'} + A^j_{j'} A^k_{k'} (\partial_\lambda g_{jk})(\partial_i \lambda) A_{i'} - A_{i'} A_{j'} (\partial_\lambda g_{ij})(\partial_k \lambda) H^k_{k'}$$

which cancels with the corresponding term on the right of (15), verifying the previous statement that ∂_i could replace D_i. Hence (15) reduces to simply

$$(16) \qquad \{i'j', k'\} = A^i_{i'} A^j_{j'} A^k_{k'} \{ij, k\} + A^i_{i'j'} A^j_{k'} g_{ij} \ ,$$

where $\{ij, k\}$ is the Christoffel symbol of the first kind $g_{k\alpha}\gamma^\alpha_{ij}$. Equation (16) is identical to the Riemannian case, notwithstanding the fact that $\{ij, k\}$ is a function of λ. Hence (16) may be solved for $A^i_{i'j'}$ obtaining

$$(17) \qquad A^i_{j'k'} = \frac{\partial^2 x^i}{\partial x^{j'} \partial x^{k'}} = A^i_{i'} \gamma^{i'}_{j'k'} + A^j_{j'} A^k_{k'} \gamma^i_{jk} .$$

Covariant differentiation may now be defined using (17), and since the formula is identical to the Riemannian case it follows that covariant differentiation is also given by the classical formulas. If a tensor V^i depends on λ, two covariant derivatives may be distinguished.

Conservative differentiation

$$V^i_{|j} = \partial_j V^i + V^\alpha \gamma^i_{\alpha j}$$

Dissipative differentiation

$$V^i_{,j} = D_j V^i + V^\alpha \gamma^i_{\alpha j} = V^i_{|j} + \partial_\lambda V^i \bullet_j \lambda .$$

In view of (11) it follows that

$$g_{ij|k} = 0 \qquad g_{ij,k} = \partial_\lambda g_{ij} \partial_k \lambda .$$

Corresponding to conservative differentiation one may introduce

Auto-Parallel Curves

$$\frac{\delta \dot{x}^i}{\delta t} = \ddot{x}^i + \gamma^i_{jk} \dot{x}^j \dot{x}^k = 0 .$$

Since λ is kept fixed relative to the $|$ operation, it is clear that a curvature tensor $R^{\alpha}_{\cdot oijk}$ may be defined as in Riemannian geometry,

$$A_{i|jk} - A_{i|kj} = R^{\alpha}_{oijk} A_\alpha ,$$

and that $R^{\alpha}_{\cdot oijk}$ satisfies the usual identities relative to the $|$ operation. Taking $D_{r'}$ of both sides of (17) one finds that

$$A^i_{r'j'k'} = A^i_{k'j'r'}$$

if and only if

$$A^i_{\sigma'} R^{\sigma'}_{j'r'k'} = A^\alpha_{j'} A^\beta_{k'} A^\sigma_{r'} R^i_{\alpha\sigma\beta} ,$$

where

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$$R^k_{\ell ij} = \begin{vmatrix} D_i & D_j \\ \gamma^k_{\ell i} & \gamma^k_{\ell j} \end{vmatrix} + \begin{vmatrix} \gamma^k_{\beta i} & \gamma^k_{\beta j} \\ \gamma^\beta_{\ell i} & \gamma^\beta_{\ell j} \end{vmatrix} .$$

If we define $F_{ij} = R^\alpha_{\alpha ij} = D_i \gamma^\alpha_{\alpha j} - D_j \gamma^\alpha_{\alpha i}$, and expand D_i, the terms corresponding to ∂_i and ∂_j cancel as in the Riemannian case since (12) holds. Hence

$$(18) \qquad F_{ij} = \partial_\lambda \gamma^\alpha_{\alpha j} \partial_i \lambda - \partial_\lambda \gamma^\alpha_{\alpha i} \partial_j \lambda = -F_{ji} .$$

It was shown in J. Bazinet's thesis [2] that the Bianchi identity still holds in Riemann-fatigue geometry,

$$R^k_{\ell ij,r} + R^k_{\ell jr,i} + R^k_{\ell ri,j} = 0 ,$$

from which it readily follows that the analogue of Maxwell's first equations hold, namely

$$(19) \qquad F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 .$$

Equation (19) can be proved directly as follows. By (11)

$$\partial_\lambda \gamma^\alpha_{\alpha i} = \partial_i \partial_\lambda \ell n \sqrt{g} = \partial_i \phi$$

while, since λ is a function of the x's, $(\partial_i \lambda)_{,j} = (\partial_i \lambda)_{|j}$ $= (\partial_j \lambda)_{|i} = (\partial_j \lambda)_{,i}$. Writing $F_{ij} = \partial_j \phi . \partial_i \lambda - \partial_i \phi . \partial_j \lambda$, then

$$F_{ij,k} + F_{jk,i} + F_{ki,j} = \{(\partial_j \phi)_{,k} - (\partial_k \phi)_{,j}\} \partial_i \lambda + \{(\partial_k \phi)_{,i} - (\partial_i \phi)_{,k}\} \partial_j \lambda + \{(\partial_i \phi)_{,j} - (\partial_j \phi)_{,i}\} \partial_k \lambda .$$

But $(\partial_\alpha \phi)_{,\beta} - (\partial_\beta \phi)_{,\alpha} = \partial_\alpha \partial_\lambda \phi . \partial_\beta \lambda - \partial_\beta \partial_\lambda \phi . \partial_\alpha \lambda$, and substituting yields (19). The definition (18), or its equivalent

(18')
$$F_{ij} = \gamma^{\alpha}_{\alpha j, i} - \gamma^{\alpha}_{\alpha i, j} ,$$

is clearly analogous to the definition of the electro-magnetic field $F_{ij} = \partial_i A_j - \partial_j A_i$ as in classical texts [6].

The fact that F_{ij} is not trivially zero can be seen from the example $\int \{ (\dot{x}^1)^2 + (\dot{x}^2)^2 + (\dot{x}^3)^2 - e^{\lambda x^4} (\dot{x}^4)^2 \}^{1/2} dt$ for which

$$F_{ij} = \begin{pmatrix} 0 & 0 & 0 & \partial_1 \lambda \\ 0 & 0 & 0 & \partial_2 \lambda \\ 0 & 0 & 0 & \partial_3 \lambda \\ -\partial_1 \lambda & -\partial_2 \lambda & -\partial_3 \lambda & 0 \end{pmatrix} .$$

Thus far the tensors R^{α}_{oijk}, R^{α}_{ijk} and $F_{ij} = R^{\alpha}_{\alpha ij}$ have been introduced. Since R^{α}_{oijk} is identical to the Riemannian case, as is also the conservative operation "|" differentiation, the conservative Ricci and Einstein tensors can be defined

$$R_{oij} = R^{\alpha}_{oij\alpha} , \quad \underset{o}{R} = g^{\alpha\beta} R_{\alpha\beta} , \quad \underset{o}{G}^{ij} = \underset{o}{R}^{ij} - \frac{1}{2} g^{ij} \underset{o}{R} ,$$

and will satisfy the standard equations relative to conservative differentiation.

Hence
$$g_{ij|k} = 0 ,$$

$$R_{o\alpha ijk} = -R_{o\alpha ikj} = -R_{oi\alpha jk} = R_{ojk\alpha i} ,$$

$$R^{i}_{ojk\ell} + R^{i}_{ok\ell j} + R^{i}_{o\ell jk} = 0 ,$$

$$R^{i}_{ojk\ell|m} + R^{i}_{oj\ell m|k} + R^{i}_{ojmk|\ell} = 0 , \quad \underset{o}{G}^{ij}{}_{|j} = 0 .$$

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If these equations are taken in conjunction with auto-parallel curves, one obtains the standard Riemannian geometry in which appears a parameter λ. The conservative Riemann-fatigue geometry is obtained in other words by considering λ as locally fixed.

For the dissipative aspects of the geometry we have so far

$$g_{ij,k} = \partial_\lambda g_{ij} \partial_k \lambda ,$$

$$R^k_{\ell ij} = -R^k_{\ell ji} ,$$

$$R^k_{\ell ij,r} + R^k_{\ell jr,i} + R^k_{\ell ri,j} = 0 \quad \text{(proof not given)} ,$$

(20) $$F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 .$$

This list is now to be extended. There is no difficulty in verifying

$$R^\alpha_{ijk} + R^\alpha_{jki} + R^\alpha_{kij} = 0 .$$

Hence contracting on α and k one obtains

$$R_{ij} - R_{ji} = F_{ji} = R^\alpha_{\alpha ji}$$

so that the tensor F_{ij} is (except for a factor of 2) the non-symmetric part of the Ricci tensor R_{ij}.

If T^{ij} is an anti-symmetric tensor, then $T^{ij}{}_{,ji} = T^{ij} R_{ij}$, for

$$2T^{ij}{}_{,ji} = T^{ij}{}_{,ji} + T^{ji}{}_{,ij} = T^{ij}{}_{,ji} - T^{ij}{}_{,ij}$$

$$= T^{\alpha j} R^i_{\alpha ji} + T^{i\alpha} R^j_{\alpha ji} = T^{\alpha\beta} R^i_{\alpha\beta i} - T^{\beta\alpha} R^j_{\alpha\beta j} = 2T^{\alpha\beta} R_{\alpha\beta}$$

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The above derivation holds equally well for $T^{ij}{}_{|ji} = T^{\alpha\beta} R_{o\alpha\beta}$, and since $R_{o\alpha\beta}$ is symmetric we have

$$F^{ij}{}_{,ji} = F^{\alpha\beta} R_{\alpha\beta} \qquad F^{ij}{}_{|ji} = 0 .$$

Also, from (20), using the fact that $A^{\cdot\cdot}{}_{\ldots,i} = A^{\cdot\cdot}{}_{\ldots|i} + \partial_\lambda A^{\cdot\cdot\cdot}{}_{\ldots} . \partial_i \lambda$ it follows that

$$\{ F_{ij|k} + F_{jk|i} + F_{ki|j} \} + \{ (\partial_\lambda F_{ij}) \partial_k \lambda + (\partial_i F_{jk}) \partial_i \lambda + (\partial_\lambda F_{ki}) \partial_j \lambda \} = 0 .$$

However, substituting for F_{ij} from (18), that is

$$F_{ij} = (\partial_\lambda \gamma^\alpha_{\alpha j}) \partial_i \lambda - (\partial_\lambda \gamma^\alpha_{\alpha j}) \partial_j \lambda$$

the second bracketed term becomes zero ($\partial_\lambda \partial_i \lambda = 0$ since not a function of λ) so that we also have

$$F_{ij|k} + F_{jk|i} + F_{ki|j} = 0 .$$

Hence, recapitulating the properties of F_{ij} ,

$$(21) \qquad \begin{cases} F_{ij} = R^\alpha_{\alpha ij} = R_{ji} - R_{ij} , \\ F_{ij,k} + F_{jk,i} + F_{ki,j} = 0 , \quad F^{ij}{}_{,ji} = F^{\alpha\beta} R_{\alpha\beta} , \\ F_{ij|k} + F_{jk|i} + F_{ki|j} = 0 , \quad F^{ij}{}_{|jk} = 0 . \end{cases}$$

The following remarks may be of interest. First, since the transformation law for the Christoffel symbols is

$$\gamma^{i'}_{j'k'} = A^{i'}_i A^j_{j'} A^k_{k'} \gamma^i_{jk} + A^i_{j'k'} A^{i'}_i ,$$

they are of course not tensors. But since the transformation is

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assumed independent of λ, differentiation yields

$$\partial_\lambda \gamma^{i'}_{j'k'} = A^{i'}_{i} A^{j}_{j'} A^{k}_{k'} \partial_\lambda \gamma^{i}_{jk}$$

so that $\partial_\lambda \gamma^{i}_{jk}$ is a tensor.

Secondly, the geodesic equation and auto-parallel curves may be written

$$\frac{\delta \dot{x}^i}{\delta t} = \epsilon \{ -g^{i\alpha} \frac{\partial g_{\alpha\beta}}{\partial \lambda} + \frac{1}{2} \frac{\partial g_{\alpha\beta}}{\partial \lambda} \dot{x}^\alpha \dot{x}^i \} \dot{x}^\beta \quad , \tag{22}$$

where $\epsilon = 0$ yields auto-parallel, $\epsilon = 1$ yields geodesics. But since the right hand side is a tensor, and since it depends on the dissipative quantity $\partial_\lambda g_{ij}$, one can consider the family of curves for $0 \leq \epsilon \leq 1$, taking ϵ as a measure of the particle's reaction to the dissipative field, (somewhat like a charge).

Finally, the auto-parallel curves ($\epsilon = 0$), while dependent on λ, do not depend on the dissipative fields formed from $\partial_\lambda g_{ij}$. They are geodesics in the Riemannian geometry defined by the curvature tensor R^{α}_{oijk}. Hence classical gravitational field theory is applicable to them.

<u>Conformal Riemann Fatigue.</u> No restrictions have been placed on the geometry other than that it be symmetric Riemann fatigue. In this section a condition is imposed on the variation of g_{ij} with λ. This restriction will be written in the form

$$0 = \partial_\lambda \left\{ \frac{g_{i\alpha} \dot{x}^\alpha}{g_{\alpha\beta} \dot{x}^\alpha \dot{x}^\beta} \right\} \tag{23}$$

and clearly if $g_{ij}(x^\alpha, \lambda) = f(x^\alpha, \lambda) g_{ij}(x^\alpha, 0)$, where $f(x^\alpha, \lambda)$ acts as a gauge function, (23) is satisfied. Expanding (23) and using the fact that along geodesics the parameter is such that $g_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 1$ one obtains

(24) $$\partial_\lambda g_{\alpha\beta}\dot{x}^\beta = g_{\alpha\beta}\dot{x}^\alpha \partial_\lambda g_{\sigma\gamma}\dot{x}^\sigma \dot{x}^\gamma .$$

THEOREM 3. If the space is conformal in the sense of (23) then the geodesics and auto-parallel curves can be written

(25) $$\frac{\delta \dot{x}^i}{\delta t} = -\frac{\epsilon}{2} g^{i\alpha}(\partial_\lambda g_{\alpha\beta})\dot{x}^\beta$$

for $\epsilon = 1$ and 0 respectively. Further, let λ be a distance function from P_0 while $x^i(t)$ are λ-geodesics from P_0. Then these λ-geodesics satisfy

(26) $$g^{i\alpha}_{\ \ ,\alpha} = \rho \dot{x}^i \quad \text{where} \quad \rho = \partial_\lambda g^{\alpha\beta}\partial_\alpha \lambda \partial_\beta \lambda .$$

Proof. (25) is immediate upon substitution of (24) in (22). To obtain (26), recall that if $x^i(t)$ is a λ-geodesic then $\dot{x}^i = g^{i\alpha} p_\alpha = g^{i\alpha}\partial_\alpha \lambda$. Hence (24) may be written

$$g^{\nu\beta}\partial g_{\alpha\beta}\partial_\nu \lambda = g^{\sigma j} g^{\gamma k}\partial_\lambda g_{\sigma\gamma}\partial_j \lambda \partial_k \lambda g_{\alpha\beta}\dot{x}^\beta .$$

But multiplying by $g^{i\alpha}$ yields, in view of (10)

$$\partial_\lambda g^{ij}\partial_j \lambda = (\partial_\lambda g^{jk}\partial_j \lambda \partial_k \lambda)\dot{x}^i .$$

But the left side is precisely $g^{ij}_{\ \ ,j}$ and the theorem follows.

Motion of Charges in an E. M. Field. In flat space the equations of motion for a charged particle in an electro-magnetic field can be written in the form [6]

(27) $$\frac{\delta \dot{x}^i}{\delta t} = \epsilon\, g^{i\alpha} F_{\alpha\beta}\dot{x}^\beta + f^i \qquad \epsilon = \frac{e}{m}, \quad c = 1$$

where

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$$(28) \qquad f^i = \frac{2\epsilon e}{3}\left\{\frac{\delta^2 \dot{x}^i}{\delta t^2} - g_{\alpha\beta}\dot{x}^\alpha \frac{\delta^2 \dot{x}^\beta}{\delta t^2}\right\}.$$

If as an approximation one takes equation (27) with $f^i = 0$, substitution in (28) leads to the expression

$$f^i = \frac{2\epsilon^2 e}{3}\left(g^{i\alpha} F_{\alpha\beta,\gamma}\dot{x}^\beta \dot{x}^\gamma + \epsilon g^{i\alpha} F_{\alpha\beta} g^{\beta\sigma} F_{\sigma\gamma}\dot{x}^\gamma - \epsilon g^{\beta\sigma} F_{\alpha\beta} F_{\sigma\gamma}\dot{x}^\gamma \dot{x}^\alpha \dot{x}^i\right).$$

Substituting back in (27) leads to

$$(29) \qquad \ddot{x}^i + \left\{\gamma^i_{jk} - \frac{2\epsilon^2 e}{3} g^{i\alpha} F_{\alpha j,k}\right\}\dot{x}^j \dot{x}^k$$

$$= \epsilon g^{i\alpha}\left(F_{\alpha\beta} + \frac{2\epsilon^2 e}{3} F_{\alpha\gamma} g^{\gamma\sigma} F_{\sigma\beta}\right)\dot{x}^\beta$$

$$- \epsilon\left(F_{\alpha\beta} + \frac{2\epsilon^2 e}{3} F_{\alpha\gamma} g^{\gamma\sigma} F_{\sigma\beta}\right)\dot{x}^\alpha \dot{x}^\beta \dot{x}^i$$

where we have used the fact that $F_{\alpha\beta}\dot{x}^\alpha \dot{x}^\beta = 0$ since $F_{\alpha\beta}$ is skew-symmetric. A simpler expression can be formed in terms of the stress-energy tensor

$$T_{ij} = F_{i\alpha} g^{\alpha\beta} F_{\beta j} - \frac{1}{2} g_{ij} F^{\sigma}_{.\nu} F^{\nu}_{.\sigma}.$$

Substituting for $F_{i\alpha} g^{\alpha\beta} F_{\beta j}$ leads to the

Approximate equations of motion with damping

$$(30) \qquad \frac{{}'\delta \dot{x}^i}{{}'\delta t} = \epsilon g^{i\alpha}(F_{\alpha\beta} + KT_{\alpha\beta})\dot{x}^\beta - \epsilon(F_{\alpha\beta} + KT_{\alpha\beta})\dot{x}^\alpha \dot{x}^\beta \dot{x}^i$$

where $K = (2/3)\kappa^2 e$ and where $'\delta$ stresses the fact that the Christoffel symbols have been modified as indicated in (29). Here $g_{\alpha\beta}\dot{x}^\alpha\dot{x}^\beta = 1$.

In view of (25) and (26) and the similarity between (30) and (22) it seems significant to consider the case in which g_{ij} is not symmetric for possible applications to electro-magnetic and gravitational fields. This, it is hoped, will be the subject of a subsequent paper.

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