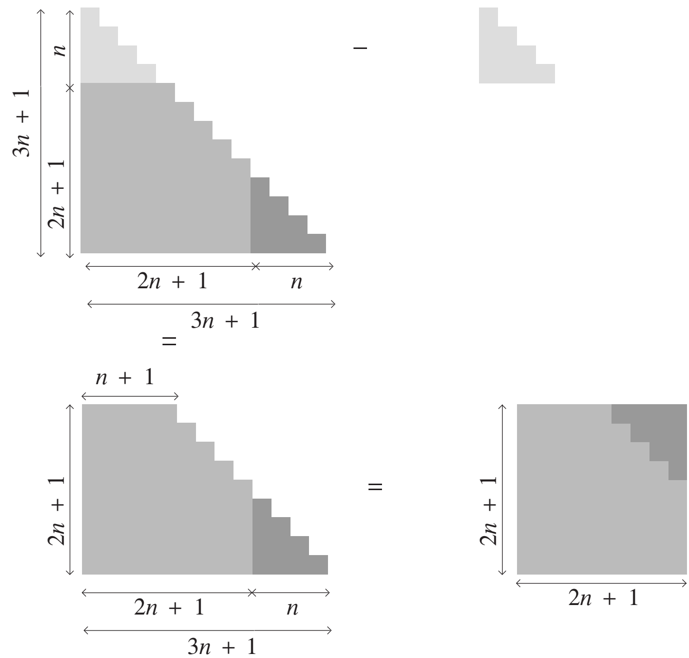
108.26 PWW: A property of triangular numbers

$$T_n = \frac{n(n+1)}{2} \Rightarrow T_{3n+1} - T_n = (2n+1)^2$$



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K. B. SUBRAMANIAM
Ganit Ashram,
135, Fine Avenue Phase-I,
Nayapura, Kolar Road,
Bhopal- 462042. India
e-mail: *kbsubramaniam.1950@gmail.com*
AMARNATH MURTHY
D-503, Sharan Residency,
New C G Road, Chandkheda,
Ahmedabad, Gujarat- 382424. India
e-mail: *amaranth.1960@gmail.com*

108.27 More on the Euler limit for *e*

The well-known *Euler limit* is defined as $\lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n = e = 2.71828\ldots$ (see for example [1]). Recently, in [2], appeared the following generalisation of the Euler limit.

Theorem 1: Let A_n be a strictly increasing sequence of positive reals satisfying $A_{n+1} \sim A_n$. Then

$$\lim_{n \to \infty} \left(\frac{A_{n+1}}{A_n} \right)^{\frac{A_n}{A_{n+1} - A_n}} = e.$$

Note that the symbol "$\sim$" means asymptotic equivalence, i.e., $x_n \sim y_n$ if $\lim_{n \to \infty} \frac{x_n}{y_n} = 1$.

Here, we offer the following generalisation.

Theorem 2: Let A_n be a strictly monotone sequence of positive reals satisfying $A_{n+1} \sim A_n$. Let B_n be any sequence of reals satisfying $B_n \sim \frac{A_n}{A_{n+1} - A_n}$. Then

$$\lim_{n \to \infty} \left(\frac{A_{n+1}}{A_n} \right)^{B_n} = e.$$

Proof: First, we consider the case of A_n monotone increasing. Theorem 1 gives

$$\lim_{n \to \infty} \left(\frac{A_{n+1}}{A_n} \right)^{B_n} = \lim_{n \to \infty} \left(\left(\frac{A_{n+1}}{A_n} \right)^{\frac{A_n}{A_{n+1} - A_n}} \right)^{\frac{B_n(A_{n+1} - A_n)}{A_n}} = e^1 = e.$$

Now we consider the other case, of A_n monotone decreasing. We set $A_n' = \frac{1}{A_n}$ and $B_n' = B_n$ to get

$$\lim_{n \to \infty} \left(\frac{A_{n+1}}{A_n} \right)^{B_n} = \lim_{n \to \infty} \left(\frac{A_n'}{A_{n+1}'} \right)^{B_n'}.$$

We conclude by observing that $B_n \sim \dfrac{A_n}{A_{n+1} - A_n} = -\dfrac{A_{n+1}'}{A_{n+1}' - A_n'} \sim -\dfrac{A_n'}{A_{n+1}' - A_n'}$, and applying the first case to B_n' and the monotone increasing A_n'. Theorem 2 is proved.

Theorem 2 allows us to compare the speed of convergence of $\left(\frac{A_{n+1}}{A_n} \right)^{B_n}$ towards e as n increases by choosing different sequences A_n and B_n. For example, let $A_n = n$, $B_n = n$, $n = 100$. This gives $\left(\frac{A_{n+1}}{A_n} \right)^{B_n} \simeq 2.7048$. If $A_n = n$, $B_n = n + \frac{1}{2}$, $n = 100$, then $\left(\frac{A_{n+1}}{A_n} \right)^{B_n} \simeq 2.7183$, which is a much better estimate. However, for these two examples, it can be seen that when n increases, the speeds of convergence in the two cases approach each other.

By changing A_n and B_n, we can further generalise Theorem 2. We take $A_{n+1} = A_n(1 + \varepsilon_n)$, where $\varepsilon_n \to 0$. Our previous assumptions of monotone increasing (decreasing) A_n now correspond to ε_n positive (negative). We have $B_n \sim \frac{1}{\varepsilon_n}$. Set r_n to be a positive sequence with $r_n \to 1$. Now, Theorem 2 is equivalent to

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\frac{r_n}{\varepsilon_n}} = e. \tag{1}$$

The sign of ε_n does not matter for this limit, so we can generalise the left-hand side of (1). For any constant k and δ_n a sequence with $|\delta_n|$ monotone decreasing to 0, we have

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\delta_n + k} = 1. \tag{2}$$

Multiplying (1) by (2) we obtain

$$\lim_{n \to \infty} (1 + \varepsilon_n)^{\frac{r_n}{\varepsilon_n} + \delta_n + k} = e. \tag{3}$$

This allows the reader to choose parameters to optimise convergence.

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2. R. Farhadian, A generalization of Euler's limit, *Amer. Math. Monthly* **129** (2022) p. 384.

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REZA FARHADIAN
Department of Statistics,
Razi University,
Kermanshah, Iran
e-mail: *farhadian.reza@yahoo.com*
VADIM PONOMARENKO
Department of Mathematics and Statistics,
San Diego State University, San Diego, USA
e-mail: *vponomarenko@sdsu.edu*

108.28 π is a mean of 2 and 4

A series of *Mathematical Gazette* contributions, [1, 2, 3, 4], deals with limits of infinite sequences where the first n entries are specified and where latter entries correspond to a specified type of average of the n preceding entries. To the list of recursively defined averages may be added also the more well-known arithmetic-geometric mean, the arithmetic-harmonic mean and the geometric-harmonic mean. We are not aware of studies of recursions where some property of the index k dictates what average to