UNIQUE CONTINUATION FOR PARABOLIC EQUATIONS OF HIGHER ORDER

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1. Let $x = (x_1, \ldots, x_n)$ be a point in the n-dimensional Euclidean space and let $\mathscr{D}$ be the unit sphere $|x| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} < 1$. In the $(n+1)$-dimensional Euclidean space with coordinate (x, t), we put

$$\Omega = \Omega_{T', T''} = \{(x, t)\ ;\ x \in \mathscr{D},\ T' \leqq t \leqq T''\}$$

and

$$S = S_{T', T''} = \{(x, t)\ ;\ x \in \dot{\mathscr{D}},\ T' \leqq t \leqq T''\},$$

where $\dot{\mathscr{D}}$ denotes the boundary of $\mathscr{D}$. We also use the following notation:

$$\mathscr{D}_T = \{(x, t)\ ;\ x \in \mathscr{D},\ t = T\}.$$

For real-valued functions $h_1 = h_1(x, t)$ and $h_2 = h_2(x, t)$ square integrable in Ω, we put

$$(h_1, h_2) = (h_1, h_2)_\Omega = \iint_\Omega h_1 h_2\, dxdt$$

and

$$\|h_1\|^2 = \|h_1\|_\Omega^2 = \iint_\Omega h_1^2\, dxdt.$$

We denote by $\mathfrak{B}$ the family of all the functions $v = v(x, t) \in C^{2s}(\Omega \cup S)$ which vanishes on $\mathscr{D}_{T'}$ and satisfies $D_x^\alpha v = 0$ $(|\alpha| \leqq s-1)$ on S. Here $C^{2s}(\Omega \cup S)$ is the class of all functions $2s$-times continuously differentiable in (a neighbourhood of) $\Omega \cup S$ and $D_x^\alpha v$ is the derivative

$$\frac{\partial^{|\alpha|} v}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$$

of v for a multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$ $(\alpha_i \geqq 0)$ of integers with length $|\alpha| = \alpha_1 + \cdots + \alpha_n$.

2. Consider a differential operator

Received May 19, 1965.

(1) $$L = A - (-1)^s \frac{\partial}{\partial t}$$

defined in $\Omega \cup S$, where A is of the form

$$A = \sum_{|\alpha| \leqq 2s} a_\alpha D_x^\alpha.$$

We assume that all the coefficients $a_\alpha = a_\alpha(x, t)$ are s-times continuously differentiable in $\Omega \cup S$ and are real-valued.

In this note, we shall prove the following theorem.

THEOREM. *Suppose that L is an operator of the form* (1) *and that A is uniformly elliptic in $\Omega \cup S$, that is, suppose that there exists a positive constant k_0 depending only on A and satisfying, at every point $(x, t) \in \Omega \cup S$,*

$$\sum_{|\alpha| = 2s} a_\alpha(x, t)\xi^\alpha \geqq k_0(\xi_1^2 + \cdots + \xi_n^2)^s$$

for any real vector $\xi = (\xi_1, \ldots, \xi_n)$, where $\xi^\alpha = \xi_1^{\alpha_1} \cdots \xi_n^{\alpha_n}$ for $\alpha = (\alpha_1, \ldots, \alpha_n)$. If in Ω

(2) $$(Lu)^2 \leqq k_1 \sum_{|\alpha| \leqq s} |D_x^\alpha u|^2$$

for some constant k_1 and if $u = 0$ on $\mathscr{D}_{T''}$ and $D_x^\alpha u = 0$ ($|\alpha| \leqq s-1$) on S, then u vanishes in Ω.

In the case when s is even, our theorem gives a backward uniqueness property of a solution of the equation $\left(A - \frac{\partial}{\partial t}\right)u = 0$. If s is odd, our theorem gives a uniqueness of a solution of the boundary value problem for $\left(A - \frac{\partial}{\partial t}\right)u = 0$.

Analogous theorems were given by many authors, Ito-Yamabe [3], Mizohata [7], Yamabe [10], Lees-Protter [5], Protter [9] and Edmunds [1]. In abstract way, such results were stated by Yosida [11], Lions-Malgrange [6] and Lees [4].

3. To prove the theorem, we prepare two lemmas which are analogous to Lees-Protter's estimates.

LEMMA 1. *Assume that A in* (1) *is uniformly elliptic in $\Omega \cup S$. If v is in $\mathfrak{B}$, if $f = f(t)$ is in $C^1([T', T''])$ and if $g = g(t)$ continuous in $[T', T'']$ has no zero, then there exist two positive constants k_2 and k_3 depending only on A such that*

$$k_2\|fv\|_s^2 \leqq \|fgLv\|^2 + ((k_3f^2 - 2ff' + f^2g^{-2})v,\ v) + \int_{\mathscr{D}_{T''}} f^2v^2dx,$$

where $\|v\|_s^2 = \sum_{|\alpha|\leqq s} \|D_x^\alpha v\|^2$.

Proof. It is obvious that

$$(3)\qquad (-1)^s 2(fv,\ fLv) \leqq \|fgLv\|^2 + \|fg^{-1}v\|^2.$$

Since A is uniformly elliptic in $\Omega \cup S$, it is easily proved in a manner quite similar to Nirenberg's [8] that Gårding's inequality [2] holds, that is, there exist two constants k_2 and k_3 depending only on A such that

$$k_2\|fv\|_s^2 \leqq (-1)^s 2(fv,\ fAv) + k_3\|fv\|^2.$$

So we have

$$(4)\qquad k_2\|fv\|_s^2 \leqq (-1)^s 2(fv,\ fLv) + k_3\|fv\|^2 + 2\left(fv,\ f\frac{\partial v}{\partial t}\right).$$

As to the last term of the right hand side of this inequality, we see by integration by parts

$$2\left(fv,\ f\frac{\partial v}{\partial t}\right) = -2(fv,\ f'v) + \int_{\mathscr{D}_{T''}} f^2v^2dx.$$

Here we have used the assumption $v \in \mathfrak{V}$. From (3), (4) and this, we have our lemma.

LEMMA 2. *Suppose that* v *is in* $\mathfrak{V}$ *and that* $f = f(t) \in C^\infty([T',\ T''])$ *and* $g = g(t)$ *continuous in* $[T',\ T'']$ *have no zero. Then for a given operator* L *in* (1), *there exists a constant* k_4 *depending only on* A *such that*

$$(fv,\ f''v) \leqq \|fLv\|^2 + k_4(\|fgv\|_s^2 + \|f'g^{-1}v\|_s^2) + \int_{\mathscr{D}_{T''}} ff'v^2dx.$$

Proof. Putting $u = fv$, we see easily

$$(5)\qquad -2\left(\frac{\partial u}{\partial t},\ f'v\right) \leqq \|fLv\|^2 - 2(-1)^s(Au,\ f'v) - \|f'v\|^2.$$

Obviously u is in $\mathfrak{V}$. Integrating by parts we get

$$(6)\qquad -2\left(\frac{\partial u}{\partial t},\ f'v\right) = ((ff'' - f'^2)v,\ v) - \int_{\mathscr{D}_{T''}} ff'v^2dx.$$

Now we estimate the integral $(a_\alpha D_x^\alpha v,\ f'v)$. Repeated use of integration by parts and Leibniz' formula gives us

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$$|(a_\alpha D_x^\alpha u,\ f'v)| = |(D_x^\beta u,\ D_x^\gamma(a_\alpha f'v))|$$
$$\leqq Mk_5 \|fgD_x^\beta v\|\ \|f'g^{-1}v\|_s,$$

where $\alpha = \beta + \gamma$, $|\beta| \leqq s$, $|\gamma| \leqq s$ and k_5 is a constant depending only on s and n and further the constant M depends only on L. Hence it holds that

$$(7) \qquad -2(-1)^s(Au,\ fv) \leqq k_4(\|fgv\|_s^2 + \|f'g^{-1}v\|_s^2)$$

for a constant k_4 depending only on A. From (5), (6) and (7) we obtain the required.

4. Now we give the proof of Theorem.

Take two numbers $\eta(>T'')$ and T_1 $(T'<T_1<T'')$ such that

$$(8) \qquad k_1\left(1+\frac{K}{2}\right)(\eta - T_1) < \frac{k_2}{4}.$$

where $K = k_3(\eta - T_1) + 1$ and k_1, k_2 and k_3 are constants appearing in Lemma 1 and the assumption of Theorem.

It is sufficient to show that u vanishes in $\Omega_{T_1,T''}$.

Let $\varphi = \varphi(t)$ be a function infinitely many times differentiable in $[T',\ T'']$ such that

$$\varphi = \begin{cases} 1, & T_2 < t < T'' \\ 0, & T' < t < T_1(<T_2) \end{cases}$$

for some T_2 fixed. Put $w = \varphi u$. It is evident that w is in $\mathfrak{B}$ and $w = 0$ on $\mathscr{D}_{T''}$. Taking an integer $m(>0)$ and applying Lemma 1 for $v = w$, $f = (\eta - t)^{-m-1/2}$ and $g = (\eta - t)^{1/2}$, we have

$$(9) \qquad k_2\|(\eta - t)^{-m-1/2} w\|_s^2 \leqq \|(\eta - t)^{-m} Lw\|^2 + K\|(\eta - t)^{-m-1} w\|^2.$$

Next we apply Lemma 2 for $v = w$, $f = (\eta - t)^{-m}$ and $g = m^{1/2}(\eta - t)^{-1/2}$ and we get

$$\|(\eta - t)^{-m-1} w\|^2 \leqq \frac{1}{m(m+1)}[\|(\eta - t)^{-m} Lw\|^2 + 2k_4 m\|(\eta - t)^{-m-1/2} w\|_s^2].$$

Substituting this into (9), we get

$$k_2\|(\eta - t)^{-m-1/2} w\|_s^2$$
$$\leqq \left(1+\frac{K}{2}\right)\|(\eta - t)^{-m} Lw\|^2 + \frac{2k_4K}{m+1}\|(\eta - t)^{-m-1/2} w\|_s^2.$$

The function w is identical with u in $\Omega_{T_2,T''}$ and the assumption (2) implies that

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$$\begin{aligned}\|(\eta-t)^{-m}Lw\|^2 &= \|(\eta-t)^{-m}Lu\|^2_{\Omega_{T_2,T''}} + \|(\eta-t)^{-m}Lw\|^2_{\Omega_{T_1,T_2}} \\ &\leqq k_1\|(\eta-t)^{-m}w\|^2_S + \|(\eta-t)^{-m}Lw\|^2_{\Omega_{T_1,T_2}},\end{aligned}$$

whence follows that

$$\begin{aligned}k_2\|(\eta-t)^{-m-1/2}w\|^2_S &\leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}} \\ &+\left[k_1\left(1+\frac{K}{2}\right)(\eta-T_1)+\frac{2k_4K}{m+1}\right]\|(\eta-t)^{-m-1/2}w\|^2_S.\end{aligned}$$

This is valid for any positive integer m. We can choose an m_0 such that

$$\frac{2k_4K}{m+1} < \frac{k_2}{4}$$

for any $m \geqq m_0$. From this and (8), we have

$$\frac{k_2}{2}\|(\eta-t)^{-m-1/2}w\|^2_S \leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

for $m \geqq m_0$. Restricting the integral of the left hand side over $\Omega_{T_3,T''}$ for such a T_3 as $T_2 < T_3 < T''$, we get

$$\frac{k_2}{2}(\eta-T_3)^{-2m-1}\|u\|^2_{S,\Omega_{T_3,T''}} \leqq \left(1+\frac{K}{2}\right)(\eta-T_2)^{-2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

or

$$\|u\|^2_{S,\Omega_{T_3,T''}} \leqq \frac{2}{k_2}\left(1+\frac{K}{2}\right)(\eta-T_3)\left(\frac{\eta-T_3}{\eta-T_2}\right)^{2m}\|Lw\|^2_{\Omega_{T_1,T_2}}$$

for $m \geqq m_0$. Making m tend to infinity, we see $u=0$ in $\Omega_{T_3,T''}$. Since T_3 is arbitrary as far as $T_2 < T_3 < T''$, it is seen that u vanishes in $\Omega_{T_2,T''}$. Further, T_2 is arbitrary as far as $T_1 < T_2 < T''$. So u vanishes throughout $\Omega_{T_1,T''}$. Thus our theorem is proved.

Remark. It is not difficult to see that, in our theorem, we can replace the assumption $u=0$ on $\mathscr{D}_{T''}$ by the condition

$$\lim_{t\to T''}\int_{\mathscr{D}_t}\sum_{i=1}^{n}\left(\frac{\partial u}{\partial x_i}\right)^2 dx = 0.$$

References

[1] D. E. Edmunds: Two uniqueness theorems associated with parabolic differential operators, Journ. London Math. Soc., **39** (1964), 345-354.

[2] A. Gårding: Dirichlet's problem for linear elliptic partial differential equation, Math. Scand., **1** (1953), 55-72.

[3] S. Ito and H. Yamabe: A unique continuation theorem for solutions of a parabolic equation, Journ. Math. Soc. Jap., **10** (1958), 314-321.

[4] M. Lees: Asymptotic behaviour of solutions of parabolic differential inequalities, Canad. Journ. Math., **14** (1962), 626-631.

[5] M. Lees and M. H. Protter: Unique continuation for parabolic differential equations and inequalities, Duke Math. Journ., **28** (1961), 369-382.

[6] J. L. Lions and B. Malgrange: Sur l'unicité rétrograde dans les problemes mixtes paraboliques, Math. Scand., **8** (1960), 277-286.

[7] S. Mizohata: Unicité du prolongement des solutions pour quelques opérateurs différentiels paraboliques, Mem. Coll. Sci., Univ. Kyoto, A **31** (1958), 219-239.

[8] L. Nirenberg: Remarks on strongly elliptic partial differential equations, Comm. Pure Appl. Math., **8** (1955), 648-674.

[9] M. H. Protter: Properties of solutions of parabolic equations and inequalities, Canad. Journ. Math., **13** (1961), 331-345.

[10] H. Yamabe: A unique continuation theorem of a diffusion equation, Ann. of Math., **69** (1959), 462-466.

[11] K. Yosida: An abstract analyticity in time for solutions of a diffusion equation, Proc. Jap. Acad., **35** (1959), 109-113.

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https://doi.org/10.1017/S0027763000011673 Published online by Cambridge University Press