On the numerical values of the roots of the equation $\cos x = x$.

By T. H. MILLER, M.A.

The angles which satisfy the equation

$$x = \cos x,$$

occur in the solution of certain problems in analytical geometry.

It is easy to show that one value of the angle is coscoscos...cos 1, when the cosine is taken successively to infinity.

The value taken ten times is ·7442, which is correct to two decimal places, but the numerical values are more easily found by the method of trial, and the following considerations facilitate the work.

First: there is only one real root. For while x increases from 0 to $\pi/2$, $\cos x$ diminishes from 1 to 0. Also $\cos x$ is positive while x lies between 0 and -1.

This root lies near $\frac{\pi}{4}$. Then let

$$\cos\left(\frac{\pi}{4} - \beta\right) = \frac{\pi}{4} - \beta$$

where β is small; that is

$$\frac{\sqrt{2}}{2}(\sin\beta + \cos\beta) = \frac{\pi}{4} - \beta$$

Then $\sin\beta$ and $\cos\beta$ may be expanded in ascending powers of β, and retaining only the first power the error is less than ·0005.

If then β be assumed equal to ·046298 we find

$$\cos \cdot 7391 = \cdot 7390751\ldots,$$

three terms of the expansions giving $\sin\beta + \cos\beta$ to seven decimal places.

Now let

$$\begin{aligned} \cos(\cdot 7391 - \gamma) &= \cdot 7391 - \gamma, \\ &= \cdot 7390751 + \delta \sin \cdot 7391. \\ \therefore \qquad \gamma &= \frac{\cdot 00002488}{1 + \sin \cdot 7391}, \\ &= \frac{\cdot 00002488}{1 \cdot 67362}, \\ &= \cdot 0000148659\ldots \end{aligned}$$

and the value of the angle is ·739085134....

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Now putting cos·739085134 into the form

$$\frac{\sqrt{2}}{2}\{\cos\cdot 046313029+\sin\cdot 046313029\}$$

three terms of the expansions give

$$\cos\cdot 739085134=\cdot 739085132....$$

Thus the real root of the equation is

$$\cdot 73908513....$$

correct to eight decimal places.

Next: To find the imaginary roots.

Let $A+B\sqrt{-1}$ be a root, then

$$A+B\sqrt{-1}=\cos A(e^{B}+e^{-B})/2+\sqrt{-1}\sin A(e^{B}-e^{-B})/2.$$

$$\therefore\ A=\cos A.(e^{B}-e^{-B})/2$$

$$\text{and } B=\sin A.(e^{B}-e^{-B})/2.$$

Now $$1+\{(e^{B}-e^{-B})/2\}^2=\{(e^{B}+e^{-B})/2\}^2$$

$$\therefore\ 1+B^2/\sin^2 A=A^2/\cos^2 A$$

$$\therefore\ B=\pm\tan A\sqrt{A^2\cos^2 A,} \quad \dots \quad \dots \quad (2)$$

and $$A=\cos A\left(e^{\tan A\sqrt{(A^2-\cos^2 A)}}+e^{-\tan A\sqrt{(A^2-\cos^2 A)}}\right)/2, \quad (3)$$

Thus A cannot be less than cosA. The values $B=0$, $\cos A=A$, satisfy these equations. This gives the real root already found. But there is another value of A between ·739... and $\pi/2$ which satisfies (3).

For $\cos A\,(e^{B}+e^{-B})/2=\infty$ when $A=\pi/2$, and equals A when $A=\cdot 739...$; and also the rate of change of that expression is less than the rate of change of A, when $A=\cdot 739....$ Therefore A and $\cos A.(e^{B}+e^{-B})/2$ must coincide for another value of A. This value is easily found by trial to be ·9623..... Thus

$$\cdot 962...+1\cdot 1096\sqrt{-1}$$

is a root of the equation.

And there is no other root between ·962... and $\pi/2$, for the rate of change of $\cos A(e^{B}+e^{-B})/2$ is greater than the rate of change of A, for these values.

The process of solving equation (3) by trial may be simplified as follows.

There can be no value of A in the second or the third quadrant, which will satisfy the equation, for cosA is negative. Between the values $3\pi/2$ and 2π, of A, the right side of equation (3) changes from ∞ to 0, and it passes through the same values between 2π and $5\pi/2$. Therefore there is a root between the values $A = 2n\pi \pm \pi/2$ and $A = 2n\pi$, where n is any integer.

Let $2n\pi + C$ be a value. Then as a first approximation C may be neglected in comparison with $2n\pi$.

Thus (3) becomes

$$2n\pi = \cos C . e^{2n\pi \tan C}/2.$$

When n is large, C becomes small, and we may put $\tan C = C$ and $\cos C = 1$.

$$\therefore \; C = \log_e 4n\pi/2n\pi, \quad \dots \quad \dots \quad \dots \quad (4)$$

If a value a, substituted for A in (3), make the right side of the equation $= a + d$, then the correction to be applied to a is very nearly

$$d/4n^2\pi^2, \dots \quad \dots \quad \dots \quad \dots \quad (5)$$

For the rate of variation of $e^{n\pi C}/2$ as C changes is $n\pi e^{2n\pi C}$, that is

$$4n^2\pi^2$$

by equation (4).

Now let x be the correction required

$$a - x = (a + d) - 4n^2\pi^2 x.$$

or $$x = d/4n^2\pi^2 \text{ nearly.}$$

By employing these methods the following are the values of the roots obtained,

$$\cdot 962 \dots\dots\dots + 1\cdot 1096 \dots \sqrt{-1},$$
$$5\cdot 86956 \dots\dots + 2\cdot 5449 \dots \sqrt{-1},$$
$$6\cdot 6663 \dots\dots\dots + 2\cdot 6607 \dots \sqrt{-1},$$
$$12\cdot 30856 \dots\dots + 3\cdot 2349 \dots \sqrt{-1},$$
$$12\cdot 817498 \dots\dots + 3\cdot 2799 \dots \sqrt{-1},$$
$$18\cdot 657 \dots\dots\dots + 3\cdot 6191 \dots \sqrt{-1},$$

etc.

For higher values of n, the value of C given by (4) is true to at least two decimal places, and the approximation given by using equation (5) is correct to at least four places.

For example, when $n = 4$, equation (4) gives $A = 24{\cdot}975\ldots$ the correct value being $24{\cdot}977\ldots$.

The Wallace line and the Wallace point.

By J. S. MACKAY, LL.D.

In what follows I propose to give the history of two theorems and to state some of the consequences that have been developed from them.

The first theorem is :

If a triangle be inscribed in a circle, and from any point in the circumference perpendiculars be drawn to the sides, the feet of these perpendiculars lie in a straight line.

This straight line is sometimes called the pedal line of the triangle, but it is much more frequently named the Simson line, from the belief that Robert Simson of Glasgow was the discoverer of the theorem. This belief is erroneous, for the theorem is not to be found in any of Simson's published works ; I have searched every one of them for it in vain. It may be worth mentioning also that no writer who has used the appellation Simson line has ever given a reference to any passage of Simson's works where the theorem is either stated or implied. How then has this appellation arisen ? The first time that the theorem is attributed to Simson is about 1814 in an article by F. J. Servois in Gergonne's *Annales de Mathématiques*, IV. 250. Servois merely says he believes (*je crois*) the theorem is Simson's. Poncelet in his *Propriétés Projectives*, published in 1822, remarks (§ 468) that Servois attributes the theorem to Simson, and it is, I conjecture, this reproduction of Servois's belief by Poncelet on which succeeding geometers have relied when they bestowed the name Simson line.

If the credit of the discovery of the line may not then be given to Simson, to whom does it belong ? In the *Proceedings of the Edinburgh Mathematical Society*, III. 104 (1885) Dr Thomas Muir mentions the fact that the theorem in question occurs in an article by William Wallace in Leybourn's *Mathematical Repository* (old series), II. 111. Apart from the circumstance that I have not met

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