CHAPTER 1

Proto-arithmetical Abilities

In the Introduction, I stressed the importance of distinguishing between proto-arithmetical and proper arithmetical abilities. I suggested that the term 'number' should be reserved for developed mathematical abilities dealing with exact number concepts. In particular, talk of natural numbers should be restricted to the exact number concepts associated with arithmetic. Any quantitative ability that is not properly arithmetical or mathematical should be described in terms of treating *numerosities*. In Section 1.3, I will make the distinction between proto-arithmetical and arithmetical fully explicit. But, in order to do that, we first need to have a good understanding of what kind of abilities comprise proto-arithmetical cognition.

1.1 Object Tracking System and Subitising

Recall the experiment reported by Wynn (1992), in which five-month-old infants reacted to 'unnatural' numerosity of the dolls by a longer looking time (Figure I.1). This experiment has been replicated many times in different variations and it is strongly established that it is indeed the numerosity of the objects that the infants react to, instead of other variables such as total visible surface area (for a review study of the replications see Christodoulou et al., 2017). The methodology of Wynn's experiment is also widely accepted. Infants have a limited range of reacting, so longer looking times are generally interpreted as a reaction to them having seen something surprising (Cantrell & Smith, 2013). And the infants indeed did show significantly longer looking times in the trials in which one doll and one doll put behind the screen revealed only one doll in the end. They were apparently surprised by the 'unnatural' numerosity.[1] Indeed, one replication of the original experiment showed that the

[1] By 'unnatural' I mean nothing more than behaviour of objects that would generally be considered to conflict with physical laws concerning macro-level objects.

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infants were more surprised by the unnatural numerosity than they were by the actual dolls changing (from the *Sesame Street* character Elmo to Ernie, or vice versa) (Simon et al., 1995). The unnatural quantity of objects seemed to be more surprising than the unnatural changes in the *identity* of the objects. This suggests that not only is numerosity one factor in how infant observations are processed in their minds, but it is a particularly important factor. But the question is, were the infants really doing arithmetic – as claimed by Wynn – and if not, what was the cognitive capacity they were applying?

The first experiment to establish an infant ability with numerosities was reported by Starkey and Cooper (1980). They used the standard method of *habituation* to test whether infants (twenty-two weeks old) are sensitive to changes in numerosity. The infants were habituated to an array of two or three dots and then shown an array where the quantity of the dots had changed (either from three to two or from two to three). The children reacted by longer looking times. The experiment was controlled for other variables, such as the spatial arrangement of the dots, and Starkey and Cooper concluded that it was the numerosity that the infants were sensitive to. Importantly, they did not show the same ability to discriminate between the numerosity of the dots when it was increased to four and six. These results prompted Starkey and Cooper to conclude that the infants were *subitising* when discriminating between the numerosity of the dots.

The term 'subitising' was coined by Kaufman and colleagues (1949) and it refers to making fast, accurate judgements about the numerosity of observed items. The existence of this ability has been confirmed in many experiments and in adult humans it gives an alternative method to counting for determining the numerosity of objects reliably. Trick and Pylyshyn (1994) studied reaction times in determining the quantity of dots and saw a clear difference between one and four dots when compared to larger numerosities. While four and fewer items typically take 40–200 ms/item to enumerate, more than four items take 250–350 ms/item. As seen in Figure 1.1, there is a clear difference when the numerosity increases beyond four. This is because five is already beyond the *subitising range*. For numerosities from one to four, the subjects were able to subitise the answer, which is a much faster process. For numerosities larger than four, they needed to count to enumerate the objects.

What Starkey and Cooper (1980) established was that infants can already subitise. This ability seems to be behind the infant behaviour also in the experiment of Wynn (1992). Infants lack the resources to count, as

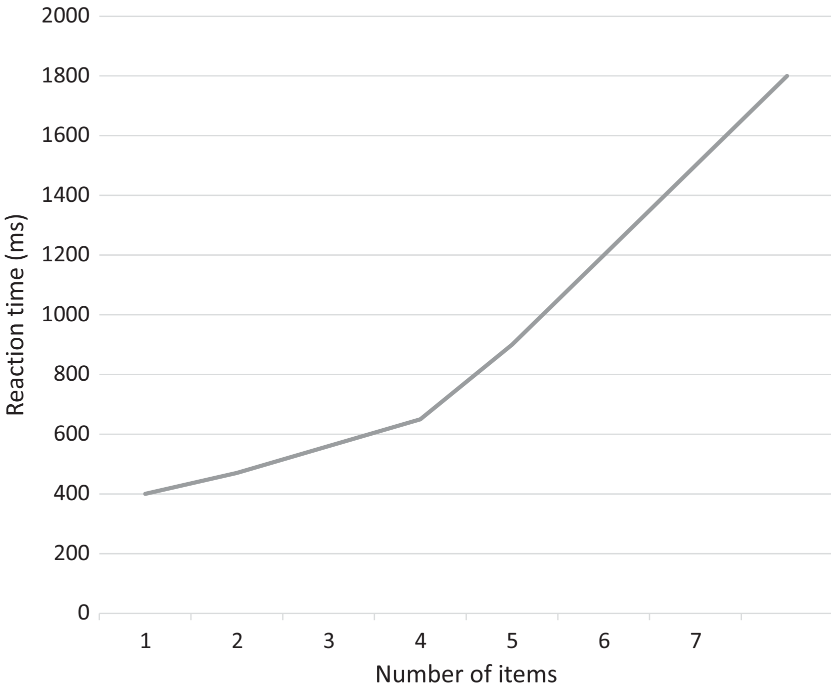

Figure 1.1 Reaction time in enumeration tasks.*
* We can see a clear difference in the reaction time (y-axis, milliseconds) when the number of dots increases over the subitising range (one to four).
Data from Trick and Pylyshyn (1994).

already suggested by their inability to discriminate between four and six dots in the experiment of Starkey and Cooper (1980), but they are able to subitise. So, what the infants could be doing in the doll experiment is subitising the numerosity of the visible dolls and then conducting calculations to create expectations concerning the numerosity of the dolls when the screen is lifted. This certainly appears to be Wynn's (1992) interpretation of the data: that subitising is a process that encodes numerical information. In short, according to her, the infants subitise the numerosity of the dolls and they possess numerical concepts that they can use to calculate expectations.

Others, however, are more sceptical. Uller et al. (1999) have suggested that a cognitively simpler explanation can explain the infant behaviour.

In contrast to the numerical concept model proposed by Wynn, they have suggested that an *object file* model can explain the same data. Object files refer to the way visual experience is processed cognitively to identify persisting objects (Kahneman et al., 1992; Noles et al., 2005). Kahneman and colleagues (1992) introduced object files as theoretical concepts to explain how successive states of objects are connected in visual experiences. Noles and colleagues (2005) have proposed that these object files can explain how our everyday perceptual experiences are formed in terms of persisting objects. The object files are closely connected to the ability to track multiple objects at once (Spelke, 2000; Trick & Pylyshyn, 1994). Three dots in our field of vision, for example, are not represented as some kind of observed 'threeness' but rather in three distinct object files.

Susan Carey (2009) has argued that object tracking is a *core cognitive* ability. Core cognition refers to the way human cognition is thought to begin with 'highly structured innate mechanisms designed to build representations with specific content' (Carey, 2009, p. 67). In this core cognitive framework, the *object tracking system* (OTS) is then considered to be the cognitive system responsible for the ability to subitise (Knops, 2020). Each observed object occupies an object file and this makes it possible to determine the numerosity of observed items without counting them. Since the object tracking ability is closely connected to the ability to individuate objects in a parallel fashion, the OTS is often also referred to as the *parallel individuation system* in the literature (e.g., Carey, 2009; Cheung & Le Corre, 2018; Hyde, 2011).

In Chapter 2, I will present in detail Carey's account of how natural number concepts are acquired on the basis of the OTS, but for now the important point is that the object tracking system and the associated object files provide an alternative explanation for the infant behaviour in Wynn's (1992) experiment. What if the dolls simply occupied object files in the minds of the infants? They expected the dolls to be persisting objects and, when there was only one doll instead of two, this violated their object file occupancy. As I see it, this explanation can accommodate the Wynn experiment in a satisfactory manner. Importantly, there is no need to presuppose that the infants were engaging in arithmetical calculations, nor that they process true numerical concepts. They could simply expect to see the dolls as persisting objects.

But what about the change from Elmo to Ernie (or vice versa), as reported by Simon and colleagues (1995)? If the above suggestion about object files is along the right lines, should the children not have been more surprised by the changing identity of the dolls than the changing numerosity of the dolls?

I believe that there is a good explanation also for this. The object tracking system, and hence the object files, are thought to be cognitively between low-level sensory processing and higher levels of cognition (Noles et al., 2005). The object files are not the kind of cognitive apparatus that is used for identifying dolls. They are much cruder, with the (putative) purpose of tracking objects across time and movement. Indeed, given the proposed character of the OTS, we would expect that infants are more sensitive to the changing numerosity than they are to the changing identity, at least as long as the latter is not more significant than a change from one doll to another.

The OTS, as should be clear by now, is not numerosity-specific. It is thought to be responsible for tracking objects and individuating them parallelly as persisting objects. However, since it makes the subitising ability possible, it is highly relevant for the topic of this book. Indeed, the OTS appears to be responsible for much of what is misleadingly called 'infant arithmetic' and 'animal arithmetic' in the literature. When infants in Wynn's (1992) experiment are supposed to do arithmetical calculations, what they could really be doing is simply applying the OTS in tracking parallel individual objects. When Hauser and colleagues observed similar behaviour in wild rhesus monkeys and Garland and Low with New Zealand robins, again the data could be explained as based on the OTS without postulating numerical concepts or arithmetical ability (Garland & Low, 2014; Hauser et al., 1996).

I do not mean to downplay the cognitive abilities of infants and non-human animals, nor to suggest that they neither observe cardinalities nor discriminate between numerosities. While the OTS is not a numerosity-specific system, it enables the subitising ability that I consider to be *proto-arithmetical*. As such, as we will see, it is highly important for explaining the development of arithmetical cognition. But it is important not to postulate cognitive abilities beyond what is needed for explaining the observed behaviour. The problem is that the ascription of arithmetical abilities can initially seem so fitting for the kind of behaviour we have been discussing. The explanation according to which infants first calculate 1 + 1 = 2, and the unfolding situation then violates their expectations, describes their behaviour in a very palpable, dare I say *exciting*, way. Yet the foundation of my methodology is that we should fight every temptation to propose such explanations if there is another, cognitively less demanding explanation available.[2] In the OTS, there is such an

[2] In comparative psychology, this principle is known as *Morgan's Canon* and it states roughly that animal activity should never be interpreted in terms of higher psychological processes if it can also be

explanation available for a lot of the observed infant and non-human animal behaviour. But not all, as we will see in Section 1.2.

1.2 Approximate Numerosity System

In Section 1.1, I have suggested that the infant behaviour in the experiment reported by Wynn (1992) and its replications can be explained by the activation of the object tracking system. However, McCrink and Wynn (2004) designed a computerised version of the experiment in which, instead of one and two dolls, five or ten blobs showed on the screen. In their study, nine-month-old infants showed similar longer looking times in cases of unnatural numerosity (equivalent, for example, to the unnatural arithmetic of 5 + 5 = 5) as the infants did in the original Wynn (1992) experiment and its many replications. But now the OTS hypothesis fails, because five and ten are both beyond the OTS range of numerosities. Certainly, the infants were not counting the items, so what was the cognitive capacity they were employing?

This area of research goes back to the nineteenth century, when the pioneering psychologists Ernst Weber and Gustav Fechner studied the way humans perceive physical stimuli. What they established was that the perception of many types of stimuli, related, for example, to brightness, loudness and weight, follows a similar law, according to which the noticeable difference between two stimuli is related to the overall magnitudes of the stimuli (Fechner, 1860). For example, humans are able to detect the difference between weights of 100 and 120 grams, but not between 1,000 and 1,020 grams, although the absolute difference in the weights is the same. They can, however, distinguish between the weights of 1,000 grams and 1,200 grams, where the ratio is the same as in the former case (100 : 120 = 1000 : 1200). This logarithmic relationship between actual physical intensity and perceived intensity has become known in the literature as *Weber-Fechner Law* (or nowadays, more commonly, simply Weber's Law) (Knops, 2020).

What does the Weber-Fechner Law have to do with the McCrink and Wynn experiment? As it turns out, one of the stimuli that humans (and many non-human animals) perceive according to the Weber-Fechner Law concerns numerosity. Data show that there is an evolutionarily developed non-verbal capacity to estimate numerosities beyond the OTS range (Dehaene, 2011). Dehaene has called this cognitive capacity *number sense* but more commonly in the literature it is referred to as the *approximate*

interpreted in terms of lower processes (Morgan, 1894). While Morgan's Canon can be problematic when used as a blunt instrument, I believe that some form of this kind of parsimony principle is justified.

number system (ANS) (e.g., Spelke, 2000). Here I will refer to this cognitive system as the *approximate numerosity system* (while retaining the same acronym ANS), in order to distinguish between the pre-verbal ANS-representations and exact number concepts.[3]

In an influential experiment, Xu and Spelke (2000) tested infants (six months of age) for the perception of numerosity stimuli. The infants were habituated to arrays of either eight or sixteen dots (Figure 1.2). When habituated to eight dots, infants reacted with surprise when presented with four or sixteen dots, but not when the array had twelve dots. When habituated to sixteen dots, they reacted to arrays of eight and thirty-two dots, but not twenty-four. This suggests that six-month-olds are sensitive to difference in numerosity when the ratio is 1:2. It has been shown that this ratio gets smaller with development. Lipton and Spelke (2003) showed that the ratio is 2:3 for nine-month-olds. This development in ANS-acuity continues all the way through pre-school and school years, normally peaking around thirty years of age (Halberda et al., 2012). However, at all stages the ANS-based numerosity estimations remain approximate and follow the Weber-Fechner Law. This can be seen in the two standard signatures of ANS-based estimation (Dehaene, 2003). First is the *distance effect*, meaning that distinguishing between numerosities becomes easier as the numerical distance between them increases. Second is the *size effect*, according to which the estimations become less accurate as the numerosities become larger.

In addition to visual stimuli, the ANS also activates from other kinds of sensory stimuli. Izard and colleagues (2009) habituated neonates with a series of tones and then presented them with a visual array of items. As in experiments with purely visual stimuli, the infants looked longer when the visual array was different in numerosity from the tones. This suggests that the ANS representations are indeed about abstract numerosities, in the sense that they are not specific to any particular type of sensory stimulus.

Not everyone agrees with this. It has been argued that, instead of evoking abstract representations numerosities, the data can be explained by cognitive systems of sensory integration involving continuous magnitudes (Gebuis et al., 2016; Leibovich et al., 2017). The main reason for this criticism of the ANS is due to what Clarke and Beck (2021) call the *numerical congruency effect*, which shows that judgements of numerosity are often influenced also by the perception of non-numerical magnitudes, such as object size. An instructive example of this was presented by Henik

[3] In the literature, ANS is also used as an acronym for *analogue* number system, referring to the same capacity (see, e.g., Geary et al., 2014).

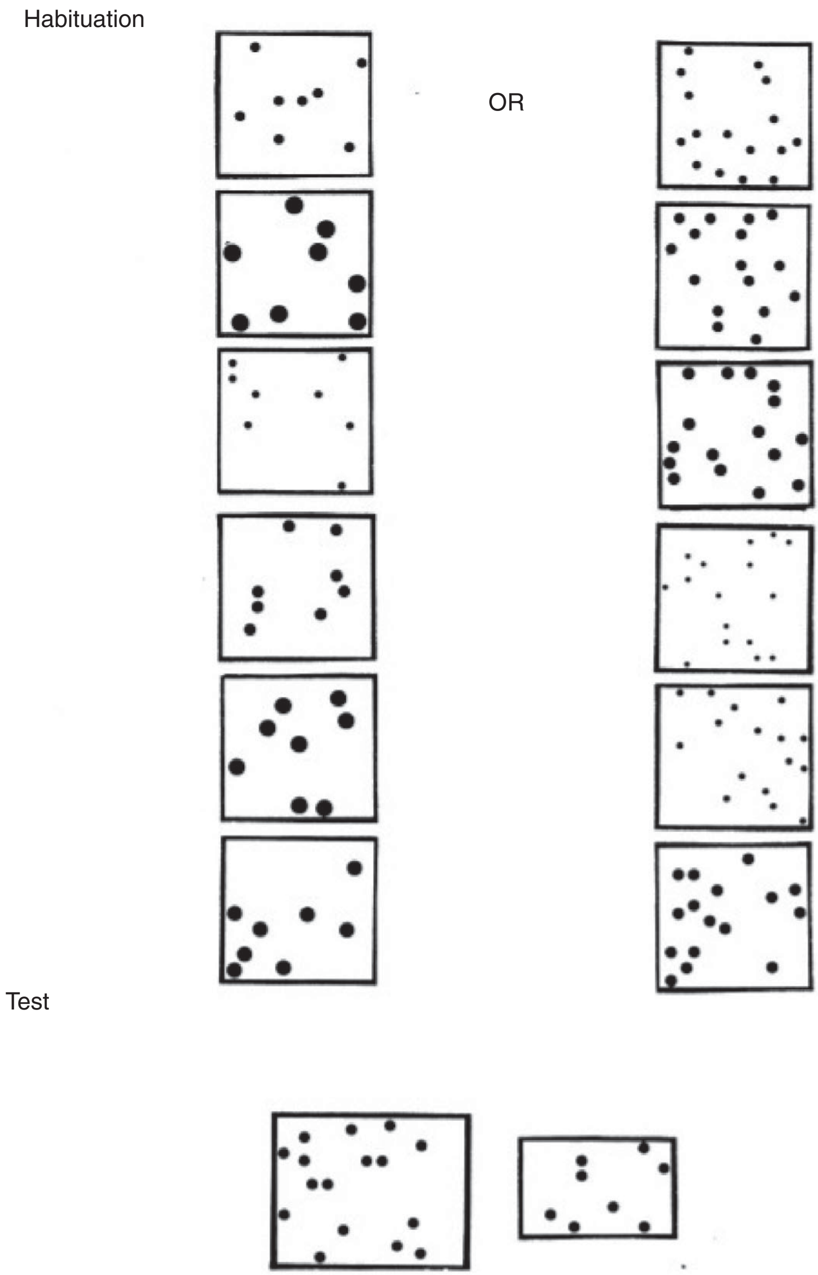

Figure 1.2 Numerosity estimation by infants.*
* Xu and Spelke (2000) habituated six-month-old infants to arrays of either eight or sixteen dots and tested their reactions to changing numerosity of dots.
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and Tzelgov (1982), who showed that (adult) subjects show a numerical *Stroop effect*, that is, they have longer reaction times and lower accuracy in estimating which number is larger when the size of the physical numeral symbol was incongruent to the numerical value (e.g., the pair 5 3 was more difficult to process than the pair 5 3). However, as argued by Clarke and Beck (2021), this kind of argument against the ANS is problematic, since congruency effects are also present in many judgements of magnitudes. For example, judgements of duration show a congruency effect on size, length and distance (Casasanto & Boroditsky, 2008; Sarrazin et al., 2004; Xuan et al., 2007). Yet, duration is uncontroversially accepted as being perceptible and not derived from other magnitudes.

Another argument against the ANS is what Clarke and Beck (2021) call the argument from *confounds*. As pointed out by Leibovich and colleagues (2017), in the experiments with arrays of dots, for example, there are always confounding variables. When the array of dots is changed and one non-numerical variable (e.g., item size or cumulative area) is controlled for, at the same time some other variable (e.g., the cumulative area or density) also changes. How is it possible to control for all non-numerical variables and thus be certain that it is the numerosity that the subjects are sensitive to? The simple answer is that it is not. The best we can hope for is to get enough corroborating cross-modal evidence to support the hypothesis that the ANS is sensitive to numerosity. Here I cannot go into the details of the cumulating evidence, but it seems clear that in many criticisms the ANS hypothesis is put under unreasonable criteria. There are very few experiments that can be said to test sensitivity to only one property of sensory stimuli. The arguments from congruency and confounds are problematic because by the same criteria many other cognitive capacities in detaching one property (duration, size, luminance, etc.) would be in doubt. In this, the ANS does not fare any worse and, hence, I do not see a convincing reason to doubt that the ANS is indeed sensitive to numerosity.[4]

Now the question is, if the ANS represents numerosity, *how* does it do it? As we have seen, the OTS is thought to represent numerosity implicitly in object files. What is the mechanism underlying the ANS? Following the work of Dehaene (2003), it is common to speak of the ANS as representing numerosities in a *mental number line*. Again, I want to correct the terminology to *mental numerosity line*. Dehaene argued that the Weber-Fechner Law of numerosity estimations can be explained from a neural

[4] See Clarke and Beck (2021) for a more thorough discussion on the topic. However, it is important to keep in mind that they explicitly argue that the ANS is sensitive to *number*, which I disagree with.

basis by postulating an innate, logarithmic mental line on which numerosity estimations are represented (Dehaene, 2003).[5]

On a logarithmic number line, the distances between numbers become smaller as the numbers become larger. On a *linear* number line, which most students become familiar with on the primary level, numbers are evenly spaced. This question of 'logarithmic or linear' has been an important topic in research on numerical cognition. However, it is important to note that there are two different questions involved. First is whether the ANS-based *estimations* follow a logarithmic rather than linear structure. The second concerns how the ANS-based *representations* are realised in the brain. When it comes to the first question, there is little doubt (see, e.g., Nieder & Dehaene, 2009). In general, the ANS-based estimations are characterised by their logarithmic manner, following the Weber-Fechner Law: establishing the difference between, say, sets of six and seven objects is easier than that between sixteen and seventeen objects. Importantly, this is a property of the proto-arithmetical processing of numerosities. Empirical data show that, when asked to place numerosities on a line, people in cultures with limited or no proper numeral systems (such as the Pirahã and the Munduruku of the Amazon) place the numerosities in a way that is best modelled as logarithmic (Dehaene et al., 2008; Pica et al., 2004).[6]

But the second question, how the ANS-based representations are realised in the brain, is more controversial.[7] The way the ANS represents numerosities has been debated ever since Meck and Church (1983) proposed their 'accumulator' model in which numerosity information extracted from sensory input is represented on a linear scale. This account was first contested by Dehaene and Changeux (1993), who proposed a neuronal model in which the ANS-based representations are logarithmic.

[5] Interestingly, there is new research suggesting that already the pupil of the eye is sensitive to numerosity. According to a study reported by Castaldi and colleagues (2021), numerosity modulates an automatic reflex in pupil dilation, 'suggesting that numerosity is a spontaneously encoded visual feature' (p. 1). If this is confirmed, it is reason to not approach ANS in a brain-centric manner but consider it in a wider context of our neural architecture connected to sensory organs.

[6] However, this is not always the case (Núñez, 2011). Perhaps most importantly, Núñez points out that 37 per cent of the experimental runs on the Munduruku reported in Dehaene et al. (2008) showed a bimodal response that used only the endpoints of the number line. Thus, the results reported by Dehaene and colleagues are better interpreted in a conditional manner: if people place numerosities on the entire number line in non-arithmetical cultures, it is in a logarithmic rather than linear manner.

[7] Instead of representations, some authors write about 'coding schemes' (see, e.g., Dehaene, 2001b). As mentioned earlier, I am not committed to a representationalist model of the brain, so a non-representationalist is welcome to interpret this part to concern coding schemes.

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This model has since received important support from new experiments, which show that the neuronal activity in the prefrontal cortex in primates is best described by a 'nonlinear compressed scaling of numerical information' (Nieder & Miller, 2003). In further research, Andreas Nieder (2016) has established that the primate brain has dedicated 'number neurons' in the intraparietal sulcus (IPS), which are thought to encode the numerosity of items in a sensory stimulus. These neurons – better called 'numerosity neurons' in the present terminology – appear to activate in association with the same numerosity independently of the sensory modality, suggesting that they represent numerosity in an abstract manner.[8] The numerosity neurons carry some 'noise', that is, when the neurons associated with the observation of seven items activate, so do some of the neurons associated with six and eight. This noise increases as the numerosities become larger, thus reflecting the Weber-Fechner Law (Nieder, 2016).[9]

If Nieder is correct, then ANS-based representations can be located in the brain, more specifically in the intraparietal sulcus. But how are the representations coded and organised neurally? A lot of further work is needed on both questions, but the data reported by Nieder suggests that the numerosity representations are in some way located in a spatially ordered fashion (Nieder, 2006). Thus, the metaphor of a mental numerosity line may not be, to some degree, only a metaphor. The well-known phenomenon of *spatial-numerical association of response codes* (SNARC) seems to support this. The SNARC effect shows that Western subjects standardly respond more quickly to small numerosities if they are on the left extrapersonal hemiside of their perceived environment, whereas they respond faster to larger numerosities if they are on the right extrapersonal hemiside of their perceived environment. This suggests that there is a spatial association of small numerosities being on the left side and

[8] Not everybody agrees. Vandervert, for example, argues that given the association of the prefrontal and parietal brain regions also with movement in space and time, it 'might therefore be more appropriate to suggest that the many research findings leading to Nieder's "number neurons" are simply about differential magnitudes related to the dynamics and kinematics of cognition and movement associated with the perceptual–cognitive control of body movement that occurred in humans, for example during stone-tool evolution' (Vandervert, 2021, p. 125).

[9] In addition to primate studies, brain imaging on humans show activation in the IPS connected to numerical tasks. Further evidence for the location of non-verbal numerosity representations in the brain comes from malfunctioning proto-arithmetical abilities in subjects who have suffered brain injury localised in the left parietal lobe, including the left intraparietal sulcus (Ashkenazi et al., 2008; Lemer et al., 2003). For more details on this topic, as well as a good general exposition of the research on non-verbal ability with numerosities, I refer the reader to Nieder (2019).

larger on the right side, as with the standard number line (Dehaene et al., 1993).[10]

The SNARC effect has also been reported in non-human animals, even in new-born chicks that, like humans, seemed to place smaller numerosities on the left side and larger ones on the right (Rugani et al., 2015). This has been seen as evidence for the early evolutionary origins of the mental number line (see, e.g., Knops, 2020, p. 61). However, Beran and colleagues (2008) failed to observe any space-numerosity association effect in rhesus and capuchin monkeys. It could well be that there is a missing factor in the experiment on chicks that made them associate smaller numerosities with the left side, and a mental number line was not responsible for this behaviour. Thus, with the present evidence, we should be careful about making inferences from the SNARC effect.

Finally, it should be noted that not all researchers agree that the ANS and the OTS are separate core cognitive systems. While this is the view held by most researchers (e.g., Agrillo, 2015; Carey, 2009; Dehaene, 2011; Feigenson et al., 2004; Hyde, 2011; Spelke, 2011), recently Cheyette and Piantadosi (2020) have proposed that there could in fact only be one innate numerosity system. One common argument for the existence of two systems is the discontinuity between identifying items in collections in the OTS range and those beyond that range. While collections of up to four items are identified almost perfectly, following the Weber-Fechner Law larger collections are identified increasingly inaccurately. Cheyette and Piantadosi, however, present a mathematical model which suggests that this difference between small and large numerosities would be part of an optimal representation of cardinal numerosities also in a single cognitive system. This question requires further study, but it is by no means certain that the OTS and ANS are indeed separate cognitive systems.[11]

The question of whether there are one or two core cognitive systems sensitive to numerosity of course prompts the question of just what a core cognitive system *is*. Every cognitive system comparable to the OTS and the ANS is a higher-level abstraction of neuronal activity. As such, we should not be ontologically committed to cognitive systems beyond what is

[10] The direction of SNARC effect does not vary with handedness, but it does change with the direction of writing. Iranian and Arabic subjects accustomed to right-to-left writing responded faster to larger numerosities on the left extrapersonal hemiside (Dehaene et al., 1993; Zebian, 2005). Thus, the evidence points to spatial association with numerosity size being universal, but the direction depending on the cultural custom of writing.

[11] In Pantsar (2014), I supported the one-system theory myself, but in the face of the state of the art in empirical research, in later works I have focused on the two-system model.

necessary for explanations. If Cheyette and Piantadosi are correct and there can be a mathematical model that necessitates only one system, with an Occam-like principle it would seem to be preferable over two-system models. However, a mathematical model should also be consistent with brain-imaging data. Supporting the two-system model, subitising and estimating have been reported to have different neural correlates in an fNIRS (functional near-infrared spectroscopy) study (Cutini et al., 2014).[12]

In this book, I will follow the two-system model, but I recognise that this remains an open question. As of now, the evidence still appears to favour the existence of the OTS and the ANS as distinct systems, and the association of the subitising ability with the former and the estimation ability with the latter. It could be that this understanding of the core cognitive abilities needs to be adjusted, but for the present approach that would not make a dramatic difference. The account I will be developing is compatible also with a single-system account. Instead of the particular core cognitive systems, the present account makes important use of the subitising and estimating *abilities*. That these are two different abilities is much less controversial than the two cognitive systems being different.[13]

1.3 Proto-arithmetic and Arithmetic

In Sections 1.1 and 1.2 we have seen that many experiments with infants and non-human animals can be explained by them employing the object tracking system or the approximate numerosity system (or perhaps both), with no reason to postulate more sophisticated numerical abilities to explain the behaviour. However, not all experimental data can be explained this way. Cantlon and Brannon (2007) conducted an experiment in which macaques were shown two configurations of dots on a screen (for 500 milliseconds each, corresponding to, for example, 1 + 1 = 2 or 2 + 2 = 4) and then were made to choose between two options on a

[12] Recently there have also emerged results from AI research that have been interpreted to question the existence of a numerosity-specific system such as ANS. These results show that generic artificial neural networks going through unsupervised learning develop a human-like acuity in numerosity estimation, suggesting that, instead of ANS being numerosity-specific, it could be a more general characteristic of learning through visual observations (Stoianov & Zorzi, 2012; Testolin et al., 2020). While these results are very early and tentative, they could point to an interesting new research direction in the development of numerical cognition. For more on this topic, see Pantsar (2023a).

[13] This state of affairs was very different at the turn of the century. Dehaene (1997), in the first edition of *The Number Sense*, for example, was a proponent of the view that the subitising ability is fundamentally a part of the estimation ability. See Section 2.3 for more.

screen for the correct sum. What they observed was that the macaque performance was largely comparable to that of college students in the same non-verbal task. The conclusion of Cantlon and Brannon was that the 'data demonstrate that nonverbal arithmetic is not unique to humans but is instead part of an evolutionarily primitive system for mathematical thinking shared by monkeys' (Cantlon & Brannon, 2007, p. 1).

These data are difficult to explain by simple mental numerosity representations of cardinality. What the macaque were reported to be doing seem to be genuine *operations* on numerosities. What this suggests is that the OTS- and ANS-based numerosity representations can be manipulated mentally. This is an important finding and is corroborated by many experiments. Indeed, the macaque behaviour is far from the only reported infant and non-human animal behaviour concerning numerosities that is truly remarkable (see, e.g., Agrillo, 2015; and Pepperberg, 2012 for studies on new-born chicks and parrots, respectively). Furthermore, it seems that numerosity is the decisive factor in most of the reported cases. Yet, again, we should be cautious not to postulate more sophisticated cognitive abilities than is needed to account for the behaviour. That is why, also in this case, it is important to stick to the distinction between proto-arithmetic and arithmetic.

Due to the short reaction times required in the experiment reported by Cantlon and Brannon, the college students could not revert to their arithmetical ability. With more time, they could have easily counted the dots and outperform the macaques. One main point of the experiment was to deny them that chance, in order to test their 'nonverbal arithmetic' skills against those of the macaques. However, while it is possible that there are nonverbal arithmetical skills, the experiment of Cantlon and Brannon did not test them. What it tested was the way macaques and humans use their proto-arithmetical, evolutionarily developed, skills in cardinality processing tasks. As such, however, the results are highly important. They show that monkeys can process operational transformations of numerosities. Indeed, another study by Cantlon and colleagues showed that this monkey proto-arithmetical ability carries some of the same signatures as human (proper) arithmetical ability, like the problem size effect, according to which reaction times and error rates increase as the numerosity sizes become larger (Cantlon et al., 2016).

Yet even this should not make us think of human arithmetical ability merely as a specification of evolutionarily developed proto-arithmetical abilities. The problem size effect, for example, is a common characteristic of both proto-arithmetic and arithmetic, but this can be due to different factors. While with the ANS the problem size effect is due to the

Weber-Fechner Law, with mental arithmetic it can be due to the increasing computational and cognitive *complexity* of the task (Buijsman & Pantsar, 2020; Pantsar, 2019a, 2021b). Due to limitations in working memory and attention span, for most people arithmetical calculations quickly become prohibitively complex to be carried out mentally (see, e.g., Imbo et al., 2007). It is possible that there are some common grounds for the problem size effect for proto-arithmetical tasks and arithmetical tasks. But it is also likely that, due to the important differences in the tasks – for example, only the latter including verbal or symbolic processing – the two effects have partly different causes. For this reason, we cannot assume that arithmetical calculations are a developmental continuation of proto-arithmetical transformations of numerosities.

Nevertheless, there are important connections between arithmetical and proto-arithmetical processes. For example, consider the following extremely simple task: Which one of the numbers below is bigger?

4 5

Compare that to the task of finding out which of these numbers are bigger:

4 9

One would not expect arithmetically skilled humans to show differences in reaction times, but the data show that the top pairing takes considerably longer than the bottom pairing. This is consistent with the distance effect: the larger the numerical distance between two numbers, the faster we are in solving the problem (Dehaene, 2011, pp. 62–64). It is interesting that the distance effect remains even when the test subjects are trained to solve the problem. It even remains when the quickest way of solving the problem only requires comparing the first digits of multi-digit numbers. For example, consider the pairing:

71 65

Now compare it to the pairing:

79 65

In this case, one would certainly expect the reaction times to be similar, given that in both cases it only needs to be established that 7 is bigger than 6. Yet the distance effect remains (Hinrichs et al., 1981; Pinel et al., 2001). It even remains when the task is to determine whether two numbers are the *same*, in which case it suffices to only determine that two physical shapes are different, as well as when numeral words replace numeral symbols (Dehaene & Akhavein, 1995).

The generally accepted explanation for all these results is that seeing numeral symbols (or numeral words) automatically activates the ANS and makes us process the numerosities as magnitudes (Dehaene, 2011, p. 16). Conforming to the Weber-Fechner Law, when the numerical distance is larger, the task of choosing the bigger number gets easier. It is important to note that the distance effect, unlike the problem size effect, is not a characteristic of arithmetical cognition. But even in tasks where arithmetical skills would be all that is needed, and the proto-arithmetical skills indeed only delay the correct solution, it seems that we cannot escape our proto-arithmetical origins. Even when trained for particular tasks that do not require assessing the numerical size, subjects cannot help processing it.

What these, and many other, experimental data suggest is that there is a connection between proto-arithmetical abilities and arithmetical abilities that endures also in arithmetically skilled individuals. Thus, the question becomes how the proto-arithmetical abilities are employed in the development of arithmetical abilities and arithmetical knowledge. In early literature on the development of numerical cognition, it was thought that arithmetic is made possible by first forming semantic representations of numerals and numeral symbols (e.g., McCloskey, 1992; McCloskey & Macaruso, 1995). This was contrasted by the 'triple-code' model of Dehaene and Cohen (1995), in which there are thought to be different representations of numerosities (visual symbolic, auditory verbal and approximate numerosity estimations). In the triple-code model, these representations have different roles in numerical cognition tasks. This model has been further refined by Campbell and colleagues (Campbell, 1994; Campbell & Epp, 2004), whose 'encoding complex' model includes interactions between the different kinds of representations.[14] In Chapter 2, we will focus on the issue of how the representations in the triple-code model are made possible. Indeed, so far with regard to the ANS, we have only focused on the third type of representation, that of the approximate numerosity estimations. Thus, the big question is how these proto-arithmetical representations – as well as those provided by the

[14] That there are different representations is supported by data showing activation in different brain areas (Nieder, 2019, p. 211). In a particularly interesting finding, it has been established that seeing numeral symbols is associated with activation of a special area of the temporal lobe, the ventral occipito-temporal cortex, also called the 'number form area' in the literature (Shum et al., 2013). This area shows increased activity with numeral symbols compared to other symbols, which can also help explain why dyslexia and *dyscalculia* (difficulty in learning arithmetic) are distinct learning disabilities (Nieder, 2019, p. 188).

OTS – are employed in acquiring exact number concepts, that is, arithmetical representations of cardinalities.

Before we move on to that question, it should be noted that not everyone wants to make the present distinctions between proto-arithmetic and arithmetic, as well as numerosities and numbers, even when aware of the kind of problems I have been discussing. I assume that, by now, most empirical researchers have encountered the kind of criticism that I have presented against incongruent use of terminology, yet they do not seem to have a problem with continuing to write about 'infant arithmetic' or 'animal arithmetic'. For the most part, such issues do not seem to be seen as particularly important among empirical researchers. Some authors, however, explicitly justify their choice of terminology. Carey, for example, writes:

> In the literature on mathematical cognition, analogue magnitude number representations are sometimes called 'numerosity' representations, for they are representations of the cardinal values of sets of individuals, rather than fully abstract number representations. There is no evidence that animals or babies entertain thoughts about 7 (even approximately 7) in the absence of a set of entities they are attending to. Still, cardinal values of sets are numbers, which is why I speak of analogue magnitude number representations rather than numerosity representations. (Carey, 2009, p. 136)

I can understand the practical side of using the more familiar word 'number' instead of the technical term 'numerosity'. But Carey's reason for her use of vocabulary is hardly satisfactory. While it is correct that (cardinal) numbers are measures of cardinality of sets, so are (cardinal) numerosities in the sense used by myself and others (e.g., De Cruz et al., 2010; Dos Santos, 2022). The entire point of introducing a new technical term such as 'numerosity' is to differentiate between two ways of representing the cardinal values of sets. Hence, Carey's reasoning for sticking with the term 'number' is quite problematic.

Another explicit justification for speaking about numbers instead of numerosities in the context of the ANS has recently been given by Clarke and Beck (2021). They argue that:

> [T]he ANS represents numbers (i.e., that numbers serve as the referents of the ANS), but under a unique mode of presentation that respects the imprecision inherent in the ANS ... This will allow us to avoid a commitment to exotic entities such as 'numerosities' without losing sight of the important differences between ANS representations and the precise numerical concepts that emerge later in development. (Clarke & Beck, 2021, p. 5)

Like others – for example, Dutilh Novaes and dos Santos (2021) and Núñez et al. (2021) – I find this justification equally unsatisfactory to Carey's. In my terminology, the ANS represents *cardinality* in terms of *numerosity representations*. With these distinctions in place already in the terminology, it is impossible to lose sight of the differences between the ANS representations and precise numerical concepts. Knowing how often in the literature problems have arisen because the two have been confused, why not distinguish between them in the terminology? If the reason is simply that numerosities as entities are considered to be 'exotic', surely that is a weak justification. As Clarke and Beck concede, we need to distinguish between ANS representations and exact number concepts in any case. So, if there is an exotic entity involved, it is already included in the ANS representations. It only makes sense that we would use a different scientific term when introducing theoretically novel entities like that. Hence, to conclude this chapter, I believe it is more important than ever to distinguish between proto-arithmetic and arithmetic, as well as between numerosities and numbers. In Chapter 2, we will fully grasp why, as we tackle the question of how number concepts can arise on the basis of numerosity representations.

1.4 Summary

In this chapter, I presented the *proto-arithmetical* abilities, that is, *subitising* and *estimating*, and emphasised the importance of distinguishing them from arithmetical abilities. I reviewed the empirical literature on the cognitive basis of proto-arithmetical abilities, focusing on the *core cognitive* theory of the object tracking system (OTS) and approximate numerosity system (ANS). Although the topic requires further research, I proceed with the view that the OTS and ANS are different cognitive systems and responsible for subitising and estimating abilities, respectively.