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ON THE PETTIS MEASURABILITY THEOREM

DIETRICH HELMER

It is shown that, in Pettis's criterion for Bochner measurability of a vector-valued function $f\colon S \to X$, scalar measurability of f can be weakened to requiring that $u \circ f$ be measurable for u in some subset of the dual X^* separating the points of X. Even then, the separability hypotheses in Pettis's Theorem can be weakened as well.

Throughout, (S, Σ, μ) denotes a positive measure space. Then Σ_0 stands for the collection of all μ-null sets and Σ_b for $\{E \in \Sigma \mid 0 < \mu(E) < \infty\}$. Moreover, X is a Fréchet space over $\mathbb{K}$, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$. We fix a function $f\colon S \to X$. Recall that f is *μ-measurable* (in the sense of Bochner) if and only if $\chi_E f$ is, for every $E \in \Sigma_b$, the limit μ-almost everywhere in S of a sequence of μ-simple functions. The set of all μ-measurable functions $S \to X$ is denoted by $\mathcal{M}(\mu, X)$.

One of the most frequently employed and useful criteria for μ-measurability is the Pettis Measurability Theorem for Banach spaces X [**12**, p.278] (compare [**2**, p.149]): $f \in \mathcal{M}(\mu, X)$ if (and only if) every $E \in \Sigma_b$ admits an $N \in \Sigma_0$ such that $f(E \setminus N)$ is separable and f is *scalarly μ-measureable*, that is,

$$X^* \circ f := \{u \circ f \mid u \in X^*\} \subseteq \mathcal{M}(\mu, \mathbb{K}) =: \mathcal{M}(\mu).$$

A discussion of the importance of the theorem for Measure Theory and the theory of Banach spaces has been given by Uhl in [**19**]. It is known (compare [**19**] [**1**, p.43] for finite μ) that scalar μ-measurability of f in Pettis's Theorem can be replaced by the weaker condition that $U \circ f \subseteq \mathcal{M}(\mu)$ for some norming $U \subseteq X^*$. We shall show in this note that a minimal requirement is already sufficient here: namely, that $U \circ f \subseteq \mathcal{M}(\mu)$ for some $U \subseteq X^*$ separating sufficiently many points of X. (A key argument has already been utilised implicitly in [**4**] for Haar measures μ and, for general σ-finite μ, in [**7**].) It is often easy to identify point-separating subsets U of X^*; and, depending on X^* and f, there may be flexibility in tailoring U so as to facilitate the verification of $U \circ f \subseteq \mathcal{M}(\mu)$.

Let $\mathbb{I} := [0, 1]$. If $B \subseteq X$, then $\overline{span}\, B$ stands for the closed linear span of B in X and B_σ denotes B with the subspace topology it inherits from the weak topology of X.

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Similarly, if $V \subseteq X^*$, then V_σ is V equipped with the restricted w^*-topology. $(S, \widehat{\Sigma}, \mu)$ denotes the Lebesgue extension of (S, Σ, μ). We call $\Sigma_c \subseteq \widehat{\Sigma}$ *μ-covering* if and only if for every $E \in \Sigma_b$, there exists some $T \in \Sigma_c$ such that $E \cap T \notin \Sigma_0$, which is the case if and only if every $E \in \Sigma_b$ admits a sequence T_n in Σ_c with $E \setminus \bigcup_{n=1}^{\infty} T_n \in \Sigma_0$. And $\Gamma \subseteq \widehat{\Sigma}$ is said to be a *μ-regularity set* if and only if $\mu(E) = \sup\{\mu(G) \mid G \in \Gamma, G \subseteq E\}$ for every $E \in \Sigma_b$. Any such Γ is μ-covering.

THEOREM 1. *$f \in \mathcal{M}(\mu, X)$ if and only if there exists a μ-covering set Σ_c such that every $T \in \Sigma_c$ admits some $N \in \Sigma_0$ with $f(T \setminus N)$ separable and some $U \subseteq X^*$ separating the points of $\overline{\mathrm{span}}\, f(T \setminus N)$ with $U \circ f \subseteq \mathcal{M}(\mu)$.*

PROOF: If f is μ-measurable, then the conditions are satisfied with $\Sigma_c := \Sigma_b$ and $U := X^*$. Conversely, suppose now that there exists a μ-covering collection Σ_c as above. First, using routine arguments, one reduces the proof to the case where μ is finite, X is separable, and where $U \circ f \subseteq \mathcal{M}(\mu)$ for some subset U of X^* separating the points of X. In this situation, it follows by means of Egoroff's Theorem that the linear subspace $D := \{u \in X^* \mid u \circ f \in \mathcal{M}(\mu)\}$ of X^* is sequentially closed in X^*_σ. For every 0-neighbourhood B in X, the polar $B^\circ := \{u \in X^* \mid |u(b)| \leqslant 1 \textit{ for all } b \in B\}$ is compact and metrisable in X^*_σ; and, consequently, $D \cap B^\circ$ is closed in $(B^\circ)_\sigma$. According to the Krein-Šmulian Theorem [**9**, 22.6], therefore, D itself is closed in X^*_σ. On the other hand, D is dense in X^*_σ since it separates the points of X. Thus, $D = X^*$, that is f is scalarly μ-measurable. If X is a Banach space, an appeal to Pettis's Theorem finishes the proof. In the general case, the usual arguments need adaptation: Fix a sequence B_n of closed convex circled subsets of X constituting a 0-neighbourhood base in X such that $B_{n+1} \subseteq B_n$ for all n. Thereafter, pick, for every n, a sequence u_{nk} that is dense in $(B_n^\circ)_\sigma$. Moreover, let x_m be a dense sequence in X. Then

$$h_{nm} : s \longmapsto \sup\{|u_{nk}(f(s) - x_m)| \mid k \in \mathbb{N}\} : S \longrightarrow \mathbb{K}$$

is μ-measurable for all n, m, and, consequently, $h_{nm}^{-1}(\mathbb{I}) \in \widehat{\Sigma}$. But $h_{nm}^{-1}(\mathbb{I}) = f^{-1}(x_m + B_n)$ by means of the Bipolar Theorem. Now the classical arguments carry over mutatis mutandis to show that f is the uniform limit of a sequence f_n of countably-valued functions with $\{f_n^{-1}(x) \mid n \in \mathbb{N}, x \in X\} \subseteq \widehat{\Sigma}$ and, finally, to show that f is μ-measurable. ∎

EXAMPLE. *Suppose S is also a topological space and Γ is a μ-regularity set consisting of closed subsets or of open subsets of S. Let D be an open connected set in $\mathbb{C}$ and $h : S \times D \to \mathbb{C}$ a function that is analytic in the second argument such that $\{h(-, z) \mid z \in A\} \subseteq \mathcal{M}(\mu)$ for some $A \subseteq D$ with an accumulation point in D. Then, given $\varepsilon > 0$, every $E \in \Sigma_b$ contains a $G \in \Gamma$ with $\mu(E \setminus G) < \varepsilon$ such that h is continuous on $G \times D$.*

PROOF: Consider $\psi\colon s \mapsto h(s, -)\colon S \to H(D)$, where $H(D)$ denotes the separable Fréchet space consisting of all analytic functions $D \to \mathbb{C}$ equipped with the topology of uniform convergence on compacta. The subset U of $H(D)^*$, made up of the evaluation functionals at points of A, separates the points of $H(D)$ and satisfies $U \circ \psi \subseteq \mathcal{M}(\mu)$. So ψ is μ-measurable by Theorem 1; whence some sequence of countably-valued μ-measurable functions $S \to H(D)$ converges to ψ uniformly. Adapting standard arguments (compare [**16**, p.27]), given ε and E, we find some $G \in \Gamma$ with $G \subseteq E$ and $\mu(E \setminus G) < \varepsilon$ on which ψ is continuous. To complete the proof, it therefore suffices to utilise the continuity of the evaluation $H(D) \times D \to \mathbb{C}$. $\square$

We now turn to the second ingredient of the Pettis Theorem, separability, and replace it by weaker conditions, while keeping our minimal hypotheses on scalar μ-measurability.

We call a topology $\mathcal{T}$ on $R \subseteq S$ a *μ-topology* if and only if $\mathcal{T} \cap \Sigma_0 \subseteq \{\emptyset, R\}$. If R is σ-finite in $\widehat{\Sigma}$ and θ is a lifting of $\Delta := \{D \in \widehat{\Sigma} \mid D \subseteq R\}$, or just any lower density of Δ, then the density topology

$$\mathcal{T}_\theta := \{\theta(D) \setminus M \mid D \in \Delta,\, M \in \Delta \cap \Sigma_0\} = \{F \in \Delta \mid F \subseteq \theta(F)\}$$

(compare [**8**, p.54]) is a μ-topology on R. Generally: if $R \in \widehat{\Sigma} \setminus \Sigma_0$ and $\mathcal{O}$ is any topology on R with $\mathcal{O} \cap \Sigma_0 = \{\emptyset\}$, then $\{Q \setminus M \mid Q \in \mathcal{O},\, M \in \Sigma_0\}$ is a basis of a μ-topology. If P is a topological space and $V \subseteq C(P)$, then V is considered equipped with the topology of pointwise convergence and $Alg\, V$ denotes the smallest closed $\mathbb{K}$-subalgebra of $C(P)$ containing V. Recall that P satisfies the Souslin *countable chain condition* CCC if and only if every family of non-empty, pairwise disjoint, open subsets of P is countable. We say that a collection $\mathcal{B}$ of Borel subsets of $\mathbb{K}$ is a *Borel subbase for* $\mathbb{K}$ if and only if for every open $W \subseteq \mathbb{K}$ and every $\alpha \in W$, there are $B_1, \dots, B_m$ in $\mathcal{B}$ with $\alpha \in \bigcap_{k=1}^{m} B_k \subseteq W$.

THEOREM 2. *$f \in \mathcal{M}(\mu, X)$ if there is a μ-covering set Σ_c such that every $T \in \Sigma_c$ admits pseudo-compact sets $P_1, P_2, \dots$ in X_σ, μ-null sets $N_0, N_1, \dots$ with*

$$f(T \setminus N_0) \subseteq \overline{span}\left(\bigcup_{i=1}^{\infty} f(T \setminus N_i) \cap P_i\right) =: Y,$$

and a $U \subseteq X^*$ separating points of Y with $U \circ f \subseteq \mathcal{M}(\mu)$ satisfying, for every $k \geqslant 1$, one of these conditions:

(1) *There is a sequence V_{km} of pseudo-compact CCC-subspaces of $C(P_{k\sigma})$ with $U|_{P_k} \subseteq Alg\left(\bigcup_{m=1}^{\infty} V_{km}\right)$.*

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(2) *There is a Borel subbase $\mathcal{B}_k$ for $\mathbb{K}$ so that every pairwise disjoint collection in*

$$\left\{T \cap \bigcap_{i=1}^{m} f^{-1}(P_k \cap u_i^{-1}(B_i)) \setminus N_k \mid m \in \mathbb{N},\, u_i \in U,\, B_i \in \mathcal{B}_k\right\}$$

is countable.

(3) *$T \cap f^{-1}(P_k)$ is in $\widehat{\Sigma}$ and σ-finite; and there is a μ-topology $\mathcal{T}_k$ on $T \cap f^{-1}(P_k)$ such that $u \circ f$, restricted to $(T \cap f^{-1}(P_k)) \setminus N_k$, is $\mathcal{T}_k$-continuous for $u \in U$.*

Conversely, these conditions are necessary for $f \in \mathcal{M}(\mu, X)$.

PROOF: Suppose that there exists a μ-covering collection Σ_c with the properties listed above. Fix $T \in \Sigma_c$, and then let the P_j's, the N_j's and U be as guaranteed by the hypotheses. In view of Theorem 1, it suffices to show that, for every fixed $k \geqslant 1$, the weak closure C_k of $f(T \setminus N_k) \cap P_k$ is separable with respect to the weak topology. Note that $C_k \subseteq Y$, as Y is closed in X_σ by Mazur's Theorem [**9**, 17.1]. The class of Eberlein compacta (that is, those topological spaces that are homeomorphic with weakly compact subsets of Banach spaces) is well-known to be closed under the formation of countable products. And according to a result of Preiss and Simon [**13**], every weakly pseudo-compact subset of a Banach space is weakly compact. Thus, as X admits a linear homeomorphic embedding into a product of countably many Banach spaces, $P_{k\sigma}$ is Eberlein compact.

First, suppose that (1) is satisfied. Fix m. Evaluation $g : V_{km} \times P_{k\sigma} \to \mathbb{K}$ is separately continuous. Let $h \in C(\mathbb{K}, \mathbb{I})$. Then an argument given by Pták [**14**, p.572] shows that

$$\lim_{i\to\infty} \lim_{j\to\infty} h(v_i(p_j)) = \lim_{j\to\infty} \lim_{i\to\infty} h(v_i(p_j)) \quad \text{in } \mathbb{I}$$

for every sequence (v_i, p_i) in $V_{km} \times P_k$ for which all limits involved exist. According to [**5**, 2.1.(2)], therefore, $h \circ g \colon V_{km} \times P_{k\sigma} \to \mathbb{I}$ admits a separately continuous extension to $\beta V_{km} \times P_{k\sigma}$, where β denotes the Stone-Čech compactification operator. Furthermore, if A, B are any two disjoint closed subsets of $\mathbb{K}$, then $h'(A) \subseteq \{0\}$ and $h'(B) \subseteq \{1\}$ for a suitable $h' \in C(\mathbb{K}, \mathbb{I})$. Consequently, g admits a separately continuous extension $\beta V_{km} \times P_{k\sigma} \to \beta\mathbb{K}$ by [**5**, 2.1.(7)]. Moreover, all subspaces of $\mathbb{K}$ of the form $V_{km}(p)$ with $p \in P_k$ and of the form $v(P_k)$ with $v \in V_{km}$ are compact. It therefore follows from [**5**, 2.5.(1), 2.8] that V_{km} has Eberlein compact closure $\overline{V}_{km}$ in $C(P_{k\sigma})$. On the other hand, $\overline{V}_{km}$ satisfies CCC. Consequently, $\overline{V}_{km}$ is separable. This follows from the Rosenthal Separability Theorem [**15**, 4.6] and is also a rather direct consequence of Namioka's continuity Theorem 4.2 in [**11**]. Thus, $A := Alg\left(\bigcup_{m=1}^{\infty} V_{km}\right)$ is separable in

$C(P_{k\sigma})$, whence $A|_{C_k}$ is separable in $C(C_{k\sigma})$. But $A|_{C_k}$ separates the points of C_k since it contains $U|_{C_k}$. Consequently, $C_{k\sigma}$ is separable, indeed.

In cases (2), (3), it suffices to show that $C_{k\sigma}$ satisfies CCC because it is Eberlein compact. So let $(Q_\lambda)_{\lambda\in\Lambda}$ be any family of non-empty, open, pairwise disjoint subsets of $C_{k\sigma}$. Fix $\lambda \in \Lambda$, and choose some $t_\lambda \in T \cap \big(f^{-1}(P_k \cap Q_\lambda) \setminus N_k\big)$. Since $U|_{C_k}$ separates the points of C_k, it generates the topology of $C_{k\sigma}$. So there exist $u_1, \ldots, u_m \in U$ and open subsets $B_1, \ldots, B_m$ in $\mathbb{K}$ such that

$$f(t_\lambda) \in C_k \cap \bigcap_{i=1}^{m} u_i^{-1}(B_i) \subseteq Q_\lambda.$$

But then $f(D_\lambda) \subseteq Q_\lambda$ for

$$D_\lambda := T \cap f^{-1}(P_k) \cap \bigcap_{i=1}^{m} (u_i \circ f)^{-1}(B_i) \setminus N_k.$$

And $D_\lambda \neq \emptyset$, as $t_\lambda \in D_\lambda$.

Suppose now that (3) holds. Then D_λ is an open $\mathcal{T}_k$-neighbourhood of t_λ in $\big(T \cap f^{-1}(P_k)\big) \setminus N_k$. Since $\mathcal{T}_k$ is a μ-topology on $T \cap f^{-1}(P_k)$, it follows that $D_\lambda \notin \Sigma_0$, provided that $D_\lambda \neq \big(T \cap f^{-1}(P_k)\big) \setminus N_k$. Moreover, in view of the hypothesis that $U \circ f \subseteq \mathcal{M}(\mu)$ and that $T \cap f^{-1}(P_k)$ is σ-finite in $\widehat{\Sigma}$, we obtain that $D_\lambda \in \widehat{\Sigma}$ (compare [**2**, p.148]). Consequently, the usual summability argument shows that Λ must be countable.

Finally, suppose that (2) is satisfied. Then we may assume that $B_1, \ldots, B_m$ are, actually, members of $\mathcal{B}_k$ (though no longer open, perhaps). Consequently, (2) guarantees countability of Λ in this case as well.

Conversely, as for the necessity of the conditions, suppose that f is μ-measurable. Take $\Sigma_c := \Sigma_b$ and $U := X^*$. Fix $T \in \Sigma_c$ and, thereafter, $N_0 \in \Sigma_0$ such that $\bigcup_{i=1}^{\infty} P_i$ is dense in $f(T \setminus N_0)$ for some sequence P_k of singletons. Let $N_k := \emptyset$ for $k \geqslant 1$. Then (1) and (2) are, trivially, satisfied for every k. As is (3) since $T \cap f^{-1}(P_k) \in \widehat{\Sigma}$ (compare [**2**, p.148]). ∎

REMARKS. **1.** The proof of Theorem 1 essentially consisted in showing that, if $U \circ f \subseteq \mathcal{M}(\mu)$ for some $U \subseteq X^*$ separating the points of X, then f is scalarly μ-measurable, provided that X is separable. Scalar measurability has received considerable attention, in particular in the context of the Pettis integral (compare [**18**] and references there). The Krein-Šmulian Theorem can be applied to the scalar measurability problem as well. Even if X is non-separable, the argument for scalar μ-measurability of f in the proof of Theorem 1 goes through, provided that $(B^\circ)_\sigma$ is sequential for every 0-neighbourhood B in X. The latter condition is satisfied, for instance, if X is a closed

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linear subspace of a Fréchet space Z such that $\bigcup_{n=1}^{\infty} P_n$ is total in Z for some sequence P_n of pseudo-compact subsets of Z_σ. But some additional assumption on X is, generally, indispensable to arrive at the conclusion. Consider, for instance, the map $g\colon \mathbb{I} \to \ell^\infty$, where $g(s)$ is the sequence making up the dyadic expansion of $s \in \mathbb{I}$ such that $g(s)$ is not 1 eventually. Let $U \subseteq (\ell^\infty)^*$ consist of the evaluation functionals at points of $\mathbb{N}$. For every $u \in U$, then $u \circ g$ is measurable with respect to Lebesgue measure λ (in fact, $u \circ g$ is of Baire class 1). It can be shown, however, that g is not scalarly λ-measurable.

2. If S is a Souslin space and μ a Borel measure, then the set $\mathcal{K}(S)$ of all compact subsets of S is a μ-regularity set (compare [**16**, p.122]).

3. If S is a completely regular space and μ a Baire measure, then it can be shown that the collection $\mathcal{Z}(S)$ of zero-sets of S (that is, sets $g^{-1}(0)$ with $g \in C(S, \mathbb{I})$) is a μ-regularity set. This, in turn, can be used to show that, if S is a locally compact group and μ a Haar measure, then $\mathcal{Z}(S) \cap \mathcal{K}(S)$ is a μ-regularity set; if, moreover, the connected components of S are metrisable, then those members of $\mathcal{Z}(S)$ that are homeomorphic with $\{0, 1\}^w$ form a μ-regularity set, where w is the local weight of S [**6**, p.337].

4. In Theorems 1, 2, $U \circ f \subseteq \mathcal{M}(\mu)$ can be replaced by $U \circ f|_T \subseteq \mathcal{M}(\mu_T)$, where μ_T is the restriction of μ to T.

5. For certain measures μ, a much stronger version of the σ-compactness condition for $f(T \setminus N_0)$ in Theorem 2 is a necessary by-product of μ-measurability. Suppose S is also a topological space and has a μ-regularity set Γ consisting of closed pseudo-compact sets of S. (For example, μ a Radon measure.) If $f \in \mathcal{M}(\mu, X)$, then every $T \in \Sigma_b$ admits some $N_0 \in \Sigma_0$ such that $f(T \setminus N_0)$ is σ-compact, even in X. This follows from the fact (compare the Example) that T contains, for every $\varepsilon > 0$, some $G \in \Gamma$ with $\mu(T \setminus G) < \varepsilon$ such that $f|_G\colon G \to X$ is continuous.

6. Given the P_k's in Theorem 2, condition (1) is satisfied if U is pointwise bounded on X and U_σ satisfies CCC. For instance, if Z is a Banach space for which $B := \{z \in Z \mid \|z\| \leq 1\}_\sigma$ satisfies CCC — such spaces have been studied and examples exhibited by Wheeler in [**20**] — and if X is the strong dual of Z, then one may take the set $U \subseteq X^*$ corresponding to B or any sufficiently large subset thereof.

7. Concerning the countable chain condition in (1), it would not be good enough to suppose that $\bigcup_{m=1}^{\infty} V_{km}$ be a CCC-space. Consider $X := \ell^2(\mathbb{I})$, $U := X^*$, $P_k := \{x \mid \|x\| \leqslant 1\}$, and $V_{km} := \left\{v|_{P_k} \,\middle|\, v \in X^*, \|v\| \leqslant m\right\}$. Since X_σ^* is homeomorphic with a dense subset of some product $\mathbb{K}^\Lambda$, it satisfies CCC. But the scalarly Lebesgue measurable function $f\colon s \mapsto \chi_{\{s\}}\colon \mathbb{I} \to X$ is not measurable (as Birkhoff observed long ago).

8. In Theorem 2: if $T \cap f^{-1}(P_k)$ is in $\widehat{\Sigma}$ and σ-finite and the set system in (2) has only $\emptyset$ in common with Σ_0 for some $\mathcal{B}_k$, then (2) is satisfied.

9. If (3) is satisfied for all k, if $T \setminus \bigcup_{k=1}^{\infty} f^{-1}(P_k) \in \Sigma_0$, and $\bigcup_{k=1}^{\infty} \mathcal{T}_k \subseteq \widehat{\Sigma}$, then the hypothesis $U \circ f \subseteq \mathcal{M}(\mu)$ is redundant since $U \circ f|_T \subseteq \mathcal{M}(\mu_T)$ in this situation.

10. Let $\mathcal{F}$ be the class of those Fréchet spaces in which all weakly compact, hence all weakly pseudo-compact, subsets are separable. (1), (2), (3) in Theorem 2 are redundant if $X \in \mathcal{F}$. — Now let $X = L^\infty(\nu, Y)$, where ν is a σ-finite measure and Y a Banach space. Clearly, $X \in \mathcal{F}$, if $Y \in \mathcal{F}$ and $L^1(\nu)$ is separable. It is shown in [7] that large subspaces of X are in $\mathcal{F}$ if Y admits a continuous linear injection into some Banach space Z such that W_σ satisfies CCC for some bounded $W \subseteq Z^*$ separating the points of Z. More elementary is the fact that $X \in \mathcal{F}$ if Y_σ^* is separable. For a compact space L, the Banach space $C(L)$ may have a weak*-separable dual, even when L itself is not separable; compare [10] and [17]. In [3, 5.6], an example is given of a Banach space X with X_σ^* separable (equivalently: with a countable subset of X^* separating the points of X) such that no countable subset of X^* is norming.

Corollary. *Let μ be σ-finite. If $f \in \mathcal{M}(\mu, X)$, then there are $N \in \Sigma_0$, some $U \subseteq X^*$ separating points of $\overline{\operatorname{span}}\, f(S \setminus N)$, and a topology $\mathcal{T}$ on S with $\mathcal{T} \setminus \{\emptyset\} \subseteq \widehat{\Sigma} \setminus \Sigma_0$ rendering $u \circ f|_{S \setminus N}$ $\mathcal{T}$-continuous for all $u \in U$. The converse holds if $f(S \setminus M)$ is relatively compact in X_σ for some $M \in \Sigma_0$.*

Proof: Let $f \in \mathcal{M}(\mu, X)$ and then Z a separable closed linear subspace of X such that $N_1 := f^{-1}(X \setminus Z) \in \Sigma_0$. Choose a countable subset U of X^* that separates the points of Z. Moreover, let θ be a lifting of $\widehat{\Sigma}$. (Such a lifting exists: if (S, Σ, μ) is not finite and complete, instead we may consider any measure $\nu\colon \widehat{\Sigma} \to \mathbb{I}$ producing the same null sets as μ. Compare [8, p.46].) But then every $u \in U$ admits some $N_u \in \Sigma_0$ for which $u \circ f|_{S \setminus N_u}$ is continuous with respect to the density topology $\mathcal{T}_\theta$ on S [8, p.59]. Consequently, $N := N_1 \cup \bigcup_{u \in U} N_u$ does the job. Conversely, suppose that, in addition to the conditions listed, there exist a weakly compact subset P of X and some $M \in \Sigma_0$ such that $f(S \setminus M) \subseteq P$. Then the μ-measurability of f follows by means of Theorem 2.(3) with $N_0 := N \cup M =: N_k$, $\Sigma_c := \{S \setminus N_0\}$, $P_k := P$, and $\mathcal{T}_k := \{Q \setminus N_0 \mid Q \in \mathcal{T}\}$; compare Remark 9. ∎

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Department of Mathematics
University of Bahrain
PO Box 32038
Isa Town
Bahrain