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108.05 Ramanujan's proof of Bertrand's postulate

Introduction

In this Note we adhere closely to Ramanujan's original paper [1]. We think it should be inspirational for mathematics students to see an accurate reproduction of a short but significant work by a great mathematician with perhaps some of the pitfalls of trying to understand that work smoothed over. Our main contribution is to remove any mention of the gamma function or Stirling's formula. Simply to invoke a technical device without explaining how it can be used in a proof is insufficient. Instead of referring to Stirling's formula we give a direct proof in Lemma 3 of two inequalities which are unique and central to Ramanujan's proof. The assertions of Lemma 3 are essential for the validity of Ramanujan's argument and conclusions, but the proof of Lemma 3 bears no relation to the rest of the paper. It would be feasible just to assume the conclusions of Lemma 3, essentially as Ramanujan has done, but we have chosen to give a proof. The binomial coefficent $\binom{2n}{n}$ first occurred in a proof of Bertrand's postulate in Ramanujan's paper. In his proof of Bertrand's postulate [2, 3], Erdős also used this binomial coefficient. Aside from our direct proof of the two inequalities of Lemma 3 and our preliminaries, which prepare the reader for Ramanujan's context, we do not change Ramanujan's argument. Perhaps interested readers will note that Ramanujan comes back to [4], connecting to asymptotic distributions of primes, whereas Erdős, following his proof of Bertrand's postulate, turns toward Sylvester's Theorem [5], which generalises Bertrand's postulate in another direction.

The following are the opening sentences of Ramanujan's paper [1] (or google “Ramanujan's Proof of Bertrand's Postulate” to find Ramanujan's article scanned into the net.):

“Landau in his *Handbuch* [4, pp 89–92], gives a proof of a theorem the truth of which was conjectured by Bertrand: namely that there is at least one prime p such that $x < p \leqslant 2x$, if $x \geqslant 1$. Landau's proof is substantially the same as that given by Tschebyschef. The following is a much simpler one.”

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Ramanujan introduces terminology, with some details added by the author of this paper

Let $\mathbb{N}$ denote the set of natural numbers and $\mathbb{P}$ the subset of prime numbers.

Lemma 1: For every $n \in \mathbb{N}$, $\log n! = \sum_{p \in \mathbb{P}} \sum_{j=1}^{\infty} \left\lfloor \frac{n}{p^j} \right\rfloor \log p$.

What has to be proved is that for every $p \in \mathbb{P}$ the exponent α_p in the representation $n! = \prod_{p \in \mathbb{P}} p^{\alpha_p}$ as a product of prime powers is precisely $\alpha_p = \sum_{j=1}^{\infty} \lfloor n/p^j \rfloor$. For every i, $p \leqslant i \leqslant n$, there is a maximum integer $m_i \geqslant 0$ such that i/p^{m_i} is an integer. It is clear that the greatest common divisor $\gcd\left(i/p^{m_i}, p\right) = 1$ and $\alpha_p = \sum_{i=p}^{n} m_i$. Now note that, for $j \geqslant 1$, $\lfloor n/p^j \rfloor$ equals the number of multiples of p^j which are less than or equal to n and, next, that every $p^{m_i} \leqslant n$ is counted precisely m_i times, since it is counted once for each j, $1 \leqslant j \leqslant m_i$. Thus, $\sum_{i=p}^{n} m_i = \sum_{j \geqslant 1} \lfloor n/p^j \rfloor$, so $\alpha_p = \sum_{j \geqslant 1} \lfloor n/p^j \rfloor$.

Following Landau (1909) and Ramanujan (1919) (probably Tschebyschef earlier), we introduce the functions

$$\theta(x) = \sum_{p \leqslant x} \log p \qquad \text{and} \qquad \psi(x) = \sum_{m=1}^{\infty} \theta\left(x^{1/m}\right). \tag{1}$$

(Ramanujan writes $\nu(x)$ for $\theta(x)$; $\theta(x)$, Landau's notation, is the modern notation for this function.) As a rationale for introducing these functions we may point out that Bertrand's postulate is equivalent to the assertion that

$$\theta(2x) - \theta(x) > 0 \qquad (x \geqslant 1). \tag{2}$$

The proof of this assertion is the main goal of this paper. See the last paragraph of this Note, where Ramanujan states consequences of his argument which are more general than Bertrand's postulate.

Lemma 2: $\log \lfloor x \rfloor! = \sum_{\ell=1}^{\infty} \psi(x/\ell)$.

Observing that $\theta(x^{1/m}) = \sum_{p \in \mathbb{P}\,:\,p^m \leqslant x} \log p$, we also see that

$$\theta\left((x/\ell)^{1/m}\right) = \sum_{p \in \mathbb{P}\,:\,p^m \leqslant x/\ell} \log p = \sum_{p\,:\,\ell p^m \leqslant x} \log p. \tag{3}$$

Since, by (1), $\sum_{t=1}^{\infty} \psi\left(\frac{x}{\ell}\right) = \sum_{m,\ell} \theta\left(\left(\frac{x}{\ell}\right)^{1/m}\right)$, since, by (3), the coefficient of $\log p$ in $\sum_{\ell} \theta\left((x/\ell)^{1/m}\right)$ is the number of positive integers ℓ such that $\ell p^m \leqslant x$ and since that number is the same as $\lfloor x/p^m \rfloor$, it therefore follows that the coefficient of $\log p$ in $\sum_{\ell} \psi(x/\ell)$ is $\sum_{m=1}^{\infty} \lfloor x/p^m \rfloor$, as required by Lemma 1.

Ramanujan's argument begins

From (1) we see that

$$\psi(x) - 2\psi\left(\sqrt{x}\right) = \sum_{m=1}^{\infty} (-1)^{m-1} \theta\left(x^{1/m}\right) \tag{4}$$

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and from Lemma 2 we have

$$\log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor! = \sum_{\ell=1}^{\infty}(-1)^{\ell-1}\psi(x/\ell). \tag{5}$$

Since $\theta(x)$ and $\psi(x)$ are monotone non-decreasing functions, (1) and (4) imply that

$$\psi(x) - 2\psi(\sqrt{x}) \leqslant \theta(x) \leqslant \psi(x) \tag{6}$$

and from (5) we have

$$\psi(x) - \psi(x/2) \leqslant \log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor! \leqslant \psi(x) - \psi(x/2) + \psi(x/3). \tag{7}$$

Two inequalities

Lemma 3: $\frac{2}{3}x < \log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor! < \frac{3}{4}x$. The left inequality holds for $x \geqslant 300$ and the right inequality holds for $x > 0$.

The initial setting for our proof of Lemma 3 is the relation

$$\binom{2n}{n} \leqslant \sum_{j=0}^{2n}\binom{2n}{j} = (1+1)^{2n} = 2^{2n} \leqslant (2n+1)\binom{2n}{n}, \tag{8}$$

which is true for all integers $n \geqslant 0$, since $\binom{2n}{n}$ is the unique largest of the $2n+1$ binomial coefficients of the expansion $(x+y)^{2n}$. Based on (8) but restated in logarithmic form we may write

$$2n\log 2 - \log(2n+1) < \log(2n)! - 2\log n! < 2n\log 2 \qquad (n \geqslant 1). \tag{9}$$

For the case $\lfloor x\rfloor$ even, $\lfloor x\rfloor = 2n$, $\lfloor x/2\rfloor = n$ and $2n \leqslant x < 2n+1$. In this case, Lemma 3 has the formulation

$$\frac{2}{3} \overset{?}{<} \frac{2n}{x}\log 2 - \frac{\log(2n+1)}{x} < \frac{\log(2n)! - 2\log n!}{x} < \frac{2n}{x}\log 2 \overset{?}{<} \frac{3}{4}, (2n \leqslant x < 2n+1) \tag{10}$$

with the "?"s to be resolved. Noting that $2n/x \leqslant 1$ and $\log 2 < 3/4$, we see that the right side question mark of (10) can be erased for $n \geqslant 0$. The left side inequality of (10) holds for $2n \geqslant 300$, since $\frac{300}{301}\log 2 - \frac{\log 301}{301} > \frac{2}{3}$. Thus the left side inequality of (10) is true for $n \geqslant 150$.

We turn to the case $2n+1 \leqslant x < 2n+2$. In this case, $\lfloor x\rfloor! = (2n+1)!$ and $\lfloor x/2\rfloor! = \lfloor n+\frac{1}{2}\rfloor! = n!$ and (9) implies the inequalities

$$2n\log 2 < \log(2n+1)! - 2\log n! < \log(2n+1) + 2n\log 2. \tag{11}$$

To prove Lemma 3 for the case $2n+1 \leqslant x < 2n+2$ equation (11) suggests that we should check the "?"s in (12)

$$\frac{2}{3} \overset{?}{<} \frac{2n}{x}\log 2 < \frac{\log(2n+1)! - 2\log n!}{x} < \frac{\log(2n+1)}{x} + \frac{2n}{x}\log 2 \overset{?}{<} \frac{3}{4},$$

$$(2n+1 \leqslant x < 2n+2). \tag{12}$$

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For $2n \geqslant 300$ we see that $2n \log 2/(2n + 2) > 2/3$, which implies that the left question mark of (12) may be removed. However, we find that the right side of (12),

$$\log(2n+1) + 2n\log 2 \overset{?}{<} \frac{3}{4}(2n+1), \qquad (2n+1 \leqslant x < 2n+2), \qquad (13)$$

is true for $2n \geqslant 60$ and false for $2n < 60$. As we need the full strength ($n \geqslant 0$) of the right side of Lemma 3 for (18) and (19), we give an inductive proof of (14) which demands that the reader check

$$\log(2n + 1)! - 2\log n! < \tfrac{3}{4}(2n + 1) \qquad (14)$$

only for $0 \leqslant n \leqslant 4$. To prove (14) for $n \geqslant 5$ we take as an induction hypothesis for $n > 4$

$$\log(2n - 1)! - 2\log(n - 1)! < \tfrac{3}{4}(2n - 1). \qquad (15)$$

To prove that for $n > 4$ equation (15) implies (14) it suffices to check that $\log\big(2n(2n + 1)\big) - 2\log n < 3/2$, which is true for all $n \geqslant 5$. Thus, for all integers $n \geqslant 0$ and $2n + 1 \leqslant x < 2n + 2$

$$\log\lfloor x\rfloor! - 2\log\lfloor x/2\rfloor! = \log(2n+1)! - 2\log n! < \tfrac{3}{4}(2n+1) \leqslant \tfrac{3}{4}x. \qquad (16)$$

Ramanujan's presentation continues

From (7) and Lemma 3 we obtain

$$\psi(x) - \psi(x/2) < \tfrac{3}{4} \ (x > 0); \ \psi(x) - \psi(x/2) + \psi(x/3) > \tfrac{2}{3}x \ \big(x \geqslant 300\big). \ (17)$$

Using the left side of (17) and summing the series

$$\big(\psi(x) - \psi(x/2)\big) + \big(\psi(x/2) - \psi(x/2^2)\big) + \ldots + \big(\psi(x/2^n) - \psi(x/2^{n+1})\big) + \ldots \ (18)$$

produces the result

$$\psi(x) < \tfrac{3}{2}x \qquad (x > 0). \qquad (19)$$

From (6) and (19) we obtain

$$\psi(x) - \psi(x/2) + \psi(x/3) \leqslant \theta(x) + 2\psi(\sqrt{x}) - \theta(x/2) + \psi(x/3)$$

$$< \theta(x) + 3\sqrt{x} - \theta(x/2) + x/2. \qquad (20)$$

From (20) and the right side of (17) we have for all $x > 300$ that

$$\theta(x) - \theta(x/2) > 2x/3 - x/2 - 3\sqrt{x} = x/6 - 3\sqrt{x}. \qquad (21)$$

Clearly, $x/6 - 3\sqrt{x} \geqslant 0$ for $x \geqslant 324$. Therefore, for all $x \geqslant 162$ we have proved (2).

> "In other words, there is at least one prime between x and $2x$ for $x \geqslant 162$. Thus, Bertrand's postulate is proved for all values of x not less than 162; and, by actual verification, we find that it is true for smaller values."

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A generalisation of Bertrand's postulate

Let $\pi(x)$ denote the number of prime numbers not exceeding x. Then since $\pi(x) - \pi(x/2)$ is the number of prime numbers between $x/2$ and x and $\theta(x) - \theta(x/2)$ is the sum of the logarithms of the primes between $x/2$ and x, it is obvious that

$$\theta(x) - \theta(x/2) \leqslant \big(\pi(x) - \pi(x/2)\big) \log x \tag{22}$$

for all positive x. It follows from (21) and (22) that

$$\pi(x) - \pi(x/2) > \frac{1}{\log x}\left(\frac{x}{6} - 3\sqrt{x}\right), \tag{23}$$

if $x > 300$. From this we easily deduce that

$$\pi(x) - \pi(x/2) \geqslant 1, 2, 3, 4, 5, \ldots , \tag{24}$$

if $x \geqslant 2, 11, 17, 29, 41, \ldots$, respectively.

Acknowledgement

I would like to thank Professor Rasul Khan for helpful discussions.

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https://doi.org/10.1017/mag.2024.22 Published online by Cambridge University Press