Canad. Math. Bull. Vol. **43** (2), 2000 pp. 145–156

On the 2-Parallel Versions of Links

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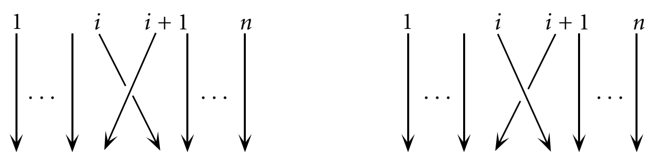
Abstract. In this paper, we show that the absolute value of the signature of the 2-parallel version of a link is less than or equal to the nullity of it and show that the signature, nullity, and Minkowski units of the 2-parallel version of a certain class of links are always equal to 0, 2, and 1 respectively.

1 Introduction

The Artin's braid group B_n on n strings has a standard presentation as a group with generators $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1}$ $(1 \le i \le n-2)$ and $\sigma_i\sigma_j = \sigma_j\sigma_i$ $(|i-j| \ge 2)$. The generator σ_i and its inverse σ_i^{-1} are represented as the oriented diagrams:



σ_i $\qquad$ σ_i^{-1}

By (b, n) we mean a braid b in B_n. The *closure* of a braid (b, n), denoted by $(b, n)^\wedge$ or simply by $b^\wedge$, is the link obtained by joining the n points at the top of the braid (b, n) to the corresponding n points at the bottom without further crossings as in Figure 1.1. It is well known that any oriented link is ambient isotopic to the closure of some braid [1].

In [8], J. Murakami defined the parallel versions of links in S^3 and showed that it is possible to distinguish links by using polynomial invariants of their parallel versions though these invariants coincide for the links themselves. So there is a motivation to study parallel versions of knots or links and their invariants.

In this paper, we define an integral matrix $B(\beta)$ for a braid word β and give the Goeritz matrix of the 2-parallel version $\left(\phi_n^{(2)}(\beta), 2n\right)^\wedge$ of a closed braid $(\beta, n)^\wedge$ in terms of the matrix $B(\beta)$. Using this we give a necessary condition for a given link ℓ to be the 2-parallel version of a knot or link by means of the signature and the nullity of ℓ. In fact we prove that if a link ℓ of μ-components is the 2-parallel version of a knot or link, then $|\sigma(\ell)| \le n(\ell) \le \mu$. This confirms that a certain class of links cannot be obtained by the 2-parallel

Received by the editors April 22, 1998; revised June 11, 1999.

The first author was supported by the Research Fund of Dongguk University. The second author was partially supported by Korea Science and Engineering Foundation. The third author was partially supported by the Korea Research Foundation made in the program year of 1998 and by TGRC-KOSEF.

AMS subject classification: 57M25.

Keywords: braid, Goeritz matrix, Minkowski unit, nullity, signature, 2-parallel version.

β $\qquad$ $\beta^\wedge$ = the closure of β

Figure 1.1

versions of knots or links. We also show that the signature $\sigma(\ell^{(2)})$, the nullity $n(\ell^{(2)})$, and the Minkowski units $C_p(\ell^{(2)})$ for any prime integer p, including $p = \infty$, of the 2-parallel version $\ell^{(2)}$ of a knot ℓ such that ℓ has a braid representative β with $\det\big(B(\beta)\big) \neq 0$ are always equal to 0, 2, and 1 respectively.

2 Matrices for Braid Words

Let $\beta = \sigma_{i_1}^{\tau_{i_1}} \sigma_{i_2}^{\tau_{i_2}} \cdots \sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1)$ be a braid word in B_n, which involves all of the generators $\sigma_1, \dots, \sigma_{n-1}$. For each $i \in \{1, 2, \dots, n-1\}$, let s_i denote the number of the letters $\sigma_i^{\pm 1}$ occurring in the word β. Let us rewrite the s_i occurrence of the generators $\sigma_i^{\pm 1}$ as $\sigma_{(i,1)}^{\tau(i,1)}, \sigma_{(i,2)}^{\tau(i,2)}, \dots, \sigma_{(i,s_i)}^{\tau(i,s_i)}$ keeping the order from left to right, where $\tau(i,k)$ denotes the exponent of the generator σ_i in β which constitutes $\sigma_{(i,k)}$ and $s_1 + s_2 + \cdots + s_{n-1} = m$. The resulting word is denoted by $\bar{\beta}$. Of course $\bar{\beta}$ and β represent the same braid in B_n.

For each $i = 1, 2, \dots, n-1$, we denote $\bar{W}_p^i (p = 1, 2, \dots, s_i)$ to be the subword of $\bar{\beta}$ whose initial letter is $\sigma_{(i,p)}^{\tau(i,p)}$ and terminal letter is $\sigma_{(i,p+1)}^{\tau(i,p+1)}$ cyclically (here, $s_i + 1$ is identified with 1). Define W_p^i to be the word obtained from $\bar{W}_p^i$ by replacing all $\sigma_{(k,q)}^{\tau(k,q)}$ $(k \neq i-1, i, i+1)$ by the empty word.

Definition 2.1 Let $\beta = \sigma_{i_1}^{\tau_{i_1}} \sigma_{i_2}^{\tau_{i_2}} \cdots \sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1)$ be a braid word in B_n, which involves all of the generators $\sigma_1, \dots, \sigma_{n-1}$ and $\bar{\beta}$, W_p^i as above.

Let $B(\beta) = (B_{ij})_{1 \le i,j \le n-1}$ be the blockwise tridiagonal $m \times m$ integral matrix defined as follows: Each diagonal block $B_{ii} (1 \le i \le n-1)$ of $B(\beta)$ is defined to be the $s_i \times s_i$ matrix given by $B_{ii} = \big(2\tau(i,1)\big)$ for $s_i = 1$ and

$$B_{ii} = \begin{pmatrix} \tau(i,1) & 0 & 0 & \cdots & 0 & \tau(i,1) \\ \tau(i,2) & \tau(i,2) & 0 & \cdots & 0 & 0 \\ 0 & \tau(i,3) & \tau(i,3) & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \tau(i,s_i - 1) & 0 \\ 0 & 0 & 0 & \cdots & \tau(i,s_i) & \tau(i,s_i) \end{pmatrix} \quad (s_i \ge 2).$$

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For $i \neq j$, the block B_{ij} is the $s_i \times s_j$ matrix defined by

$$B_{ij} = \begin{cases} O_{s_i \times s_j},\ s_i \times s_j \text{ zero matrix} & \text{if } |i-j| \neq 1 \\ (b^{ij}_{pq})_{1 \le p \le s_i; 1 \le q \le s_j} & \text{if } |i-j| = 1, \end{cases}$$

where

$$b^{ij}_{pq} = \begin{cases} 0 & \text{if } \sigma^{\tau(i,p)}_{(i,p)} \text{ is not in the word } W^j_q \\ -\tau(i,p) & \text{if } \sigma^{\tau(i,p)}_{(i,p)} \text{ is in the word } W^j_q. \end{cases}$$

Example 2.2 Let $\beta_1 = \sigma_1\sigma_2\cdots\sigma_n \in B_{n+1}$ $(n \ge 1)$ and let $\beta_2 = \sigma_1^n \in B_2$ $(n > 1)$. Then $\bar\beta_1 = \sigma_{(1,1)}\sigma_{(2,1)}\cdots\sigma_{(n,1)}$ and $\bar\beta_2 = \sigma_{(1,1)}\sigma_{(1,2)}\cdots\sigma_{(1,n)}$. Thus

$$B(\beta_1) = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & 0 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ 0 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}, \quad B(\beta_2) = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 1 \end{pmatrix}.$$

Example 2.3 Let $\beta = \sigma_1^{-1}\sigma_3\sigma_3\sigma_3\sigma_2\sigma_1\sigma_1\sigma_3^{-1}\sigma_2 \in B_4$. The rewriting word of β is

$$\bar\beta = \sigma^{-1}_{(1,1)}\sigma_{(3,1)}\sigma_{(3,2)}\sigma_{(3,3)}\sigma_{(2,1)}\sigma_{(1,2)}\sigma_{(1,3)}\sigma^{-1}_{(3,4)}\sigma_{(2,2)}$$

and

$$W^1_1 = \sigma^{-1}_{(1,1)}\sigma_{(2,1)}\sigma_{(1,2)}, \quad W^1_2 = \sigma_{(1,2)}\sigma_{(1,3)}, \quad W^1_3 = \sigma_{(1,3)}\sigma_{(2,2)}\sigma^{-1}_{(1,1)},$$
$$W^2_1 = \sigma_{(2,1)}\sigma_{(1,2)}\sigma_{(1,3)}\sigma^{-1}_{(3,4)}\sigma_{(2,2)}, \quad W^2_2 = \sigma_{(2,2)}\sigma^{-1}_{(1,1)}\sigma_{(3,1)}\sigma_{(3,2)}\sigma_{(3,3)}\sigma_{(2,1)},$$
$$W^3_1 = \sigma_{(3,1)}\sigma_{(3,2)}, \quad W^3_2 = \sigma_{(3,2)}\sigma_{(3,3)}, \quad W^3_3 = \sigma_{(3,3)}\sigma_{(2,1)}\sigma^{-1}_{(3,4)}, \quad W^3_4 = \sigma^{-1}_{(3,4)}\sigma_{(2,2)}\sigma_{(3,1)}.$$

Hence the matrix $B(\beta)$ is given by

$$B(\beta) = \begin{pmatrix} -1 & 0 & -1 & 0 & 1 & & & & \\ 1 & 1 & 0 & -1 & 0 & & & & \\ 0 & 1 & 1 & -1 & 0 & & & & \\ -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 & 0 & 0 & -1 \\ & & & 0 & -1 & 1 & 0 & 0 & 1 \\ & & & 0 & -1 & 1 & 1 & 0 & 0 \\ & & & 0 & -1 & 0 & 1 & 1 & 0 \\ & & & 1 & 0 & 0 & 0 & -1 & -1 \end{pmatrix}.$$

Remark 2.4 A braid b in B_n can be represented by many equivalent braid words β. That is, $B(\beta)$ depends on the braid word representation β of the braid b. In particular, by inserting extra unnecessary crossings, we can always arrange that β satisfies the hypotheses of Definition 2.1.

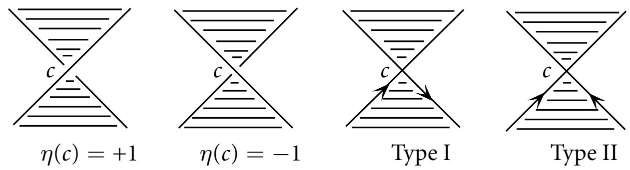

Figure 3.1

3 The Signature and Nullity

Let ℓ be an oriented link in S^3 and let L be its link diagram in the plane $\mathbb{R}^2$. Colour the regions of $\mathbb{R}^2 - L$ alternately black and white. Denote the white regions by $W_0, W_1, \ldots, W_k$ (we always take the unbounded region to be white and denote it by W_0). Assign an incidence number $\eta(c) = \pm 1$ to each vertex c of L and define a vertex c to be of *type I* or *type II* as indicated in Figure 3.1.

Let $G'(L)$ be the symmetric integral matrix defined by $G'(L) = (g_{ij})_{0 \le i,j \le k}$, where $g_{ij} = -\sum_{c \in C_L(W_i, W_j)} \eta(c)$ for $i \neq j$ and $g_{ii} = \sum_{c \in C_L(W_i)} \eta(c)$, here $C_L(W_i) = \{c \mid c \text{ is a vertex incident to } W_i\}$ and $C_L(W_i, W_j) = \{c \mid c \text{ is a vertex incident to both } W_i \text{ and } W_j\}$. The principal minor $G(L) = (g_{ij})_{1 \le i,j \le k}$ of $G'(L)$ is called the *Goeritz matrix* of ℓ associated to L [2], [3], [6].

Let $C_{II}(L) = \{c_1, c_2, \ldots, c_p\}$ denote the set of all crossings of type II in L and let $A(L) = \operatorname{diag}\big(-\eta(c_1), -\eta(c_2), \ldots, -\eta(c_p)\big)$ be the $p \times p$ diagonal matrix. Then the *modified Goeritz matrix* $H(L)$ of ℓ associated to L is defined as the block sum $H(L) = G(L) \oplus A(L) \oplus B(L)$, where $B(L)$ is the $\big(\beta_0(L) - 1\big) \times \big(\beta_0(L) - 1\big)$ zero matrix and $\beta_0(L)$ denotes the number of connected components of L. The signature $\sigma(\ell)$ and the nullity $n(\ell)$ of an oriented link ℓ in S^3 [9] are given by the formulas: $\sigma(\ell) = \sigma\big(H(L)\big)$ and $n(\ell) = n\big(H(L)\big) + 1$, where $\sigma\big(H(L)\big)$ and $n\big(H(L)\big)$ denote the signature and the nullity of the matrix $H(L)$ respectively [11].

Let B_n be Artin's (geometric) braid group on n-strings and let $\phi_n^{(2)} \colon B_n \to B_{2n}$ be the group homomorphism defined by, for each $1 \le i \le n-1$,

$$\phi_n^{(2)}(\sigma_i) = \sigma_{2i-1}^{-2} \sigma_{2i} \sigma_{2i+1} \sigma_{2i-1} \sigma_{2i}.$$

Let (β_1, n_1) and (β_2, n_2) be two braids. If the closures $(\beta_1, n_1)^\wedge$ and $(\beta_2, n_2)^\wedge$ are ambient isotopic, then the links $\big(\phi_{n_1}^{(2)}(\beta_1), 2n_1\big)^\wedge$ and $\big(\phi_{n_2}^{(2)}(\beta_2), 2n_2\big)^\wedge$ are ambient isotopic. Let ℓ be an oriented link in S^3 of μ-components and let $(\beta, n) \in B_n$ be a braid representative of the link ℓ. Then *the 2-parallel version* $\ell^{(2)}$ of ℓ is defined to be the closed braid $\big(\phi_n^{(2)}(\beta), 2n\big)^\wedge$ [8].

Theorem 3.1 *Let ℓ be a nonsplittable oriented link in S^3 and let $\beta = \sigma_{i_1}^{\tau_{i_1}} \sigma_{i_2}^{\tau_{i_2}} \cdots \sigma_{i_m}^{\tau_{i_m}}$ ($\tau_{i_k} = \pm 1$) be a braid word in B_n representing the link ℓ and $\bar{\beta}$ as defined in Section 2. Then the Goeritz matrix of the 2-parallel version $\ell^{(2)}$ of ℓ associated to $\big(\phi_n^{(2)}(\beta), 2n\big)^\wedge$ is given by the*

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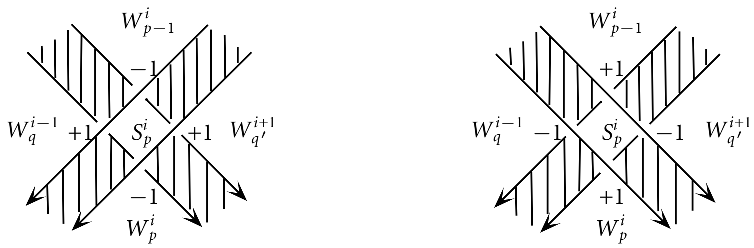

Figure 3.2

$(2m+1)\times(2m+1)$ *integral matrix of the form:*

$$G(\beta)=\begin{pmatrix} W & B(\beta)^t & O \\ B(\beta) & O_{m\times m} & [r]^t \\ O & [r] & e_{n-1}\end{pmatrix},$$

where $[r] = \big(0\ 0\cdots 0\ \ -\tau(n-1,1)\ \ -\tau(n-1,2)\cdots-\tau(n-1,s_{n-1})\big)$, $e_{n-1} = \sum_{i=1}^{s_{n-1}}\tau(n-1,i)$*, and* W *is an* $m\times m$ *symmetric integral matrix.*

Proof Let $L^{(2)}$ be the closure of the braid word $\big(\phi_n^{(2)}(\beta),2n\big)$. Colour the regions of $\mathbb{R}^2-L^{(2)}$ alternately black and white so that the unbounded region is a white region, denoted by W_1^0, and hence the region which meets the braid axis A is also a white region, denoted by W_1^n. For $i=1,2,\dots,n-1$, each letter $\sigma_{(i,p)}^{\tau(i,p)}(p=1,2,\dots,s_i)$ in $\bar\beta$ creates one white region in $L^{(2)}$, denoted by S_p^i. All the other white regions in $\mathbb{R}^2-L^{(2)}$ can be identified with the words $\{W_p^i \mid 1\le i\le n-1, 1\le p\le s_i\}$ in such a way that the vertices incident to the boundary of the white regions are the letters in the word W_p^i and we denote them by the same notation W_p^i.

3.1.1

For $1\le i\le n-1$, $0\le j\le n$, let $X_{ij}=(x_{pq}^{ij})_{1\le p\le s_i;1\le q\le s_j}$, where $s_0=s_n=1$, $x_{pq}^{ij}=-\sum_{c\in C_{L^{(2)}}(S_p^i,W_q^j)}\eta(c)$. Note that the white region S_p^i $(i=1,2,\dots,n-1)$ is incident to only four regions W_{p-1}^i, W_p^i, W_q^{i-1}, and $W_{q'}^{i+1}$ for some q,q' at one and only one vertex of incidence number $-\tau(i,p)$, $-\tau(i,p)$, $\tau(i,p)$, and $\tau(i,p)$, respectively (*cf.* Figure 3.2).

It is clear that $X_{ij}=O_{s_i\times s_j}$ for $|i-j|\ge 2$, $X_{10}=\big(-\tau(1,1)-\tau(1,2)\cdots-\tau(1,s_1)\big)^t$, and $X_{n-1n}=\big(-\tau(n-1,1)\ -\tau(n-1,2)\cdots-\tau(n-1,s_{n-1})\big)^t$.

For $1\le i,j\le n-1$ with $|i-j|=1$,

$$\begin{aligned} x_{pq}^{ij} &= -\sum_{c\in C_{L^{(2)}}(S_p^i,W_q^j)}\eta(c) \\ &= \begin{cases} 0 & \text{if } \sigma_{(i,p)}^{\tau(i,p)} \text{ is not in the word } W_q^j \\ -\tau(i,p) & \text{if } \sigma_{(i,p)}^{\tau(i,p)} \text{ is in the word } W_q^j \end{cases} \\ &= b_{pq}^{ij}. \end{aligned}$$

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Thus $X_{ij} = (x^{ij}_{pq})_{1\le p\le s_i;1\le q\le s_j} = (b^{ij}_{pq})_{1\le p\le s_i;1\le q\le s_j} = B_{ij}$ $(1 \le i, j \le n-1)$.

For $i = j$ $(1 \le i \le n-1)$, if $s_i = 1$, then S^i_1 is incident to W^i_1 at two vertices of incidence number $-\tau(i,1)$. So $X_{ii} = (x^{ii}_{11}) = \big(2\tau(i,1)\big)$. If $s_i \ge 2$, then $x^{ii}_{pq} = \tau(i,p)$ for $q = p$ or $p-1$ (if $p = 1$, $q = 1$ or s_i), otherwise, all $x^{ii}_{pq} = 0$. Hence $X_{ii} = (x^{ii}_{pq})_{1\le p,q\le s_i} = B_{ii}$ $(i = 1, 2, \ldots, n-1)$.

3.1.2

For $1 \le i, j \le n-1$, let $S_{ij} = (s^{ij}_{pq})_{1\le p\le s_i;1\le q\le s_j}$, where

$$s^{ij}_{pq} = \begin{cases} -\sum_{c\in C_{L^{(2)}}(S^i_p,S^j_q)} \eta(c) & \text{if } i \ne j \text{ or } p \ne q \\ \sum_{c\in C_{L^{(2)}}(S^i_p)} \eta(c) & \text{if } i = j \text{ and } p = q. \end{cases}$$

It is obvious that $S_{ij} = O_{s_i\times s_j}$ $(1 \le i, j \le n-1)$.

Now the matrix $G'(L^{(2)})$ associated to $L^{(2)}$ is given by

$$G'(L^{(2)}) = \begin{pmatrix} W & X^t \\ X & S \end{pmatrix},$$

where $X = (X_{ij})_{1\le i\le n-1,0\le j\le n}$, $S = (S_{ij})_{1\le i,j\le n-1}$, and $W = (W_{ij})_{0\le i,j\le n}$, where $W_{ij} = (w^{ij}_{pq})_{1\le p\le s_i;1\le q\le s_j}$, here $s_0 = s_n = 1$ and

$$w^{ij}_{pq} = \begin{cases} -\sum_{c\in C_{L^{(2)}}(W^i_p,W^j_q)} \eta(c) & \text{if } i \ne j \text{ or } p \ne q \\ \sum_{c\in C_{L^{(2)}}(W^i_p)} \eta(c) & \text{if } i = j \text{ and } p = q. \end{cases}$$

By (3.1.1), (3.1.2) and by deleting the first row and the first column of $UG'(L^{(2)})U^t$, where U is a permutation matrix, we obtain the result. ■

Corollary 3.2 *Let ℓ be a nonsplittable oriented link in S^3 and let $\beta = \sigma_{i_1}^{\tau_{i_1}}\sigma_{i_2}^{\tau_{i_2}}\cdots\sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the link ℓ. Then the modified Goeritz matrix of the 2-parallel version of $\ell^{(2)}$ of ℓ associated to $\big(\phi^{(2)}_n(\beta), 2n\big)^\wedge$ is given by*

$$H(\beta) = G(\beta) \oplus A(\beta),$$

where $G(\beta)$ is the Goeritz matrix of $\ell^{(2)}$ given in Theorem 3.1 and $A(\beta) = \operatorname{diag}(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_m}) \otimes I_2 \oplus \operatorname{diag}(-\tau_{i_1}, -\tau_{i_2}, \ldots, -\tau_{i_m}) \otimes I_2$.

Proof Let $L^{(2)} = \big(\phi^{(2)}_n(\beta), 2n\big)^\wedge$. Then the modified Goeritz matrix $\ell^{(2)}$ associated to $L^{(2)}$ is $H(L^{(2)}) = G(L^{(2)}) \oplus A(L^{(2)}) \oplus B(L^{(2)})$. By Theorem 3.1, $G(L^{(2)}) = G(\beta)$. Now each vertex corresponding to $\sigma^{\mp 2}_{2i-1}$ of $\phi^{(2)}_n(\sigma_i^{\pm 1})$ is a vertex of type II of incidence number ∓ 1 and each letter of the braid word β produces two vertices of type II in $L^{(2)}$ whose incidence numbers are equal to the exponent of the letter. So $A(L^{(2)}) = \operatorname{diag}(\tau_{i_1}, \tau_{i_2}, \ldots, \tau_{i_m}) \otimes I_2 \oplus \operatorname{diag}(-\tau_{i_1}, -\tau_{i_2}, \ldots, -\tau_{i_m}) \otimes I_2 = A(\beta)$. Since the diagram $L^{(2)}$ is connected, $B(L^{(2)})$ is the empty matrix. The result follows. ■

Corollary 3.3 *Let k be a knot in S^3 and let $k^{(2)}$ be the 2-parallel version of k. Then $\sigma(k^{(2)}) = 0$ and $n(k^{(2)}) = 2$.*

Proof Let u_2 be the trivial link of 2-components contained in an unknotted solid torus T in S^3 such that each component is parallel to the core of T and let $f: T \to S^3$ be a faithful embedding such that $f(T)$ is a tubular neighborhood of the knot k. Then it follows that the link $f(u_2)$ in S^3 is the 2-parallel version $k^{(2)}$ of the knot k. By Theorem 12 [3], we obtain that $\sigma(k^{(2)}) = \sigma(u_2) = 0$. Now let $\beta = \sigma_{i_1}^{\tau_{i_1}} \sigma_{i_2}^{\tau_{i_2}} \dots \sigma_{i_m}^{\tau_{i_m}} (\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the knot k and let $H(\beta)$ be the modified Goeritz matrix of $k^{(2)}$ given by Corollary 3.2. Then we have that $\sigma\big(H(\beta)\big) = 0$. Since $H(\beta)$ is a $(6m+1) \times (6m+1)$ matrix and $(6m+1)$ is an odd integer, the nullity $n\big(H(\beta)\big)$ must be odd. By Lemma 6.1 [9], $n(k^{(2)}) \le 2$. Thus $n\big(H(\beta)\big) = 1$ and so $n(k^{(2)}) = 2$. ∎

The following Example 3.4 shows that in general Corollary 3.3 is not true for the 2-parallel versions of links.

Example 3.4 Let $\beta = \sigma_1\sigma_1$. Then the closure $\ell = \beta^\wedge$ of β is the Hopf link and $B(\beta) = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right)$. Now the modified Goeritz matrix $H(\beta)$ of the 2-parallel version $\ell^{(2)} = (\sigma_1^{-2}\sigma_2\sigma_3\sigma_1\sigma_2\sigma_1^{-2}\sigma_2\sigma_3\sigma_1\sigma_2)^\wedge$ of ℓ is the matrix:

$$H(\beta) = \begin{pmatrix} -4 & 0 & 1 & 1 & 0 \\ 0 & -4 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & -1 \\ 1 & 1 & 0 & 0 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{pmatrix} \oplus I_4 \oplus -I_4.$$

Hence $n\big(B(\beta)\big) = 1, \sigma(\ell^{(2)}) = \sigma\big(H(\beta)\big) = -1$, and $n(\ell^{(2)}) = n\big(H(\beta)\big) + 1 = 3$. On the other hand, $\sigma(\ell) = n(\ell) = 1$. Thus by Corollary 3.3, the Hopf link is not the 2-parallel version of any knot.

Theorem 3.5 *Let ℓ be a nonsplittable oriented link in S^3 of μ-components and let $\ell^{(2)}$ be the 2-parallel version of ℓ. Let β be a braid word representing the link ℓ. Then*

(1) $0 \le n(B(\beta)) \le \mu$ and

(i) If $n\big(B(\beta)\big) = 0$, then $1 \le n(\ell^{(2)}) \le 2$.

(ii) If $n\big(B(\beta)\big) = \mu$, then $\mu \le n(\ell^{(2)}) \le 2\mu$.

(iii) If $1 \le n\big(B(\beta)\big) \le \mu - 1$, then $n\big(B(\beta)\big) \le n(\ell^{(2)}) \le 2\Big(n\big(B(\beta)\big) + 1\Big)$.

In particular, we have, in all cases, $n\big(B(\beta)\big) \le n(\ell^{(2)})$.

(2) If $n\big(B(\beta)\big) = 0$, then $0 \le |\sigma(\ell^{(2)})| \le n(\ell^{(2)}) \le 2$ and if $n\big(B(\beta)\big) \ne 0$, then $0 \le |\sigma(\ell^{(2)})| \le n\big(B(\beta)\big) \le n(\ell^{(2)}) \le 2\mu$.

Proof Let $\beta = \sigma_{i_1}^{\tau_{i_1}} \sigma_{i_2}^{\tau_{i_2}} \cdots \sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the link ℓ. It follows from Corollary 3.2 that $n(\ell^{(2)}) = n\big(H(\beta)\big) + 1 = n\big(G(\beta)\big) + n\big(A(\beta)\big) + 1$ and

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$\sigma(\ell^{(2)}) = \sigma\big(H(\beta)\big) = \sigma\big(G(\beta)\big) + \sigma\big(A(\beta)\big)$. Since $n\big(A(\beta)\big) = \sigma\big(A(\beta)\big) = 0$, $n(\ell^{(2)}) = n\big(G(\beta)\big) + 1$ and $\sigma(\ell^{(2)}) = \sigma\big(G(\beta)\big)$.

Now we suppose that the matrix $B(\beta)$ has the rank $r(0 \le r \le m)$, *i.e.*, $n\big(B(\beta)\big) = m - r$. Then there are unimodular rational matrices U and V satisfying $\mathrm{UB}(\beta)V = \left(\begin{smallmatrix} S & O \\ O & O \end{smallmatrix}\right)$, where S is a nonsingular $r \times r$ rational matrix. Let $P = V^t \oplus U \oplus (1)$. Then P is an unimodular rational matrix, $\det(P) = \pm 1$, and

$$\mathrm{PG}(\beta)P^t = \begin{pmatrix} V^t W V & \big(\mathrm{UB}(\beta)V\big)^t & O \\ \mathrm{UB}(\beta)V & O & U[r]^t \\ O & [r]U^t & e_{n-1} \end{pmatrix}.$$

Let us write $V^t W V = \left(\begin{smallmatrix} E_1 & E_2^t \\ E_2 & E_3 \end{smallmatrix}\right)$ and $[r]U^t = (X_1\ X_2)$, where E_1 is an $r \times r$ symmetric matrix, E_3 is an $(m-r) \times (m-r)$ symmetric matrix, X_1 is an $1 \times r$ row matrix, and X_2 is an $1 \times (m-r)$ row matrix. Let Q be the unimodular rational matrix defined by

$$Q = \begin{pmatrix} I_r & O & O & O & O \\ O & O & I_r & O & O \\ O & I_{m-r} & -E_2 S^{-1} & O & O \\ -X_1 (S^{-1})^t & O & X_1 (S^{-1})^t E_1 S^{-1} & O & 1 \\ O & O & O & I_{m-r} & O \end{pmatrix}.$$

Denote $M = Q\big(\mathrm{PG}(\beta)P^t\big)Q^t$. Then

$$M = \begin{pmatrix} E_1 & S^t \\ S & O \end{pmatrix} \oplus \begin{pmatrix} E_3 & -E_2 S^{-1} X_1^t & O \\ (-E_2 S^{-1} X_1^t)^t & e_{n-1} + X_1 (S^{-1})^t E_1 S^{-1} X_1^t & X_2 \\ O & X_2^t & O \end{pmatrix}.$$

Notice that the signature and the nullity of the matrix $\left(\begin{smallmatrix} E_1 & S^t \\ S & O \end{smallmatrix}\right)$ are zero.

3.5.1

If $m = r$, *i.e.*, $n\big(B(\beta)\big) = 0$, then $S = B(\beta)(U = V = I_m)$ and so the matrix E_3 and X_2 in M are the empty matrix. Hence

$$M = \begin{pmatrix} W & B(\beta)^t \\ B(\beta) & O \end{pmatrix} \oplus \Big(e_{n-1} + [r]\big(B(\beta)^{-1}\big)^t W B(\beta)^{-1} [r]^t\Big).$$

Thus $\sigma(M) = 0$ or ± 1 according as $[r]\big(B(\beta)^{-1}\big)^t W B(\beta)^{-1}[r]^t)$ is equal to $-e_{n-1}$ or not. If $\sigma(M) = 0$, then $n(M) = 1$ and so $n(\ell^{(2)}) = 2$. If $\sigma(M) = \pm 1$, then $n(M) = 0$ and so $n(\ell^{(2)}) = 1$. Thus $1 \le n(\ell^{(2)}) \le 2$ and $|\sigma(\ell^{(2)})| \le n(\ell^{(2)}) \le 2$.

3.5.2

If $X_2 = O$, then it is obvious that $m - r \le n(M) \le 2(m-r) + 1$ and $|\sigma(M)| \le m - r + 1$. Since $n(\ell^{(2)}) = n(M) + 1$, $n\big(B(\beta)\big) + 1 \le n(\ell^{(2)}) \le 2\Big(n\big(B(\beta)\big) + 1\Big)$. Since $n(\ell^{(2)}) \le 2\mu$ [9, Lemma 6.1], $0 \le n\big(B(\beta)\big) \le \mu - 1$. On the other hand, $|\sigma(\ell^{(2)})| = |\sigma(M)| \le n\big(B(\beta)\big) + 1 \le n(\ell^{(2)}) \le 2\mu$.

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3.5.3

If $X_2 \neq O$, then $n\big(B(\beta)\big) \geq 1$ and there exists an unimodular rational matrix R and $a(\neq 0) \in \mathbb{Q}$ such that

$$RMR^t = \begin{pmatrix} E_1 & S^t \\ S & O \end{pmatrix} \oplus \begin{pmatrix} E_3 & O & O \\ O & 0 & a \\ O & a & 0 \end{pmatrix} \oplus O_{m-r-1}.$$

So $m - r - 1 \leq n(M) \leq 2(m-r) - 1$ and $|\sigma(M)| = |\sigma(E_3)| \leq m - r$. Hence $n\big(B(\beta)\big) \leq n(\ell^{(2)}) \leq 2n\big(B(\beta)\big)$ and $1 \leq n\big(B(\beta)\big) \leq \mu$. Furthermore, $|\sigma(\ell^{(2)})| = |\sigma(M)| \leq n\big(B(\beta)\big) \leq n(\ell^{(2)}) \leq 2\mu$. Combining (3.5.1), (3.5.2) and (3.5.3), we obtain the result. ■

Corollary 3.6 *Let ℓ be a nonsplittable oriented link in S^3 of μ-components and let $\ell^{(2)}$ be the 2-parallel version of ℓ. Let β be a braid word representing ℓ.*

(1) If $n\big(B(\beta)\big) > 1$, then $\Delta_{\ell^{(2)}}(-1) = 0$, where $\Delta_{\ell^{(2)}}(t)$ is the reduced Alexander polynomial of $\ell^{(2)}$.

(2) If $n\big(B(\beta)\big) = 0$, then

(i) $n(\ell^{(2)}) = 1$ if and only if $\Delta_{\ell^{(2)}}(-1) \neq 0$ if and only if $|\sigma(\ell^{(2)})| = 1$.

(ii) $n(\ell^{(2)}) = 2$ if and only if $\Delta_{\ell^{(2)}}(-1) = 0$ if and only if $\sigma(\ell^{(2)}) = 0$.

(3) If $n\big(B(\beta)\big) = 1$, then

(i) $n(\ell^{(2)}) = 1$ or 3 if and only if $|\sigma(\ell^{(2)})| = 1$.

(ii) $n(\ell^{(2)}) = 2$ or 4 if and only if $|\sigma(\ell^{(2)})| \in \{0, 2\}$.

Proof (1) It follows from (4.9) in [9] that $n(\ell^{(2)}) = 1$ if and only if $\Delta_{\ell^{(2)}}(-1) \neq 0$. By Theorem 3.5(1), $n\big(B(\beta)\big) \leq n(\ell^{(2)})$ and so $n(\ell^{(2)}) > 1$. Thus $\Delta_{\ell^{(2)}}(-1) = 0$.

(2) If $n\big(B(\beta)\big) = 0$, then $n(\ell^{(2)}) = 1$ or 2. By (3.5.1), we have that $\Delta_{\ell^{(2)}}(-1) \neq 0$ if and only if $n(\ell^{(2)}) = 1$ if and only if $\sigma(\ell^{(2)}) = \pm 1$. Also $\Delta_{\ell^{(2)}}(-1) = 0$ if and only if $n(\ell^{(2)}) = 2$ if and only if $\sigma(\ell^{(2)}) = 0$.

(3) If $n\big(B(\beta)\big) = 1$, then $1 \leq n(\ell^{(2)}) \leq 4$ and the matrix E_3 and X_2 in M, in the proof of Theorem 3.5, are 1×1 matrices. So M is equal to the matrix of the form: for $a, b, c, d \in \mathbb{Q}$,

$$M = \begin{pmatrix} E_1 & S^t \\ S & O \end{pmatrix} \oplus N, \quad \text{where} \quad N = \begin{pmatrix} a & b & 0 \\ b & c & d \\ 0 & d & 0 \end{pmatrix}.$$

Hence if $n(\ell^{(2)}) = 1$, then $a, d \neq 0$ and so $\sigma(M) = \pm 1$, *i.e.*, $\sigma(\ell^{(2)}) = \pm 1$. Also if $n(\ell^{(2)}) = 3$, then $\operatorname{rank}(N) = 1$ and so $\sigma(M) = \pm 1$, *i.e.*, $\sigma(\ell^{(2)}) = \pm 1$. Conversely, if $\sigma(\ell^{(2)}) = \pm 1$, then $\operatorname{rank}(N) = 1$ or 3. So $n(M) = 0$ or 2, *i.e.*, $n(\ell^{(2)}) = 1$ or 3. The case (ii) follows from the fact that $\sigma(M) = 0$ or 2 if and only if $\operatorname{rank}(N) = 0$ or 2. ■

Corollary 3.7 *No torus link $T(n, q)$ $(n, q > 1)$ with $n = q$ or $2q \leq n$ is the 2-parallel version of a knot or link.*

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Proof Let $T(n,q)$ be the torus link of type (n,q) and let d be the greatest common divisor of n and q $(n,q>1)$. Then $T(n,q)$ is a link of d-components. By Theorem 5.2 [4] we have that $|\sigma(T(n,q))| > d$ for any $n,q>1$ with $n=q(\neq 2)$ or $2q \leq n$. Since $n(T(n,q)) \leq d$, the result follows from Theorem 3.5(2) and Example 3.4 for $n=q=2$. ■

4 The Minkowski Units

Two integral matrices A_1 and A_2 are said to be *R-equivalent* if they can be transformed into each other by a finite number of the following two types of transformations and their inverses:

$Q_1\colon A \to RAR^t$, where R is a nonsingular rational matrix,

$Q_2\colon A \to A \oplus \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$.

Any $n \times n$ nonzero symmetric rational matrix A can be transformed by Q_1 into a matrix of the form:

$$\begin{pmatrix} B & O \\ O & O \end{pmatrix},$$

where B is a nonsingular matrix. In particular, if A is an integral matrix, then B may be an integral matrix. The matrix B is called a *nonsingular matrix associated to A*.

Let A be an $n\times n$ symmetric integral matrix of rank r and B a nonsingular integral matrix associated to A. Then there is a sequence $B_1, B_2, \dots, B_r$, called the *σ-series*, of principal minors of B such that

(1) B_i is of order i and is a principal minor of B_{i+1},
(2) No consecutive matrices B_i and B_{i+1} are both singular $(i=1,2,\dots,r-1)$.

Let us denote $D_i = \det(B_i)$. Then for any prime integer p, we define [5]

$$c_p(B) = (-1,-D_r)_p \prod_{i=1}^{r-1} (D_i, -D_{i+1})_p,$$

where $(a,b)_p$ denotes the *Hilbert symbol*. If $D_{i+1}=0$, then $(D_i,-D_{i+1})_p(D_{i+1},-D_{i+2})_p$ is interpreted to be $(D_i,-h)_p(h,-D_{i+2})_p$, h being an arbitrary nonzero integer. Note that $c_p(B)$ is independent of the choice of σ-series of B.

Definition 4.1 Let B be an $r\times r$ nonsingular integral matrix. Then the *Minkowski unit* $C_p(B)$ for B is defined as follows: for any odd prime integer p,

$$C_p(B) = c_p(B)\big(\det(B), p\big)_p^{\alpha} \quad \text{and} \quad C_2(B) = c_2(B)(-1)^{\lambda},$$

where α denotes the exponent of p occurring in $\det(B)$ and

$$\lambda = \left[\frac{r}{4}\right] + \left\{1 + \left[\frac{r}{2}\right]\right\}\frac{(d+1)}{2} + \frac{(d^2-1)m}{8},$$

where [ ] denotes the Gaussian symbol, m the exponent of 2 occurring in $\det(B)$, and $d = 2^{-m}\det(B)$.

Finally, for $p=\infty$, $C_\infty(B) = \prod C_p(B)$, where the product extends over all prime integers p.

Remark 4.2 Let ν denote the number of odd primes of the form $4s+3$ occurring with odd powers in the prime factor decomposition of $\det(B)$, then $C_\infty(B) = (-1)^\gamma$, where $\gamma = [\frac{\sigma(B)-2\nu}{2}] + [\frac{\sigma(B)-2\nu}{4}]$ [10].

Let A be a symmetric integral matrix and let B be a nonsingular matrix associated to A. Then we define $\delta(A)$ to be the square free part of $|\det(B)|$ and the *Minkowski units* $C_p(A)$ of A to be $C_p(B)$ for any prime integer p, including $p=\infty$. Now let ℓ be an oriented link in S^3 and let L be a link diagram of ℓ. Let $H(L)$ be the modified Goeritz matrix of ℓ associated to L. Then $\delta(H(L))$ is an invariant of the link type ℓ, denoted by $\delta(\ell)$, and the Minkowski units $C_p\big(H(L)\big)$ for any prime integer p, including $p=\infty$, are invariants of the link type ℓ, denoted by $C_p(\ell)$, which is equal to the Minkowski units for knots or links defined by K. Murasugi [7], [10].

Theorem 4.3 *Let ℓ be a nonsplittable link of μ-components in S^3 and let $\ell^{(2)}$ be the 2-parallel version of ℓ. If $n(\ell^{(2)}) = 2\mu$ and ℓ has a braid representative β such that $n\big(B(\beta)\big) = \mu - 1$. Then*

(1) $\sigma(\ell^{(2)}) = 0$.
(2) $\delta(\ell^{(2)}) = 1$.
(3) $C_p(\ell^{(2)}) = 1$ *for any prime integer p, including $p=\infty$.*

Proof Let $\beta = \sigma_{i_1}^{\tau_{i_1}}\sigma_{i_2}^{\tau_{i_2}}\cdots\sigma_{i_m}^{\tau_{i_m}}$ $(\tau_{i_k} = \pm 1)$ be a braid word in B_n representing the link ℓ with $n\big(B(\beta)\big) = \mu-1$. By Corollary 3.2, the modified Goeritz matrix of $\ell^{(2)}$ associated to the closure of the braid word $\big(\phi_n^{(2)}(\beta), 2n\big)$ is given by $H(\beta) = G(\beta)\oplus A(\beta)$. It is easy to see that $A(\beta) = J(I_{2m}\oplus -I_{2m})J^t$ for some permutation matrix J.

If $n(\ell^{(2)}) = 2\mu$, then the matrix $M = QPG(\beta)P^tQ^t$ in the proof of Theorem 3.5 becomes $M = \left(\begin{smallmatrix} E_1 & S^t \\ S & O\end{smallmatrix}\right)\oplus O_{2\mu-1}$, where S is a nonsingular $(m-\mu+1)\times(m-\mu+1)$ rational matrix. So $\sigma(\ell^{(2)}) = \sigma\big(H(\beta)\big) = \sigma(M) + \sigma(I_{2m}\oplus -I_{2m}) = 0$.

Now we define $R = \left(\begin{smallmatrix} \frac{1}{2}I+\frac{1}{4}E_1 & \frac{1}{2}I-\frac{1}{4}E_1 \\ \frac{1}{2}S & -\frac{1}{2}S\end{smallmatrix}\right)\oplus I_{2\mu-1}$. Then R is a nonsingular and $M = R(2I_{m-\mu+1}\oplus -2I_{m-\mu+1}\oplus O_{2\mu-1})R^t$. Consequently, the diagonal matrix $D = (2I_{m-\mu+1}\oplus -2I_{m-\mu+1})\oplus(I_{2m}\oplus -I_{2m})$ is a nonsingular matrix associated to $H(\beta)$. Since $|\det(D)| = 2^{2(m-\mu+1)}$, $\delta(\ell^{(2)}) = 1$.

To prove (3) it is sufficient to show that $C_p(D) = 1$ for any prime integer p, including $p=\infty$. For each $i = 1,2,\ldots,q$ $(q = 2(3m-\mu+1))$, define $B_i = \mathrm{diag}(d_1, d_2, \ldots, d_i)$, where $d_k (1\le k\le i)$ is the (k,k)-diagonal entry of D and let $D_i = \det(B_i)$. Then the sequence $B_1, B_2, \ldots, B_q$ is a σ-series of D and $D_i = \pm 2^{a_i}$ $(a_i\in\mathbb{N}, 1\le i\le q)$. By using the properties of the Hilbert symbol [5, p. 27], $c_p(D)$ is expressed as a product of finite number of $(-1,-1)_p$ and $(-1,2)_p$.

If p is an odd prime, then $(-1,-1)_p = (-1,2)_p = 1$ and the exponent α of p occurring in $\det(D) = 2^{2(m-\mu+1)}$ is zero. Hence $C_p(D) = c_p(D)\big(\det(D), p\big)_p^\alpha = 1$.

If $p=2$, then $(-1,-1)_2 = -1$, $(-1,2)_2 = 1$ and so

$$C_2(D) = c_2(D)(-1)^\lambda = \begin{cases} (-1)^{6m-2\mu+2} & \text{if } m-\mu+1 \text{ is odd} \\ (-1)^{6m-2\mu+4} & \text{if } m-\mu+1 \text{ is even.}\end{cases}$$

Thus $C_2(D) = 1$.

Finally, since the signature $\sigma(D)$ of D and the number ν of all odd primes of the form $4s + 3$ occurring with odd powers in the prime factor decomposition of $\det(D) = (-1)^{3m-\mu+1}2^{2(m-\mu+1)}$ are both zero, $C_\infty(D) = (-1)^\gamma = (-1)^0 = 1$ due to Remark 4.2. This completes the proof. ∎

Remark 4.4 Let k be a knot in S^3 and let $k^{(2)}$ be the 2-parallel version of k. By Corollary 3.3, $n(k^{(2)}) = 2$. If k has a braid representative β such that $\det\bigl(B(\beta)\bigr) \neq 0$, it follows from Theorem 4.3 that $\delta(k^{(2)}) = 1$ and $C_p(k^{(2)}) = 1$ for any prime integer p, including $p = \infty$.

On the other hand, K. Murasugi [9], [10] showed that if ℓ is a slice link, then $\sigma(\ell) = 0$, $\delta(\ell) = 1$ and $C_p(\ell) = 1$ for any prime integer p, including $p = \infty$. Therefore Theorem 4.3 gives us the question: Is the 2-parallel version of a knot which has a braid representative β such that $\det\bigl(B(\beta)\bigr) \neq 0$ a slice link? For the case of links, this is not true in general (*cf.* Example 3.4 and Corollary 3.6).

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