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PETERZIL–STEINHORN SUBGROUPS AND μ-STABILIZERS IN ACF

MOSHE KAMENSKY[1], SERGEI STARCHENKO[2] AND JINHE YE[3]

[1] *Department of Mathematics, Ben-Gurion University, Be'er-Sheva, 8410501, Israel.*
(kamenskm@math.bgu.ac.il)

[2] *Department of Mathematics, University of Notre Dame, 255 Hurley, Notre Dame, IN 46556, USA.*
(starchenko.1@nd.edu)

[3] *Institut de Mathématiques de Jussieu-Paris Rive Gauche, Sorbonne Université, 4, place Jussieu–Boite Courrier 247 75252 Paris Cedex 05, France.*
(jinhe.ye@imj-prg.fr)

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Abstract We consider G, a linear algebraic group defined over $\Bbbk$, an algebraically closed field (ACF). By considering $\Bbbk$ as an embedded residue field of an algebraically closed valued field K, we can associate to it a compact G-space $S^{\mu}_G(\Bbbk)$ consisting of μ-types on G. We show that for each $p_\mu \in S^{\mu}_G(\Bbbk)$, $\mathrm{Stab}^{\mu}(p) = \mathrm{Stab}(p_\mu)$ is a solvable infinite algebraic group when p_μ is centered at infinity and residually algebraic. Moreover, we give a description of the dimension of $\mathrm{Stab}(p_\mu)$ in terms of the dimension of p.

Keywords and Phrases: Peterzil-Steinhorn subgroups, affine algebraic groups, μ-stabilizers, definable Berkovich spaces

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1. Introduction

Let G be a group definable in an o-minimal theory, and let $\gamma : (a,b) \to G$ be a definable curve which is unbounded, in the sense that the limit at b does not exist. In [9], it was shown that one can associate to this datum a definable 1-dimensional torsion-free group $H_\gamma \subseteq G$, which can be viewed as the stabilizer of γ at ∞. The group H_γ is called the *Peterzil–Steinhorn subgroup* associated to γ. For example, when G is a Cartesian power of the additive group, H_γ is the linear subspace whose translate is the asymptote of γ at ∞.

Assume now that G is an affine algebraic group over the complex numbers, and X is an algebraic curve embedded in G. If we view $\mathbb{C}$ as the algebraic closure of a real closed field (ACF) $\mathcal{K}$, the set of complex points of X can be viewed as the set of $\mathcal{K}$-points of a $\mathcal{K}$-definable set X^{an} in the o-minimal structure $\mathcal{K}$. This set is unbounded, and we may therefore choose an unbounded curve γ inside X^{an} and consider the corresponding PS-group H_γ. Taking its Zariski closure, we obtain an algebraic subgroup G_γ of G, of (algebraic) dimension 1.

It is natural to ask, to what extent does the subgroup G_γ depend on the nonalgebraic data involved, namely, the real closed field $\mathcal{K}$ of choice and the curve γ? And if it does not depend on these, can the construction be described in a purely algebraic manner? We first note that a choice of $\gamma : (a,b) \to X^{an}$ determines an additional *algebraic* datum: the curve X (which we may assume to be smooth) has a canonical compactification $\tilde{X}$, its projective model, which is obtained from X by adding finitely many points. Viewing γ as taking values in $\tilde{X}^{an}$ rather than X^{an}, the limit of γ at b will be precisely one of these points, and curves γ corresponding to different such points definitely might give rise to different subgroups G_γ. Hence, any hope of providing an algebraic construction of G_γ should take into account the choice of such a point at infinity.

Main results

The main result of this paper provides an algebraic construction, as anticipated, once the additional datum of a limit point is chosen:

Theorem 1.1. *Let $\Bbbk$ be an algebraically closed field and G be a linear algebraic group over $\Bbbk$, and let $X \subseteq G$ be an irreducible curve over $\Bbbk$. Then there are finitely many* 1*-dimensional linear subgroups of G, naturally associated to points at infinity of a smooth projective model of the curve X. In fact, they are the μ-stabilizers of those points at infinity, as explained later.*

A more precise version is given in Theorem 3.12, and the notion of μ-stabilizers will be introduced later. The main result of the paper, Theorem 1.2, includes a generalization of this theorem to higher dimensions, and some analysis of the structure of the resulting group. These results are obtained by viewing G as a definable group in ACVF, the theory of algebraically closed valued fields, and applying some results from [4]. The relation of the μ-stabilizers to the original construction of Peterzil and Steinhorn is explained in Remark 3.13.

To state the main result, we need to introduce some additional terminology. The subgroups we are interested in were introduced in an abstract setup in [7]. There, the authors consider (suitably defined) definable topological groups. To such a group G, one associates an infinitesimal subgroup μ, the intersection of all definable neighborhoods of the identity. If P is a (partial) type on G, the set $\mu \cdot P$ can be viewed geometrically as a tube around P, and the μ-stabilizer $\mathrm{Stab}^\mu(P)$ of P is defined to be the stabilizer of this set.

In the o-minimal context, the datum of a curve γ as before determines a type at infinity p_γ, and it is easy to see that the PS-group H_γ depends only on this type. It is shown in [7] that H_γ is precisely the μ-stabilizer of p_γ. Similarly, every closed point of the projective model of a smooth curve X determines an ACVF type on X, and the associated group is defined as the μ-stabilizer of this type. To see that the definition is reasonable, it is shown that the resulting group is 1-dimensional. Furthermore, it is contained in the (algebraic) stabilizer of the corresponding point in every equivariant compactification of G (Remark 3.4).

The definition of a μ-stabilizer makes sense for types of higher Zariski dimension as well. However, two types of different Zariski dimension might have the same tube

(Example 2.17), so the dimension comparison is not straightforward. We say that a type is μ-reduced if it is of minimal dimension among all types with a given tube. With this terminology, we have the following generalization of Theorem 1.1:

Theorem 1.2 (Main theorem). *Let G be a linear algebraic group defined over $\Bbbk$ and p be a residually algebraic type. If p is centered at infinity, then* $\mathrm{Stab}^{\mu}(p)$ *is infinite. Furthermore, if p is μ-reduced, then* $\dim(\mathrm{Stab}^{\mu}(p)) = \dim p$. *And for each type p,* $\mathrm{Stab}^{\mu}(p)$ *is a solvable linear algebraic group.*

Here the term "centered at infinity" should be understood as "unbounded" in the o-minimal counterpart. Do note that one cannot hope for the group to be torsion-free, as in the result on PS-groups, since the underlying field may have positive characteristic. The three parts of the theorem are proved, respectively, as Corollary 4.13, in §4.3, and as Theorem 4.20.

Structure of the paper

The structure of the paper is as follows: In §2 we review some definitions and results related to group actions, and provide an alternative approach to μ-stabilizers. In §3 we consider the 1-dimensional case of Theorem 1.1. Though formally included in the general case, this case is considerably simpler, and sheds light on the more complicated general case. Then in §4 we deal with the general case.

2. μ-Stabilizers over ACF

Let $\Bbbk$ be an arbitrary algebraically closed field (ACF), and G a linear algebraic group defined over $\Bbbk$. In this section, we develop the theory of μ-types and their stabilizers in this context, following [7]. Before going into μ-types, we begin with some generality on definable group actions.

2.1. Definable group actions

Let us start by recalling some general facts about stabilizers of definable types in an arbitrary complete theory T, following [7, Section 2]. We fix a monster model $\mathbb{U}$ of T, and all the models of T we consider will be elementary submodels of $\mathbb{U}$.

Let $\mathbf{X}$ be a definable set (over 0), and let A be a small set of parameters. We use $\mathcal{L}_{\mathbf{X}}(A)$ to denote the set of formulas ψ over A such that $\psi(x) \Rightarrow x \in \mathbf{X}$. Such formulas will occasionally be called $\mathbf{X}$-*formulas*. And by a *(partial)* $\mathbf{X}$-*type over* A, we mean a consistent collection of formulas in $\mathcal{L}_{\mathbf{X}}(A)$.

We fix $\mathbf{H}$ to be a definable group with a definable action on $\mathbf{X}$. For an $\mathbf{H}$-formula $\phi(x)$ and $\mathbf{X}$-formula $\psi(y)$, let $(\phi \cdot \psi)(z)$ be

$$\exists x \exists y \phi(x) \wedge \psi(y) \wedge z = x \cdot y.$$

And for a partial $\mathbf{X}$-type p, $\phi \cdot p = \{\phi \cdot \psi : \psi \in p\}$.

By a *definable* $\mathbf{X}$-*type over* A, we mean an $\mathbf{X}$-type over A such that for any formula $\phi(x,y)$, $\{a \in A : \phi(x,a) \in p\} = \{a \in A : d_p\phi(a)\}$ for some formula $d_p\phi$ over A. Note that in this definition, a can be tuples in A.

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Let $\mathbb{M}$ be a model of T such that $A = \mathbf{Y}(\mathbb{M})$ for some $\mathbb{M}$-definable set $\mathbf{Y}$. Then any two such definitions $d_p\phi$ will be equivalent. Moreover, p can be extended to a unique type $p \mid \mathbb{L}$ over $\mathbf{Y}(\mathbb{L})$ determined by $\{\phi(x,c) : c \in d_p\phi(\mathbf{Y}(\mathbb{L}))\}$, for any $\mathbb{L}$ such that $\mathbb{M} \preceq \mathbb{L}$.

Convention 2.1. For the remainder of the paper, we will assume that we are working over a set of parameters A such that $A = \mathbf{Y}(\mathbb{M})$ for some model $\mathbb{M}$, and we assume further that $\mathbf{H}$ and its action on $\mathbf{X}$ are defined over A. We assume further that $\mathbf{H} \subseteq \mathbf{Y}^n$ for some Cartesian product of $\mathbf{Y}$.

Definition 2.2. Let p be a definable partial $\mathbf{X}$-type over A as in Convention 2.1. We define

$$\mathrm{Stab}(p)(\mathbb{M}) = \{h \in \mathbf{H}(\mathbb{M}) : \text{For any } \phi \in \mathcal{L}_{\mathbf{X}}(A),\ p \models h \cdot \phi \Leftrightarrow p \models \phi\},$$

where $h \cdot \phi$ stands for $(x = h) \cdot \phi$. We will occasionally denote $\mathrm{Stab}(p)(\mathbb{M})$ by $\mathrm{Stab}(p)(A)$.

The following is [7, Proposition 2.13]:

Fact 2.3. Let $\mathbf{H}$ be a definable group with a definable action on $\mathbf{X}$, and assume we are in the setting of Convention 2.1. Let p be a partial definable $\mathbf{X}$-type over A. Then $\mathrm{Stab}(p)$ is a A-type-definable subgroup of $\mathbf{H}$ in the following sense: there is a small system $\mathbf{H}_\alpha$ of A-definable subgroups of $\mathbf{H}$ such that for every elementary extension $\mathbb{M} \preceq \mathbb{L}$, for $a \in \mathbf{H}(\mathbb{L})$, we have $a \in \mathrm{Stab}(p \mid \mathbb{L})(\mathbb{L})$ if and only if $a \in \mathbf{H}_\alpha(\mathbb{L})$ for all α.

With the language set up, we will now look at the setting to talk about μ-types as in [7] over algebraically closed fields.

2.2. μ-Stabilizers over ACF

Let $\Bbbk$ be an algebraically closed field. The theory of algebraically closed fields is not rich enough to have a good notion of infinitesimal subgroups as in [7]. Hence, it is natural to work with the theory T_{loc} as introduced in [3, Section 6]. The language for T_{loc} has two sorts, a sort **VF** for the valued field and a sort Γ for the value group. The sort **VF** is has a unary predicate **RES** for an embedded copy of the residue field, which can be viewed as an additional sort. Thus we have function symbols $\mathrm{res} : \mathbf{VF}^2 \to \mathbf{RES}$ and $\mathrm{val} : \mathbf{VF} \to \Gamma$. The theory T_{loc} asserts that the **VF** sort is an algebraically closed valued field, **RES** is a subfield, val is a valuation map, and $\mathrm{res}(x,y) = \mathrm{res}(x/y)$, the residue of x/y if $\mathrm{val}(x) \geq \mathrm{val}(y)$ and 0 otherwise, with $\mathrm{res}(c,1) = c$ for $c \in \mathbf{RES}$. For notational simplicity, we will use $\mathrm{res}(x)$ to denote $\mathrm{res}(x,1)$ for $x \in \mathcal{O}$, the valuation ring.

We further assume that we have constants for the elements of the field $\Bbbk$ in **RES**. Thus, models of T_{loc} are algebraically closed valued fields with embedded residue field extending $\Bbbk$.

Fact 2.4 ([3, Lemma 6.3]). T_{loc} admits quantifier elimination in this language. The sorts Γ and **RES** are stably embedded and orthogonal to each other. The induced structures on Γ and **RES** are of divisible ordered abelian groups and algebraically closed fields.

Remark 2.5. In [3], a constant symbol 1 in the Γ-sort for some positive element was included. But the proof of quantifier elimination does not rely on the constant.

In some cases, we will work in the reduct of T_{loc} in the 3-sorted language $\mathcal{L}_{\text{val}}$, which consists of the valued field sort **VF**, the value group sort Γ, the residue field sort **RES**, and maps $\text{val}: \mathbf{VF} \to \Gamma$ and $\text{res}: \mathbf{VF} \to \mathbf{RES}$. In this language, we have constants for $\Bbbk$ in both the **VF**-sort and the **RES**-sort. The induced theory on the reduct is $\text{ACVF}_{\Bbbk}$, the theory of algebraically closed valued fields with constants for $\Bbbk$, which admits quantifier elimination in $\mathcal{L}_{\text{val}}$. Since this is a reduct, we will freely view formulas in $\mathcal{L}_{\text{val}}$ as definable sets in T_{loc}. The main point of working with this reduct is the application of topological results from [4] available for this restricted class of definable sets.

Recall that we are given a linear algebraic group G defined over $\Bbbk$. In our context, this group determines a number of distinct definable groups: the definable group $\mathbf{G}$ of **VF**-points of G, and the definable subgroups $\mathbf{G}(\mathcal{O})$ and $\overline{\mathbf{G}}$ of $\mathcal{O}$- and **RES**-points, respectively. The (pointwise) residue map $\text{res}: \mathbf{G}(\mathcal{O}) \to \overline{\mathbf{G}}$ is a definable group-theoretic retraction (since $\mathbf{G}$ is over $\Bbbk$), whose kernel we denote by μ. Note that μ is definable over $\Bbbk$ as well. Geometrically, μ can be viewed as an infinitesimal neighborhood of the identity in $\mathbf{G}$.

Let P be an algebraic variety over $\Bbbk$ on which G acts. (Our main example will be $P = G$, with G acting on itself by left multiplication, but occasionally we will need the more general setup.) Then P determines a definable set $\mathbf{P}$ in **VF** and a definable action of $\mathbf{G}$ on $\mathbf{P}$, which restricts to an action of $\overline{\mathbf{G}} \leq \mathbf{G}$. We are therefore in the setting of Convention 2.1, where we set $T = T_{loc}$, $\mathbf{X} = \mathbf{P}$, $\mathbf{Y} = \mathbf{RES}$, and $\mathbf{H} = \overline{\mathbf{G}}$. In fact, one can take $\Bbbk = \mathbf{RES}(\mathbb{M})$, where $\mathbb{M}$ is a field of Hahn series with coefficients in $\Bbbk$.

Definition 2.6. Let P and G, and the associated terminology, be as in the foregoing. We denote by $S_{\mathbf{P}}(\Bbbk)$ the space of complete $\mathcal{L}_{\text{val}}$-**P**-types over $\Bbbk$. For $p \in S_{\mathbf{P}}(\Bbbk)$, the *μ-stabilizer* $\text{Stab}^{\mu}(p)$ of p is $\text{Stab}_{\overline{\mathbf{G}}}(\mu \cdot p)$.

Note that a complete $\mathcal{L}_{\text{val}}$-**P**-type might be a partial type in T_{loc}. By quantifier elimination, such types correspond to pairs of the form $(\mathbf{Z}, v)$, where $\mathbf{Z}$ is an irreducible closed subvariety of $\mathbf{P}$ and v is a valuation on the function field of $\mathbf{Z}$ which is trivial on $\Bbbk$. By Fact 2.4, the **RES**-sort is stably embedded as an algebraically closed field. In particular, each $p \in S_{\mathbf{P}}(\Bbbk)$ is definable over $\Bbbk$ in $\mathcal{L}_{\text{val}}$.

Proposition 2.7. *Set $p \in S_{\mathbf{P}}(\Bbbk)$. Then $\mu \cdot p$ is a definable partial type over $\Bbbk$.*

Proof. Let $\mathbf{Z}$ be an $\mathcal{L}_{\text{val}}$-definable set over $\Bbbk$. Then $\mu \cdot p \models \mathbf{Z}$ if and only if

$$p(x) \models \forall \epsilon \in \mu (\epsilon \cdot x \in \mathbf{Z}).$$

The latter condition is $\mathcal{L}_{\text{val}}$-definable over $\Bbbk$, and hence the result follows from the definability of p. □

By Fact 2.3 and the foregoing discussion, $\mathrm{Stab}^{\mu}(p)$ is given by an intersection of $\mathcal{L}_{\mathrm{val}}$-definable subgroups of $\overline{\mathbf{G}}$. However, $\overline{\mathbf{G}}$ has the descending chain condition on subgroups. Hence by Fact 2.4 we have the following:

Corollary 2.8. *Let p be an $\mathcal{L}_{val}$-complete G-type over $\mathbb{k}$. Then the μ-stabilizer $\mathrm{Stab}^{\mu}(p)$ of p is a $\mathbb{k}$-definable subgroup of $\overline{\mathbf{G}}$, in the sense that there is a $\mathbb{k}$-definable subgroup $\mathbf{H}$ of $\overline{\mathbf{G}}$, such that $\mathrm{Stab}^{\mu}(p \mid \mathbb{L})(\mathbb{L}) = \mathbf{H}(\mathbb{L})$ for any model $\mathbb{L} \succeq \mathbb{M}$.*

For p and $q \in S_{\mathbf{P}}(\mathbb{k})$, define $p \sim q$ if $\mu \cdot p = \mu \cdot q$. It is easy to check that $\mu \cdot p = \mu \cdot q$ if and only if in a monster model $\mathbb{U}$ there are $a \models p$, $b \models q$, and $\epsilon \in \mu$ such that $\epsilon \cdot a = b$.

We denote by $S^{\mu}_{\mathbf{P}}(\mathbb{k})$ the quotient by this equivalence relation, and for each $p \in S_{\mathbf{P}}(\mathbb{k})$ we denote by p_{μ} its equivalence class. Since μ is normal in $\mathbf{G}(\mathcal{O})$, the $\overline{\mathbf{G}}(\mathbb{k})$-action on $S_{\mathbf{P}}(\mathbb{k})$ given in §2.1 respects the equivalence relation. Hence $\overline{\mathbf{G}}(\mathbb{k})$ acts on $S^{\mu}_{\mathbf{P}}(\mathbb{k})$, and $\mathrm{Stab}^{\mu}(p) = \mathrm{Stab}\,(p_{\mu})$, where the right-hand side is by considering $\overline{\mathbf{G}}(\mathbb{k})$ acting on $S^{\mu}_{\mathbf{P}}(\mathbb{k})$.

Lastly, we note that stabilizers for types in the same orbit are conjugate:

Lemma 2.9. *Let $g \in \overline{\mathbf{G}}(\mathbb{k})$ be such that $g \cdot p_{\mu} = q_{\mu}$. Then $\mathrm{Stab}^{\mu}(q) = g\mathrm{Stab}^{\mu}(p)g^{-1}$.*

We finish this subsection with two basic examples. These can be compared to the computations in the o-minimal setting.

Example 2.10. Let $G = \mathrm{SL}_{2,\mathbb{k}}$. Let

$$X_1 = \left\{ \begin{pmatrix} x & 1 \\ 0 & x^{-1} \end{pmatrix} : x \in \mathrm{G_m} \right\},$$

a closed subvariety of G. Since this subvariety is isomorphic (as an algebraic variety) to $\mathrm{G_m}$, the definable subset given by $\mathrm{val}(x) < 0$ isolates a complete type p on $\mathbf{G}$ (in the language $\mathcal{L}_{\mathrm{val}}$ over $\mathbb{k}$).

We claim that the left μ-stabilizer H of p is the subgroup

$$G_1 = \left\{ \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \right\}.$$

Indeed, set $g \in \mathbf{RES}$ and choose $\alpha \in \mathbf{VF}$ with $\mathrm{val}(\alpha) < 0$. Then

$$\begin{pmatrix} 1 & g \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha & 1 \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} \alpha & 1 + g\alpha^{-1} \\ 0 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} 1+\varepsilon & 0 \\ 0 & (1+\varepsilon)^{-1} \end{pmatrix} \begin{pmatrix} \beta & 1 \\ 0 & \beta^{-1} \end{pmatrix},$$

where $\varepsilon = g\alpha^{-1}$ and $\beta = (1+\varepsilon)^{-1}\alpha$. Since $\mathrm{val}(\varepsilon) > 0$, we have $\left(\begin{smallmatrix} 1+\varepsilon & 0 \\ 0 & (1+\varepsilon)^{-1} \end{smallmatrix}\right) \in \mu$ and $\left(\begin{smallmatrix} \beta & 1 \\ 0 & \beta^{-1} \end{smallmatrix}\right) \models p$.

Thus $G_1 \subseteq H$. By (the easy part of) Theorem 3.12, H is 1-dimensional, so G_1 is the connected component of H, and H has the form

$$H = \left\{ \begin{pmatrix} \xi & a \\ 0 & \xi^{-1} \end{pmatrix} : \xi^n = 1 \right\}$$

for some n. By a similar computation, such an element will take $\mu \cdot p$ to $\mu \cdot p_{\xi}$, where p_{ξ} is the type of elements $\left(\begin{smallmatrix} x & \xi \\ 0 & y \end{smallmatrix}\right)$ with $\mathrm{val}(x) < 0$, so we must have $\xi = 1$ and $G_1 = H$.

This example will be considered again in Example 3.5.

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Example 2.11. Similarly to Example 2.10, we now consider the closed subvariety

$$X_2 = \left\{ \begin{pmatrix} x & 0 \\ 1 & x^{-1} \end{pmatrix} : x \in \mathrm{G_m} \right\} \subseteq G = \mathrm{SL}_2,$$

and let q be the type on $\mathbf{G}$ determined inside it by the condition $\mathrm{val}(x) < 0$.

We claim that the left μ-stabilizer H of q is now the subgroup $G_2 = \left\{ \left(\begin{smallmatrix} a & 0 \\ 0 & a^{-1} \end{smallmatrix}\right) : a \in \mathrm{G_m} \right\}$. The computation is similar: for $g \in \mathrm{G_m}(\mathbf{RES})$ and $\alpha \in \mathbf{VF}$ with $\mathrm{val}(\alpha) < 0$, we have

$$\begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \alpha & 0 \\ 1 & \alpha^{-1} \end{pmatrix} = \begin{pmatrix} g\alpha & 0 \\ g^{-1} & (g\alpha)^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ \varepsilon & 1 \end{pmatrix} \begin{pmatrix} \beta & 0 \\ 1 & \beta^{-1} \end{pmatrix},$$

where $\beta = g\alpha$ and $\varepsilon = \frac{g^{-1}-1}{g\alpha}$.

Since $\mathrm{val}(\varepsilon) > 0$ and $\mathrm{val}(\beta) < 0$, we obtain $G_2 \subseteq H$. The only 1-dimensional algebraic subgroup of G that properly contains G_2 is its normalizer, which contains the element $w = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. This element sends q to the type of elements $\left(\begin{smallmatrix} 1 & x^{-1} \\ -x & 0 \end{smallmatrix}\right)$ with $\mathrm{val}(x) < 0$, so cannot be in H. Hence the μ-stabilizer is G_2 in this case.

2.3. A different view on μ-stabilizers

Instead of viewing the μ-stabilizers syntactically as in the previous section, we have some concrete constructions to realize them in the monster model as well. In this section, we describe the construction, following the same idea as [7, §2.4]. We work in a fixed monster model $\mathbb{U}$ of T_{loc} and identify definable sets and (partial) types with their realizations in $\mathbb{U}$. From now on, we restrict our attention to the case $\mathbf{P} = \mathbf{G}$, unless mentioned otherwise.

Definition 2.12. For $p \in S_G(\Bbbk)$ we use $\overline{\mathbf{G}}_p$ to denote the set $\left((\mu \cdot p)\cdot(\mu \cdot p)^{-1}\right) \cap \overline{\mathbf{G}}$.

Proposition 2.13. *Set $a \in \mu \cdot p$. The following are equivalent for an element $b \in \overline{\mathbf{G}}(\Bbbk)$:*

(1) $b \in \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}$;

(2) $b = \mathrm{res}\left(a_1 a^{-1}\right)$ *for some* $a_1 \models p$ *for which* $a_1 a^{-1} \in \mathbf{G}(\mathcal{O})$;

(3) $b \in \overline{\mathbf{G}}_p$;

(4) $b = \mathrm{res}\left(a_1 {a_2}^{-1}\right)$ *for some* $a_1, a_2 \models p$ *for which* $a_1 {a_2}^{-1} \in \mathbf{G}(\mathcal{O})$.

Hence, $\overline{\mathbf{G}}_p(\Bbbk) = \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}(\Bbbk)$.

Proof. The equivalence of (1) and (2) follows directly from the definitions, and likewise for (3) and (4). Hence we need to show that (4) implies (2). Assume $b = \mathrm{res}\left(a_1 a_2^{-1}\right)$. Since a and a_2 satisfy the same type over $\Bbbk$, and **RES** is stably embedded and stable, there is an automorphism τ over $\Bbbk$ such that $\tau(a_2) = a$. Then $b = \tau(b) = \mathrm{res}\left(\tau(a_1) a^{-1}\right)$, with $\tau(a_1)$ also satisfying p, showing (2) for b. □

We now have the following description of $\overline{\mathbf{G}}_p$:

Corollary 2.14. $\overline{\mathbf{G}}_p(\Bbbk) = \mathrm{Stab}^{\mu}(p)(\Bbbk)$.

Proof. Assume $g \in \overline{\mathbf{G}}$ stabilizes $\mu \cdot p$. Then for any $a \in \mu \cdot p$, $g \cdot a \in \mu \cdot p$, hence $g \in \mu \cdot p a^{-1}$, so is in $\overline{\mathbf{G}}_p$. Conversely, if $g \in \overline{\mathbf{G}}_p$ and $a \in \mu \cdot p$, writing $g = a_1 a^{-1}$ as before we obtain $g \cdot a \in \mu \cdot p$. □

Remark 2.15. We would like to have Corollary 2.14 hold for **RES**-points instead of just $\Bbbk$-points. This is not automatic, since $\overline{\mathbf{G}}_p$ is not, a priori, a definable set. However, in the special case as in Theorem 1.1, it is indeed the case.

2.4. μ-Reduced types

For $p \in S_G(\Bbbk)$ we denote by $\dim(p)$ the dimension of its Zariski closure in $\mathbf{G}$ over $\Bbbk$. Most of the rest of this paper is devoted to comparing this dimension to the dimension of $\mathrm{Stab}^\mu(p)$. We first note that if $\mathbf{X}$ is a variety over a valued field L whose valuation ring is $\mathcal{O}$, then the Zariski dimension of $\mathrm{res}(\mathbf{X} \cap \mathcal{O})$ is at most the dimension of $\mathbf{X}$ (this follows, for example, from [11, Lemma 00QK], by choosing a model of $\mathbf{X}$ over $\mathcal{O}$). Applying this observation to $\mathbf{X} = \mathbf{Y}a^{-1}$, where $\mathbf{Y}$ is a variety containing p, we obtain the following:

Proposition 2.16. *For any* $p \in S_G(\Bbbk)$, $\dim(\mathrm{Stab}^\mu(p)) = \dim \overline{\mathbf{G}}_p(\Bbbk) \leq \dim(p)$, *where* dim *means the Krull dimension in* $\mathrm{Stab}^\mu(p)$ *and* $\overline{\mathbf{G}}_p(\Bbbk)$, *and* $\dim(p)$ *is the minimal* **VF**-*dimension of the formulas* $\varphi \in p$.

In general, the bound will not be sharp, since types of different dimensions may have the same μ-type:

Example 2.17. Let $G = \mathbb{A}^2$ as an additive group. Let K be a large enough Hahn series in variable t over $\Bbbk$. Let $p = tp\left(\left(t^{-1}, t^{-1} + t^r\right)/\Bbbk\right)$, where $r > 0, r \notin \mathbb{Q}$, and tp denotes the $\mathcal{L}_{\mathrm{val}}$-type. Then $\dim(p) = 2$, since $t^{-1} + t^r$ is transcendental over t^{-1}. But $\mu \cdot p = \mu \cdot q$, where $q = tp\left(\left(t^{-1}, t^{-1}\right)/\Bbbk\right)$, since $\left(t^{-1}, t^{-1} + t^r\right)$ and $\left(t^{-1}, t^{-1}\right)$ differ by $(0, -t^r) \in \mu$, so $\dim\left(\overline{\mathbf{G}}_p\right) \leq 1$ (as we will see later, they are in fact equal). Furthermore, when $\mathrm{Char}(k) = p > 0$, we see that $\mathrm{Stab}^\mu(p)$ is not torsion-free.

This observation motivates the following definition:

Definition 2.18. For $p \in S_G(\Bbbk)$, we say that p is *μ-reduced* if p is a type of minimal dimension in p_μ. An element $a \in \mathbf{G}$ is μ-reduced over $\Bbbk$ if $a \models p$ for some μ-reduced p.

2.5. Bounded types

In this subsection, we revert to working with a general G-variety $\mathbf{P}$. We recall the following definition (e.g., from [4, §4.2]):

Definition 2.19. Let $\mathbf{V}$ be an affine variety, viewed as a definable set in ACVF, and let $\mathbf{X} \subseteq \mathbf{V}$ be an $\mathcal{L}_{\mathrm{val}}$-definable subset. We say that $\mathbf{X}$ is *bounded* if for every regular function f on $\mathbf{V}$ there is $\gamma \in \Gamma$ such that $\mathrm{val}(f(\mathbf{X})) \geq \gamma$.

For a general variety $\mathbf{V}$, a subset $\mathbf{X} \subseteq \mathbf{V}$ is bounded if it is covered by bounded subsets of an affine cover.

A partial type p in $\mathbf{V}$ is *bounded* if $p \models \mathbf{X}$ for some bounded $\mathbf{X} \subseteq \mathbf{V}$. A type in $\mathbf{V}$ is said to be *centered at infinity* if it is not bounded.

Note that the property of a definable set to be bounded depends on the ambient variety (for example, $\mathbb{A}^1$ is bounded as a subset of $\mathbb{P}^1$, but not as a subset of $\mathbb{A}^1$). However, if $\mathbf{V}$ is a closed subvariety of $\mathbf{W}$, then $\mathbf{X} \subseteq \mathbf{V}$ is bounded in $\mathbf{V}$ if and only if it is bounded in $\mathbf{W}$. Also, it suffices to check the conditions for generators of the regular functions. In particular, a subset $\mathbf{X}$ of a closed subvariety of $\mathbb{A}^n$ is bounded if and only if $\mathrm{val}(\mathbf{X}) \geq \gamma$ for some γ.

Over $\Bbbk$, we have in our situation the following:

Proposition 2.20. *A $\Bbbk$-definable set $\mathbf{X} \subseteq \mathbf{V}$ is bounded if and only if it is contained in* $\mathbf{V}(\mathcal{O})$

Proof. By definition, it suffices to prove the statement for $\mathbf{V}$ affine, and by the foregoing remarks, for $\mathbf{V} = \mathbb{A}^n$.

If $\mathbf{X} \subseteq \mathcal{O}^n$ we may take $\gamma = 0$ in the definition. Conversely, we may assume $n = 1$ by projecting. If $a \in \mathbf{X} \setminus \mathcal{O}$, then $\gamma = \mathrm{val}(a) < 0$ has the same type as any other negative value γ', so there is an automorphism of Γ taking γ to γ', and since Γ is stably embedded and Γ and $\mathbf{RES}$ are orthogonal, it extends to an automorphism over $\Bbbk$ that takes a to $a' \in \mathbf{X}$, with $\mathrm{val}(a') = \gamma'$. Thus $\mathbf{X}$ is unbounded. □

Let p be a bounded type on $\mathbf{P}$, a variety endowed with an action of $\mathbf{G}$. A realization a of p is then an $\mathcal{O}$-point of $\mathbf{P}$, and so determines a point $\overline{a}$ of $\mathbf{P}$ in the residue field. The type of $\overline{a}$ depends only on p (since it is encoded there), and we denote it by $\overline{p}$. The group $\overline{\mathbf{G}}$ acts on the set of all types in $\overline{\mathbf{P}}$, the variety $\mathbf{P}$ viewed as a definable set in $\mathbf{RES}$. In particular, we may consider the stabilizer of $\overline{p}$.

Proposition 2.21. *For any bounded type p on $\mathbf{P}$ we have* $\mathrm{Stab}^{\mu}(p) \leq \mathrm{Stab}(\overline{p})$.

Proof. Let $\overline{a}$ be a realization of $\overline{p}$, and let a be a realization of p whose residue is $\overline{a}$. Assume that for some $g \in \overline{\mathbf{G}}$ we have $g \cdot a = \epsilon \cdot b$ for some $\epsilon \in \mu$ and b realizing p (so that $g \in \mathrm{Stab}^{\mu}(p)$). Since all elements involved are in $\mathcal{O}$, we may apply the residue map and obtain $g \cdot \overline{a} = \overline{b}$. Since b realizes p, $\overline{b}$ realizes $\overline{p}$. Thus $g \cdot \overline{p} = \overline{p}$–that is, $g \in \mathrm{Stab}(\overline{p})$. □

Returning to the case $\mathbf{P} = \mathbf{G}$, we obtain the following:

Corollary 2.22. *If p is a bounded type on $\mathbf{G}$ such that $\overline{p}$ is realized in $\Bbbk$, then its μ-stabilizer is trivial.*

Proof. In this case $\overline{p}$ corresponds to a (closed) point of G, hence the stabilizer is trivial. □

Because of the last corollary, we shall concentrate on types centered at infinity.

3. Analyzing the 1-dimensional case

In this section, we prove the main theorem in dimension 1 (Theorem 1.1). We think this section worth including even though it follows from the general case, since it is relatively simple and it sheds light on the important idea in proving the general case. The result in this section was first proved by Moshe Kamensky and Sergei Starchenko in unpublished notes via the language of places.

3.1. Points on curves

Each smooth curve $\mathbf{X}$ over $\mathbb{k}$ embeds in a unique smooth projective one over $\mathbb{k}$, its projective model $\tilde{\mathbf{X}}$. Every closed point c on $\tilde{\mathbf{X}}$ corresponds to a valuation val_c on the function field $\mathbb{k}(\mathbf{X})$, given by the order of vanishing at c. In particular, val_c is trivial on $\mathbb{k}$. The projective model contains a finite number of closed points outside of $\mathbf{X}$, which we call the points at infinity.

In our case, $\mathbf{X}$ is an affine curve, embedded as a closed subvariety in a fixed affine space $\mathbb{A}^n$. To any point $c \in \tilde{\mathbf{X}}$ we associate the complete type on $\mathbf{X}$ determined by

$$p_c(a) = \left\{\mathrm{val}(f(a)) > 0 : \mathrm{val}_c\left(\overline{f}\right) > 0\right\},$$

where f runs over all elements of the local ring corresponding to $\mathbf{X}$ and $\overline{f}$ is the corresponding element in $\mathbb{k}(\mathbf{X})$.

We would like to describe the types that occur in this way intrinsically, in a way that will be helpful later. The condition that c is a closed point corresponds to the following:

Definition 3.1. An extension of (possibly trivially) valued fields is *residually algebraic* if the corresponding residue field extension is algebraic. For L a (possibly trivially) valued field, an $\mathcal{L}_{\mathrm{val}}$-type p over L is *residually algebraic* if for a (or every) realization a of p, $L(a)$ is residually algebraic over L.

Proposition 3.2. *Let $\mathbf{X}$ be a smooth curve embedded in $\mathbb{A}^n$ (viewed as a definable set in $\mathbf{VF}^n$). An $\mathcal{L}_{\mathrm{val}}$-type p over $\mathbb{k}$ on $\mathbf{X}$ is residually algebraic if and only if it is of the form p_c for a closed point c of $\tilde{\mathbf{X}}$, the smooth projective model of $\mathbf{X}$. Furthermore, $c \in \tilde{\mathbf{X}} \setminus \mathbf{X}$ if and only if p_c is unbounded.*

Proof. Let p be a residually algebraic type on $\mathbf{X}$, and let a be a realization that witnesses this. If $a \in \mathbb{k}$, p corresponds to the $\mathbb{k}$-point a of $\mathbf{X}$ and we are done. Otherwise, $\mathbb{k}(a)$ is isomorphic to $\mathbb{k}(\mathbf{X})$ as a field, and since p is residually algebraic, the valuation on $\mathbb{k}(a)$ is nontrivial. Thus we obtain a $\mathbb{k}$-point c of $\tilde{\mathbf{X}}$ by the discussion earlier in this subsection, and it is clear that the two procedures are inverse to each other. The last statement also follows. □

We have been working with smooth curves, but since we are interested in points at infinity, the assumption is immaterial, since the singularity of varieties are of at least codimension 1, and hence varieties are smooth at generic points.

Corollary 3.3. *For $\mathbf{X}$ a curve, there are finitely many residually algebraic types centered at infinity. Moreover, they are isolated by $\mathcal{L}_{\mathrm{val}}$-formulas over $\mathbb{k}$.*

Proof. It remains to prove the 'moreover' part of the statement. Since we know that there are only finitely many types on $\mathbf{X}$ centered at infinity, call them $p_1,\dots,p_m$. Without loss of generality, for each i,j there will be regular functions f_{ij}, g_{ij} such that $p_i \models \mathrm{val}(f_{ij}) < \mathrm{val}(g_{ij})$ but $p_j \models \mathrm{val}(f_{ij}) \geq \mathrm{val}(g_{ij})$. Hence some Boolean combinations of these formulas together with the formula $x \notin \mathbf{X}(\mathcal{O})$ will isolate the types in question. □

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Remark 3.4. Let $\mathbf{X}$ be an affine curve embedded in G, and assume that we are given a G-equivariant embedding of G in a G-variety P. Assume that the closure $\mathbf{X}'$ of $\mathbf{X}$ in P includes the point $c \in \tilde{\mathbf{X}}$. The type p_c is then bounded in P, and by Proposition 2.21 the μ-stabilizer of p_c is contained in the stabilizer of the residue type of p_c, which is simply c. Hence the μ-stabilizer of c is contained in the stabilizer of this point in every equivariant compactification where the point is realized.

This fact, along with the dimension equalities for the μ-stabilizers, justifies viewing the μ-stabilizer as a canonical stabilizer for the corresponding point.

Example 3.5. Example 2.10 provides an instance of the situation here, with $\mathbf{X} = X_1$. Since $X_1 = \mathrm{G_m}$ as an abstract variety, its projective model $\tilde{X_1}$ is $\mathbf{P}^1$, so has two additional points, 0 and ∞ (with 0 the one included in the chart where x is defined). Hence, the type p considered there is p_∞ in our notation, and p_0 corresponds to the type with the roles of x and x^{-1} reversed.

We may alter it a bit by considering the image of X_1 in PSL_2 (the computation remains essentially the same, but we now also have the elements $\left(\begin{smallmatrix}-1 & a\\ 0 & -1\end{smallmatrix}\right)$ in the stabilizer). The space PSL_2 can be compactified equivariantly by mapping it into $\mathbb{P}^3$, viewed as the projective space associated to the space of all linear endomorphisms of $\mathbb{A}^2$. It is easy to compute that under this embedding, the point ∞ maps to the element $c = \left[\left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)\right]$ of $\mathbb{P}^3$. The stabilizer of this point under the action of SL_2 is the subgroup of upper triangular matrices, so properly contains our μ-stabilizer.

Let $p \in S_G(\Bbbk)$ be a residually algebraic type of Zariski dimension 1 inside $\mathbf{G}$. There is then a curve $\mathbf{X}$ in $\mathbf{G}$ containing p. We explained in Proposition 2.13 and Remark 2.15 that $\mathrm{Stab}^\mu(p)(\Bbbk) = \mu \cdot p \cdot a^{-1} \cap \overline{\mathbf{G}}(\Bbbk)$ for any realization a of p (this will be shown again for residually algebraic types in Corollary 4.8). However, since p is isolated by Corollary 3.3, we see that $\mathrm{Stab}^\mu(p \mid \mathbb{L})(\mathbb{L}) = \mu \cdot p \cdot a^{-1}(\mathbb{L}) \cap \overline{\mathbf{G}}(\mathbb{L})$ for any $\mathbb{L}$ extending $\mathbb{M}$ and a. In particular, one can work with a model $\mathbb{L}$ of T_{loc} with $\mathbf{RES}(\mathbb{L}) = \Bbbk$. Working in this model, let p_i, p_j be two types as before. If $g \in G(\Bbbk)$ satisfies $g \cdot p_i \in {p_j}_\mu$ (i.e., if $\mu \cdot g \cdot p_i = \mu \cdot p_j$), then $\mu \cdot p_j \cdot a^{-1} = g \cdot \mathrm{Stab}^\mu(p_i)$, for any $a \models p_i$.

To complete the proof, we would like to show that this set is infinite for some realization a of p. This amounts to showing that $\mu \cdot p \cdot a^{-1}$ cannot be covered by a finite number of open balls. To do that, we will use topological methods from [4], which we first review.

3.2. Tame topology on definable sets

We make a slight digression into the tame topology of definable sets in ACVF, as developed in [4]. This is an important ingredient in the proof of the main result.

The results in this section can be found in [4]. In this section, the underlying theory is ACVF, and the main motivation is to study the topological structure of $\mathcal{L}_{\mathrm{val}}$-definable sets in the $\mathbf{VF}$-sort.

Definition 3.6. Let $\mathbf{V}$ be an algebraic variety over a valued field F. A subset $\mathbf{X} \subseteq \mathbf{V}$ is *v-open* if it is open for the valuative topology.

A subset $\mathbf{X} \subseteq \mathbf{V}$ is *g-open* if it is a positive Boolean combination of Zariski open, closed sets and sets of the form

$$\{x : v \circ f(x) > v \circ g(x)\},$$

where f and g are regular functions defined on $\mathbf{U}$, a Zariski open subset of $\mathbf{V}$.

If $\mathbf{Z} \subseteq \mathbf{V}$ is a definable subset of $\mathbf{V}$, a subset $\mathbf{W}$ of $\mathbf{Z}$ is said to be v-open (resp., g-open) if $\mathbf{W}$ is of the form $\mathbf{Z} \cap \mathbf{Y}$, where $\mathbf{Y}$ is v-open (resp., g-open) in $\mathbf{V}$.

The complement of v-open (resp., g-open) is v-closed (resp., g-closed). We say $\mathbf{X}$ is v+g-open (resp., v+g-closed) if it is both v-open and g-open (resp., both v-closed and g-closed).

Note that the v+g-opens do not form a topology, as it is not even closed under arbitrary union. However, it is still makes sense to talk about connectedness in this setting:

Definition 3.7. Let $\mathbf{X}$ be a definable subset of $\mathbf{V}$, an algebraic variety. We say that $\mathbf{X}$ is *definably connected* if $\mathbf{X}$ cannot be written as a disjoint union of two nonempty v+g-open subsets of $\mathbf{X}$.

We say that $\mathbf{X}$ has *finitely many definably connected components* if $\mathbf{X}$ can be written as a finite disjoint union of v+g-clopen definably connected subsets.

Definition 3.8. Let $f : \mathbf{V} \to \mathbf{W}$ be a definable function from $\mathbf{V}$ to $\mathbf{W}$. We say f is *v-continuous* if $f^{-1}(\mathbf{X})$ is v-open for $\mathbf{X}$ a v-open subset of $\mathbf{W}$, and we define *g-continuous* functions similarly. We say f is v+g-continuous if f is both v-continuous and g-continuous.

Proposition 3.9 (Hrushovski and Loeser). *If f is v+g-continuous, $\mathbf{X}$ is definably connected, and f is defined on $\mathbf{X}$, then $f(\mathbf{X})$ is definably connected.*

If $\mathbf{V}$ is a geometrically or absolutely irreducible variety, then it is definably connected.

The following is an easy corollary of [4, Theorem 11.1.1]:

Theorem 3.10 (Hrushovski and Loeser). *Given a definable subset $\mathbf{X} \subseteq \mathbf{V}$, where $\mathbf{V}$ is some quasi-projective variety, $\mathbf{X}$ has finitely many definably connected components.*

We also have the following:

Theorem 3.11. *Let $\mathbf{V} \subseteq \mathbb{A}^n$ be a closed subvariety. It is bounded if and only if $\mathbf{V}$ is 0-dimensional.*

Proof. If $\mathbf{V}$ is bounded, then it will be definably compact as in [4]. This implies that $\mathbf{V}$ is proper by [4, Proposition 4.2.30], and hence is 0-dimensional. The converse is clear. □

We may now prove the following, more precise version of Theorem 1.1 (the case of curves):

Theorem 3.12. *Let $p \in S_G(\Bbbk)$ be a residually algebraic type, centered at infinity with* $\dim(p) = 1$. *Then* $\dim(\mathrm{Stab}^{\mu}(p)) = 1$.

Proof. By Proposition 2.16 and the discussion preceding §3.2, it suffices to show that the μ-stabilizer is infinite. Let $\mathbf{X}$ be the Zariski closure of p in $\mathbf{G}$ and let a be a point realizing p. Assume, to the contrary, that $\mathrm{Stab}^{\mu}(p)$ is finite. Then $\mathrm{res}\left(\mu \cdot p \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right)$ is finite by Proposition 2.13. Note that $\mu \cdot \mathbf{X} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O}) = \bigcup_q \left(\mu \cdot q \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right)$, where q ranges over types centered at infinity on $\mathbf{X}$. By Corollary 3.3, there are only finitely many types centered at infinity on $\mathbf{X}$, so the set $\mathbf{X} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ is the intersection of $\mathbf{X} \cdot a^{-1}$ with a (disjoint) union of finitely many balls $\mu \cdot g$, for $g \in \overline{\mathbf{G}}(\Bbbk)$.

Therefore, $\mathbf{X} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ is a nonempty $v{+}g$-open subset of $\mathbf{X} \cdot a^{-1}$. However, it is also a $v{+}g$-closed subset of $\mathbf{X} \cdot a^{-1}$, since $\mathbf{G}(\mathcal{O})$ is $v{+}g$-closed. By Proposition 3.9, $\mathbf{X} \cdot a^{-1}$ is definably connected, so $\mathbf{X} \cdot a^{-1} \subseteq \mathbf{G}(\mathcal{O}) \subseteq \mathbb{A}^n$. However, this is impossible, since it implies that $\mathbf{X} \cdot a^{-1}$ as an affine curve is bounded in $\mathbb{A}^n$, contradicting Theorem 3.11. □

Remark 3.13 (Relation to o-minimal PS-subgroups). Recall that in the o-minimal context, for each definable group $\mathbf{G}$ and an unbounded semi-algebraic curve $\gamma : (a,b) \to \mathbf{G}$, we use H_γ to denote the PS-subgroup of γ (or in other words, the o-minimal μ-stabilizer of the type of γ at b).

In the case when $\Bbbk$ is $\mathbb{C}$, the μ-stabilizers of a point at infinity in Theorem 3.12 are closely related to the group G_γ, the Zariski closure of H_γ, as described in the construction in the beginning of the introduction (both viewed as definable in $\mathcal{R}$).

Namely, assume we are given a complex curve $\mathbf{X}$ embedded in the complex affine algebraic group $\mathbf{G}$, and a point $\alpha \in \tilde{\mathbf{X}} \setminus \mathbf{X}$. Let $\gamma : (a,b) \to \mathbf{X}$ be a semialgebraic curve over $\mathbb{R}$ whose limit at b is α (in the sense discussed in the introduction).

Let $\mathcal{R}$ be a sufficiently saturated real closed field extending $\mathbb{R}$, and let $\mathcal{C}$ be $\mathcal{R}^2$, viewed as an algebraic closure of $\mathcal{R}$. We may view $\mathbf{X}$, $\mathbf{G}$, and γ as definable in $\mathcal{R}$. By [7], one can compute the PS-subgroup H_γ associated to γ as the μ-stabilizer of $tp_{sa}(\alpha/\mathbb{R})$, the type of α in the theory of real closed fields.

Let $\mathcal{O}_R$ be the convex hull of $\mathbb{R}$ in $\mathcal{R}$, and let $\mathcal{O} = {\mathcal{O}_R}^2$, viewed as a subring of $\mathcal{C}$. Then $\mathcal{C}$ equipped with $\mathcal{O}$ as a valuation ring is a model of T_{loc}, and the type $q = tp_{\mathrm{ACVF}}(\alpha/\mathbb{C})$ is contained in $p = tp_{sa}(\alpha/\mathbb{R})$ because the maximal ideal can be viewed as a partial type. Hence $H_\gamma \subseteq \mathrm{Stab}^{\mu}(q)$. However, $\mathrm{Stab}^{\mu}(q)$ is 1-dimensional and H_γ is infinite, so $G_\gamma \subseteq \mathrm{Stab}^{\mu}(q)$ and the index is finite.

4. Proof of the main theorem

4.1. Residually algebraic saturation

We would like to work with saturation in a residually algebraic context, that is to say, "saturation" without extending the residue field. Thus we make the following definition:

Definition 4.1. A model K of T_{loc} is (sufficiently) Γ *-saturated* if every $\mathcal{L}_{\mathrm{val}}$-residually algebraic type over a (sufficiently) small subset of K is realised in K.

Theorem 4.2. *Let L be a (possibly trivially) valued field. Then there is a Γ-saturated extension of L.*

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Proof. Let Γ be a sufficiently saturated ordered abelian group and k the algebraically closed closure of $\mathbf{RES}(L)$. Consider the Hahn series field

$$k\left(\left(t^{\Gamma}\right)\right)=\left\{\sum_{\gamma} c_{\gamma} t^{\gamma} : c_{\gamma}\in k, \{\gamma : c_{\gamma}\neq 0\} \text{ is well ordered}\right\}.$$

Clearly L embeds into K (see [6], for example). Then, by a result of Poonen ([10, Theorem 2]), $k\left(\left(t^{\Gamma}\right)\right)$ is a Γ-saturated model (with residue field k). □

From now on, K will be a fixed, sufficiently Γ-saturated model K with residue field $\Bbbk$, and we will identify definable sets and $p\in S_G(\Bbbk)$ with their realizations in K. In particular, we will only consider residually algebraic types, unless otherwise stated.

As a first application, we note the following:

Lemma 4.3. *Let $p\in S_G(\Bbbk)$ be residually algebraic. Then there is $q\in p_{\mu}$, which is μ-reduced and residually algebraic.*

Proof. Let a be a realization of p in K. There is a variety $\mathbf{V}$ over $\Bbbk$ of minimal dimension that intersects $\mu\cdot a$. This can be expressed as an $\mathcal{L}_{\text{val}}$-formula, so it is witnessed by some element of K. Take q to be the $\mathcal{L}_{\text{val}}$-type of this element over $\Bbbk$. □

We would like to give a syntactic (or geometric) description of types realised in K. To this end, we need the following. It is a part of Lemma 9.1.1 in [4]. We will only state what is needed in the proof.

Lemma 4.4. *(Hrushovski and Loeser) Let F be a valued field, $\mathbf{V}$ be an F-variety, and $\mathbf{X}\subseteq\mathbf{V}$ be an F-definable g-open set. Then $\mathbf{X}(M_2)\subseteq\mathbf{X}(M_1)$ whenever M_1 and M_2 are algebraically closed valued field extensions of F with the same underlying field, and $\mathcal{O}_{M_1}\subseteq\mathcal{O}_{M_2}$.*

We now have the following description:

Proposition 4.5. *Let $\Phi(x)$ be a small finitely consistent collection of g-open sets, with parameters in $L\subseteq K$. Then Φ is realised in K. In addition, if p is an $\mathcal{L}_{\text{val}}$-residually algebraic type, then it is the intersection of the g-open formulas that it implies.*

In other words, every partial type Σ of g-open sets admits an extension to an $\mathcal{L}_{\text{val}}$-residually algebraic complete type p over the same set of parameters.

Remark 4.6. It is worth pointing out that Proposition 4.5 has an easy proof in the case when $L=\Bbbk=\mathbb{C}$ [8, Section 3.1].

Proof. Let b be any realisation of Φ in $\mathbb{U}$, and let k be the residue field of $L(b)$. Then k is the function field of some variety $\mathbf{X}$ over $\Bbbk$. Fix a valuation val' of k over $\Bbbk$, with residue field $\Bbbk$.

Let M_2 be the algebraic closure of $L(b)$ with the induced valuation from $\mathbb{U}$. Consider a valuation of $\mathbf{RES}(M_2)$ extending val'. Abusing notation, we call the valuation val' as well. Let $\bar{\mathcal{O}}$ be the valuation ring of $\mathbf{RES}(M_2)$. Consider $\text{res}^{-1}(\bar{\mathcal{O}})\subseteq M_2$; this is again a valuation ring of the underlying field of M_2 over $\Bbbk$. We use M_1 to denote the same field as M_2 with the valuation determined by $\mathrm{res}^{-1}(\bar{\mathcal{O}})$. Note that M_1 has residue field $\Bbbk$.

Then by Lemma 4.4, $\phi(M_2) \subseteq \phi(M_1)$ for each $\phi \in \Phi$. In particular, b is a realization of Φ in M_1. But the residue field of M_1 is $\Bbbk$, so $tp_{M_1}(b/L)$ is residually algebraic and hence realizable in K.

For the converse, let p be a complete $\mathcal{L}_{\mathrm{val}}$-residually algebraic type. By quantifier elimination in ACVF, it is given within its Zariski closure by formulas of the form $f(x) \neq 0$, $\mathrm{val}(f(x)) > \mathrm{val}(g(x))$, and $\mathrm{val}(f(x)) = \mathrm{val}(g(x)) \neq \infty$. Each formula of the last form is equivalent to $\mathrm{val}(f(x)/g(x)) = 0$, so that $f(x)/g(x)$ has nonzero residue. Since p is residually algebraic, the residue is actually a well-determined element b of $\Bbbk$, so the original formula is implied by $\mathrm{val}(f(x) - bg(x)) > \mathrm{val}(bg(x))$, which is also in p. □

We now apply this result in our context:

Corollary 4.7. *Set $p \in S_G(\Bbbk)$ and let if be residually algebraic. Then $\mu(K)\cdot p(K) = (\mu\cdot p)(K)$.*

Proof. Since K is contained in the monster model, $\mu(K)\cdot p(K) \subseteq (\mu\cdot p)(K)$. For the reverse containment, for p residually algebraic, fix $a \in (\mu\cdot p)(K)$. Recall that it means that for any $\phi \in p$, there is $\varepsilon_\phi \in \mu$ such that $\models \phi(\varepsilon_\phi \cdot a)$. Since p is residually algebraic, we may, by Proposition 4.5, assume that each such ϕ is g-open.

Consider the following partial type: $\Sigma(y) = \{\phi(y\cdot a) \wedge \mu(y) : \phi \in p \ g\text{-open}\}$. Each ϕ there is g-open, hence also $\phi(y\cdot a)$ (since the group is algebraic), and μ is given by strict inequalities, so this is a small collection of g-open sets, consistent by assumption. By the other direction of Proposition 4.5, we can find $\varepsilon \in \mu(K)$ such that $\varepsilon\cdot a$ satisfies p. □

Corollary 4.8. *Let p be a residually algebraic G-type over $\Bbbk$, and let a be a realization in K. Then $\mathrm{Stab}^\mu(p)(\Bbbk) = \mathrm{res}\left(\mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right)$.*

Proof. Since $\mathrm{res}\left(\mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right) \subseteq \mathrm{res}\left(\mu(\mathbb{U})\cdot p(\mathbb{U})\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right)$, we have $\mathrm{Stab}^\mu(p)(\Bbbk) \supseteq \mu(K)\cdot p(K)\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ by Corollary 2.14. The reverse containment follows from Corollary 4.7. □

4.2. μ-Reduced types and their stabilizers

In this section we prove Corollary 4.11, an analogue of Corollary 3.3 for types of higher dimension.

Recall that we are working within K, a Γ-saturated model, and all the elements in the statement are from K, and definable sets are identified with their realization in K.

In particular, we have the following:

Lemma 4.9. *If a is μ-reduced and $g \in \mathbf{G}(\mathcal{O})$, then $g\cdot a$ is also μ-reduced, of the same dimension.*

Proof. Assume $\varepsilon\cdot g\cdot a \in \mathbf{W}$ with $\varepsilon \in \mu$ and $\mathbf{W}$ a variety over $\Bbbk$. Since $\varepsilon\cdot g\cdot a = \bar{g}\cdot\varepsilon'\cdot a$ for some $\bar{g} \in \overline{\mathbf{G}}(\Bbbk)$ and $\varepsilon' \in \mu$, we have $\varepsilon'\cdot a \in \bar{g}^{-1}\cdot\mathbf{W}$, a variety over $\Bbbk$ of the same dimension as $\mathbf{W}$. □

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The following is an important observation about μ-reduced types:

Proposition 4.10. *Let $p \in S_G(\mathbb{k})$ be a μ-reduced residually algebraic type centered at infinity, and let $a \models p$. Let $\mathbf{V}$ be the unique irreducible $\mathbb{k}$-variety such that $a \in \mathbf{V}$ and $\dim(\mathbf{V}) = \dim(p)$. For any definable set $\mathbf{X}$, assume that $\mathbf{X} \subseteq \mathbf{G}(\mathcal{O})\cdot a \cap \mathbf{V}$ is definably connected and $a \in \mathbf{X}$. Then for every $b \in \mathbf{X}$ we have $tp(a/\mathbb{k}) = tp(b/\mathbb{k})$, where $tp(\cdot/\mathbb{k})$ denotes the $\mathcal{L}_{\mathrm{val}}$-type over $\mathbb{k}$.*

Proof of Proposition 4.10. By Lemma 4.9, b is not contained in any proper subvariety of $\mathbf{V}$, so it is nonzero when evaluated by any regular function on $\mathbf{V}$. Hence every element of the function field $\mathbb{k}(\mathbf{V})$ is well defined as a $\mathbb{k}$-definable function on $\mathbf{X}$.

Assume that the types of a and b are different. By quantifier elimination in ACVF, without loss of generality, there is $f \in \mathbb{k}(\mathbf{V})$ such that $\mathrm{val}(f(a)) < 0 \leq \mathrm{val}(f(b))$. We may further assume that the last inequality is strict, by subtracting the residue.

By [4], it can be easily checked that rational functions are $v{+}g$-continuous on their domain, so the image $f(\mathbf{X})$ is again definably connected. As a definable subset of K, it is a union of 'Swiss cheeses', and by definable connectedness, the Swiss-cheese decomposition of the image will be of the form $\mathbf{B} \setminus \cup_{i \leq m} \mathbf{C}_i$, where $\mathbf{B}$ is a ball and $\mathbf{C}_i$'s are disjoint subballs of $\mathbf{B}$.

Claim. $f(\mathbf{X})$ contains a $\mathbb{k}$-point.

Proof of claim. Since $\mathbf{B}$ contains both a point with positive valuation and a point with valuation ≤ 0, then it must contain $\mathcal{O}$. If $f(\mathbf{X})$ contains no $\mathbb{k}$-point, $\mathbb{k}$ must be covered by $\bigcup_{i \leq m} \mathbf{C}_i$. This implies that one of the $\mathbf{C}_i$'s contains at least two points in $\mathbb{k}$ and hence contains $\mathcal{O}$. But this is a contradiction, since it means that there is no point in $f(\mathbf{X})$ with positive valuation. □

Hence, we know that there must be some $c \in \mathbb{k}$ such that $c \in f(\mathbf{X})$. Note, however, that each element in $\mathbf{X}$ is a generic point of $\mathbf{V}$, by Lemma 4.9, and we know that this would imply that the rational function f is constant – a contradiction to the assumption. Hence we know that $tp(a/\mathbb{k}) = tp(b/\mathbb{k})$. □

Using a similar argument, we have the following, which is the key fact that will replace Corollary 3.3 for our proof of the main theorem:

Corollary 4.11. *Let $a \in \mathbf{V}(K)$ be μ-reduced, with $\mathbf{V}$ the Zariski closure of a over $\mathbb{k}$. Then there are finitely many types $p_1,\ldots,p_m \in S_G(\mathbb{k})$ for some m such that if $g \in \mathbf{G}(\mathcal{O})$ and $g\cdot a \in \mathbf{V}$, then $tp(g\cdot a/\mathbb{k}) = p_i$ for some i.*

Proof. From Theorem 3.10, we know that there are only finitely many definably connected components of the set $\mathbf{G}(\mathcal{O})\cdot a \cap \mathbf{V}$ – call them $\mathbf{X}_i$ for $i = 1,\ldots,n$ for some n. By Proposition 4.10, for each $b,b' \in \mathbf{X}_i$ we have $tp(b/\mathbb{k}) = tp(b'/\mathbb{k})$. Hence there are only finitely many types p_i with the property in the statement of the corollary. □

Here, we state a variant that is similar to [7]:

Corollary 4.12. *In the same setting as before, there is an $\mathcal{L}_{\mathrm{val}}$-definable set $\mathbf{X}$ over $\mathbb{k}$ containing a, such that for each $b \in \mathbf{G}(\mathcal{O})\cdot a \cap \mathbf{X}$, $tp(b/\mathbb{k}) = tp(a/\mathbb{k})$. Furthermore, $\mathbf{X}$ is $v{+}g$-open and $\mathbf{X}\cdot a^{-1}$ is $v{+}g$-clopen in $\mathbf{G}(\mathcal{O})$.*

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Proof. We see that there are finitely many regular functions f_{ij}, g_{ij} such that for the formula $\mathrm{val}(f_{ij}) < \mathrm{val}(g_{ij})$, p_i and p_j disagree. The set defined by some Boolean combinations of these formulas containing $tp(a/\mathbb{k})$ will define the set $\mathbf{X}$. □

In particular, we have the following:

Corollary 4.13. $\mathrm{Stab}^{\mu}(p)(\mathbb{k})$ *is infinite for each* p *residually algebraic and centered at infinity.*

Proof. Without loss of generality, we can assume that p is μ-reduced and let $a \models p$ be any realization and $\mathbf{V}$ denote its Zariski closure. We have $\dim \mathbf{V} > 0$, since p is centered at infinity. Also, $\mathbf{V}$ is an irreducible $\mathbb{k}$-variety, hence $v{+}g$-connected, and so is $\mathbf{V}\cdot a^{-1}$. By Proposition 2.13, if $\mathrm{Stab}^{\mu}(p)(\mathbb{k}) = \overline{\mathbf{G}}_p(\mathbb{k})$ is finite, $\mathbf{V}\cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ can be covered by finitely many $v{+}g$-open sets, and hence is $v{+}g$-open. But $\mathbf{V}\cdot a^{-1} \setminus \mathbf{G}(\mathcal{O})$ is also $v{+}g$-open by definition, a contradiction of the fact that nonzero-dimensional affine varieties are not bounded in the affine space. □

It is worth noting that this proof uses the same idea in the 1-dimensional case, where the key ingredient is the connectedness of irreducible varieties.

4.3. Dimension of the μ-stabilizers

Corollary 4.13 forms the first part of the main theorem. Before proving the other part, we need some machinery about varieties over $\mathcal{O}$. The main facts can be found in [5].

Definition 4.14. Let $\mathcal{O}$ be a valuation ring, $L = \mathrm{Frac}(\mathcal{O})$, and $k = \mathrm{res}(\mathcal{O})$. By a *variety over* $\mathcal{O}$, we mean a flat reduced scheme $\mathcal{V}$ of finite type over $\mathcal{O}$. In particular, it has a generic fiber $\mathbf{V}_L$, which is a variety over L, obtained by base change with respect to the morphism $\mathcal{O} \to L$, and a special fiber $\mathbf{V}_k$, which is a variety over k, obtained by base change with respect to $\mathcal{O} \to k$.

Remark 4.15. It is worth noting that since $\mathcal{O}$ is a valuation ring, $A = \mathcal{O}[x_1,\ldots,x_n]/I$ is flat over $\mathcal{O}$ if and only if no nonzero element in $\mathcal{O}$ is a zero divisor in A.

In particular, if S is any subset of $\mathcal{O}^n$, then

$$I = \{f \in \mathcal{O}[x_1,\ldots,x_n] : f(s) = 0 \ \forall s \in S\} \subseteq \mathcal{O}[\bar{x}]$$

is an ideal and $A = \mathcal{O}[x_1,\ldots,x_n]/I$ is flat over $\mathcal{O}$.

We use I_L and I_k to denote the ideals generated by I in $A \otimes L$ and in $A \otimes k$, respectively. Then the generic (resp., special) fiber of $\mathrm{Spec}(A)$ is $\mathrm{Spec}(A \otimes L/I_L)$ (resp., $\mathrm{Spec}(A \otimes k/I_k)$). Given an affine variety $\mathbf{V}$ over L, we may always choose a variety $\mathcal{V}$ affine over $\mathcal{O}$ whose generic fiber is $\mathbf{V}$.

The following is [2, Theorem 3.2.4]:

Theorem 4.16 (Halevi). *Let* K *be a model of* $ACVF$ *and* $\mathcal{V}$ *be an irreducible variety over* $\mathcal{O}_K$. *If* $\mathbf{V}_K$ *has an* $\mathcal{O}_K$*-point, then the* $\mathcal{O}_K$*-points are Zariski dense, and the canonical map* $\mathrm{res} : \mathbf{V}_K(\mathcal{O}_K) \to \mathbf{V}_k(k)$ *is surjective, where* res *is given by taking residue pointwise.*

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Now let us get back to the proof of the main theorem, to prove the part concerning the dimension.

Proof of Theorem 1.2. The first part of the theorem is the statement of Corollary 4.13, so it remains to show the equality of dimensions when the type p is μ-reduced. Under this assumption, let a realize p and let $\mathbf{V}$ be its Zariski closure. In particular, $\mathbf{V}$ is irreducible.

We first note that if $g \in \mathbf{G}(\mathcal{O})$ and $g \cdot a \in \mathbf{V}$, then $tp(g \cdot a/\Bbbk)$ will be one of the finitely many types p_i provided by Corollary 4.11. Hence $\mu \cdot \mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$ will be a finite union of cosets of $\mu \cdot \mathbf{G}_p$. Thus it suffices to show that $\operatorname{res}\left(\mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})\right)$ and $\mathbf{V}$ have the same dimension.

As before, the affine K-variety $\mathbf{V} \cdot a^{-1}$ can be viewed as the generic fiber of a variety $\mathcal{V}$ over $\mathcal{O}_K$. Furthermore, $\mathcal{V}$ has an $\mathcal{O}_K$-point, namely e, the identity of the group G. It follows from Theorem 4.16 that the map $\operatorname{res} : \mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O}) \to \overline{\mathbf{G}}$ is onto the special fiber of $\mathbf{V} \cdot a^{-1}$. Also, by flatness, the special fiber has the same dimension as the generic fiber, which is the dimension of p.

This completes the proof of the dimension part in the main result. The solvability of the group is proved separately, as Theorem 4.20. □

The proof also shows that the special fiber, being the image of $\mathbf{V} \cdot a^{-1} \cap \mathbf{G}(\mathcal{O})$, is a finite union of cosets of $\operatorname{Stab}^{\mu}(p)$. Therefore, we have established the following:

Corollary 4.17. *Let $\mathbf{V} \subseteq \mathbf{G}$ be a variety over $\Bbbk$ and let $a \models p$, where $p \in S_G(\Bbbk)$ is a μ-reduced residually algebraic type centered at infinity. Assume further that the Zariski closure of p over $\Bbbk$ is $\mathbf{V}$.*

Then the special fiber of $\mathbf{V} \cdot a^{-1}$ is equidimensional–that is, each irreducible component of it has the same dimension. Moreover, each irreducible component of the special fiber of $\mathbf{V} \cdot a^{-1}$ is a coset of an algebraic subgroup of $\overline{\mathbf{G}}$.

4.4. Structure of $\mathrm{Stab}^{\mu}(p)$

In this section, we analyze the structure of $\operatorname{Stab}^{\mu}(p)$. Note that due to trivial constraints on characteristic, it is not possible to show in general that such a group is torsion-free. However, in characteristic 0, we can indeed show it is torsion-free.

Lemma 4.18. *Let $p \in S_G(\Bbbk)$ be residually algebraic and let H be a $\Bbbk$-definable linear subgroup of G with $p \in H$. Then $\operatorname{Stab}^{\mu}(p)$ computed in $\mathbf{G}$ and in $\mathbf{H}$ coincide, where $\mathbf{H}$ denotes the group H viewed as a subset in $\mathbf{VF}$.*

Proof. The Zariski closure $\mathbf{V}$ of p is contained in $\mathbf{H}$ in this case, since $\mathbf{H}$ is a Zariski closed subgroup and $\mu_G \cap \mathbf{H} = \mu_H$. Hence, the arguments of computing the μ-stabilizers of p can be carried out in both $\mathbf{H}$ and $\mathbf{G}$, and the results will be the same. □

The following is the Iwasawa decomposition over non-Archimedean fields, which can be found in [1, Proposition 4.5.2]:

Theorem 4.19. *Let $\mathbf{G}$ be a reductive linear algebraic group over $\Bbbk$. There is a solvable subgroup H over $\Bbbk$ such that $\mathbf{G}(K) = \mathbf{G}(\mathcal{O}) \cdot \mathbf{H}(K)$.*

For GL_n we may take H to be the standard Borel subgroup (upper triangular matrices).

Theorem 4.20. *Let $p \in S_G(\Bbbk)$ be centered at infinity and residually algebraic. Then* $\mathrm{Stab}^{\mu}(p)$ *is solvable.*

Proof. We can embed $\mathbf{G} \subseteq \mathrm{GL}_n$ over $\Bbbk$ for some n, and use Lemma 4.18 to reduce to the case $\mathbf{G} = \mathrm{GL}_n$. Let $\mathbf{H}$ be the Borel. By the Iwasawa decomposition, we have some $g \in \mathbf{G}(\mathcal{O})$ such that $g^{-1} \cdot a = \beta \in \mathbf{H}(K)$. Let $g_1 \in \overline{\mathbf{G}}(\Bbbk)$ be such that $g_1 \cdot g^{-1} \in \mu$. Hence $g_1^{-1} \cdot a \in \mu(K) \cdot \beta$, so $\mathrm{Stab}^{\mu}\left(g_1^{-1} \cdot p\right) = \mathrm{Stab}^{\mu}(q) \subseteq \overline{\mathbf{H}}$. By Lemma 2.9, this group is conjugate to $\mathrm{Stab}^{\mu}(p)$, and hence $\mathrm{Stab}^{\mu}(p)$ is solvable. □

Corollary 4.21. *If G is not solvable and is irreducible, then there is no μ-reduced residually algebraic G-type of full dimension.*

Remark. We briefly introduce the Zariski–Riemann space of a variety over k, and explain its connection with our setting.

Definition. Let V be a variety over k. The Zariski–Riemann space of V over k is the set of valuation rings of $k(V)$ over k, denoted by $\mathbf{RZ}_k(V)$.

Note that by quantifier elimination in ACVF, for a linear algebraic group $\mathbf{G}$ over $\Bbbk$ it is not hard to see that the set $\mathbf{RZ}_{\Bbbk}(\mathbf{G})$ is canonically embeddable into the set $S_G(\Bbbk)$. Hence we can identify $\mathbf{RZ}_{\Bbbk}(\mathbf{G})$ with its image in $S_G(\Bbbk)$. Note further that since μ is Zariski dense in $\mathbf{G}$, we see that for each $p \in S_G(\Bbbk)$ there is some $q \in \mathbf{RZ}_{\Bbbk}(\mathbf{G})$ such that $p \sim_{\mu} q$.

This argument implies that we can consider the quotient of $\mathbf{RZ}_{\Bbbk}(\mathbf{G})$ under μ, which is exactly the space $S_G^{\mu}(\Bbbk)$. Note further that the equivalence relation induced by μ on $\mathbf{RZ}_{\Bbbk}(\mathbf{G})$ is independent of the $\Bbbk$-closed immersion of $\mathbf{G}$ into $\mathbb{A}^n$, since every embedding over $\Bbbk$ will respect μ. We will explore more on the relation induced by μ on $\mathbf{RZ}_{\Bbbk}(\mathbf{G})$ in a subsequent paper.

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