LATTICE PATHS IN E^3 WITH DIAGONAL STEPS

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(received May 2, 1967)

Moser and Zayachkowski [1] have discussed certain planar lattice paths from $(0, 0)$ to (m, n). In this note we consider analogous paths in three dimensional space. For basic definitions see reference [2]. Throughout this note each of m, n and k is a positive integer and if S is a finite set, $|S|$ will denote the number of elements of S.

Each path under consideration here is such that each of its steps is of one of the following types:

1) x-increasing only, e.g. $[(m, n, k), (m+1, n, k)]$,
2) y-increasing only, e.g. $[(m, n, k), (m, n+1, k)]$,
3) z-increasing only, e.g. $[(m, n, k), (m, n, k+1)]$,
4) xz-diagonal, e.g. $[(m, n, k), (m+1, n, k+1)]$,
5) xy-diagonal, e.g. $[(m, n, k), (m+1, n+1, k)]$,
6) yz-diagonal, e.g. $[(m, n, k), (m, n+1, k+1)]$,
7) cube diagonal, e.g. $[m, n, k), (m+1, n+1, k+1)]$.

We will initially consider paths each step of which is one of the first six types.

Each path from $(0, 0, 0)$ to (m, n, k) which contains no diagonal steps will contain a total of $m+n+k$ steps. Since the order in which these steps occur in the path is unrestricted, the number of such paths is $(m+n+k)!/m!n!k!$. If, in addition to steps of the first three types, a path were to contain r_1 steps of type 4 and no steps of any other type, then that path would contain only $(m-r_1)$ steps of type 1 and only $(k-r_1)$ steps of type 3. It would, of course, still contain n steps of type 2 and the total number of steps in each such path would be $m+n+k-r_1$. The total number of such paths is

$$(m+n+k-r_1)!/(m-r_1)!\,n!\,(k-r_1)!\,r_1!$$

Canad. Math. Bull. vol. 10, no. 5, 1967

https://doi.org/10.4153/CMB-1967-064-2 Published online by Cambridge University Press

Summing the above on r_1 we obtain the total number of paths from $(0, 0, 0)$ to (m, n, k) which contain only steps of the types 1, 2, 3 and 4. This number is

$$\sum_{r_1=0}^{\min(m,k)} \frac{(n+m+k-r_1)!}{n!\, r_1!\, (m-r_1)!\, (k-r_1)!}$$

Similarly, it is shown that the number of paths from $(0, 0, 0)$ to (m, n, k) which contain steps of the first five types only is

$$\sum_{r_1=0}^{\min(m,k)} \left[\sum_{r_2=0}^{\min(n,m-r_1)} \frac{(m+n+k-r_1-r_2)!}{(m-r_1-r_2)!\,(n-r_2)!\,(k-r_1)!\, r_1!\, r_2!} \right].$$

Finally, we see that the number of paths from $(0, 0, 0)$ to (m, n, k) which contain steps of the first six types only is

$$\sum_{r_1=0}^{\min(m,k)} \left\{ \sum_{r_2=0}^{\min(n,m-r_1)} \left[\sum_{r_3=0}^{\min(k-r_1,n-r_2)} \frac{(m+n+k-r_1-r_2-r_3)!}{(m-r_1-r_2)!\,(n-r_2-r_3)!\,(k-r_1-r_3)!\; r_1!\, r_2!\, r_3!} \right] \right\}.$$

Consider now a path from $(0, 0, 0)$ to (m, n, k) each step of which is either of type 1, 2, 3, or 7. The total number of such paths is easily seen to be

$$\sum_{r=0}^{\min(m,n,k)} \frac{(m+n+k-2r)!}{(m-r)!\,(n-r)!\,(k-r)!\, r!}$$

and will be denoted by $f(m, n, k)$. Note that

$$f(n, n, n) = \sum_{r=0}^{n} \frac{(3n-2r)!}{((n-r)!)^3\, r!}$$

In the remainder of this note the term diagonal step, or simply diagonal, will mean cube diagonal, i.e. type 7.

Let $Q(n)$ denote the set such that p belongs to $Q(n)$ if and only if

1) p is a path from $(0, 0, 0)$ to (n, n, n)

2) except for the points $(0, 0, 0)$ and (n, n, n), only points on the $(n, 0, 0)$ side of the plane $y = x$ are terms of p, and

3) each step of p is one of the types 1, 2, 3, or 7.

Note that if a path p belongs to $Q(n)$, then the initial step of p is $[(0, 0, 0), (1, 0, 0)]$. Hence, $|Q(n)|$ is not greater than the number of paths from $(1, 0, 0)$ to (n, n, n), which is the same as the number of paths from $(0, 0, 0)$ to $(n-1, n, n)$, that is, $f(n-1, n, n)$. Because a path belongs to $Q(n)$ only if it contains as a term no point of the plane $y = x$ other than $(0, 0, 0)$ and (n, n, n), it is noted that no path of $Q(n)$ terminates with the step $[(n-1, n-1, n-1), (n, n, n)]$. Hence, from the above upper bound on $|Q(n)|$ we may subtract the number of paths from $(1, 0, 0)$ to $(n-1, n-1, n-1)$, which is $f(n-2, n-1, n-1)$. Likewise, no path of $Q(n)$ terminates with $[(n, n, n-1), (n, n, n)]$. The number of paths from $(1, 0, 0)$ to $(n, n, n-1)$ is $f(n-1, n, n-1)$.

It is also noted that since, with the exception of the end points, no path of $Q(n)$ contains as a term a point on the non-$(n, 0, 0)$ side of the plane $y = x$, no path of $Q(n)$ terminates with the step $[(n-1, n, n), (n, n, n)]$. The number of paths from $(1, 0, 0)$ to $(n-1, n, n)$ is $f(n-2, n, n)$. Using the reflection device which Moser and Zayachkowski [1] attribute to D. André, we note that in the set of all paths from $(0, 0, 0)$ to (n, n, n) which begin with the step $[(0, 0, 0), (1, 0, 0)]$, there is a one to one correspondence between (1) the set of all paths which terminate with $[(n, n-1, n), (n, n, n)]$ and contain as a term a point of the plane $y = x$ other than $(0, 0, 0)$ and (n, n, n), and (2) the set of all paths which terminate with $[(n-1, n, n), (n, n, n)]$. Thus, we see that there are $f(n-2, n, n)$ paths which begin with $[(0, 0, 0), (1, 0, 0)]$ and terminate with $[(n, n-1, n), (n, n, n)]$ and contain as a term a point of the diagonal plane $y = x$ other than $(0, 0, 0)$ and (n, n, n). However, each path in $Q(n)$ is a path from $(0, 0, 0)$ to (n, n, n) whose initial step is $[(0, 0, 0), (1, 0, 0)]$. Hence, $|Q(n)| = f(n, n, n-1) - 2f(n, n, n-2) - f(n-1, n-1, n) - f(n-1, n-1, n-2)$; that is,

$$|Q(n)| = \sum_{r=0}^{n-1} \frac{(3n-2r-1)!}{((n-r)!)^2 (n-r-1)!\, r!}$$

https://doi.org/10.4153/CMB-1967-064-2 Published online by Cambridge University Press

$$- 2 \sum_{r=0}^{n-2} \frac{(3n-2r-2)!}{((n-r)!)^2 (n-r-2)!\, r!}$$

$$- \sum_{r=0}^{n-1} \frac{(3n-2r-2)!}{(n-r)!\, ((n-r-1)!)^2\, r!}$$

$$- \sum_{r=0}^{n-2} \frac{(3n-2r-4)!}{((n-r-1)!)^2 (n-r-2)!\, r!}$$

and after simplifying the expression for $|Q(n)|$, we have the following.

THEOREM. If $Q(n)$ is the set of all lattice paths such that p belongs to $Q(n)$ if and only if (1) p is a path from $(0, 0, 0)$ to (n, n, n), (2) except for the points $(0, 0, 0)$ and (n, n, n), p has as its terms only points on the $(n, 0, 0)$ side of the diagonal plane $y = x$, and (3) each step of p is either (a) x-increasing only, (b) y-increasing only, (c) z-increasing only, or (d) a cube diagonal, then the number $|Q(n)|$ of paths belonging to $Q(n)$ is

$$n^2 + \sum_{r=0}^{n-2} \frac{(r+1)(3n-2r-2)! - (n-r-1)(n-r)^2(3n-2r-4)!}{(n-r)^2((n-r-1)!)^3\, r!} .$$

Note that the restriction that each term of each path of the set $Q(n)$, with the exception of the end points, be a point of the $(n, 0, 0)$ side of the main diagonal plane $y = x$, implies that, with the exception of $(0, 0, 0)$ and (n, n, n), no point of the main diagonal is a term of any path in the collection $Q(n)$. Thus the paths of the set $Q(n)$ are analogous to planar β-paths [1].

An analysis similar to the above suffices to prove the following.

THEOREM. If $R(n)$ is the set of lattice paths such that p belongs to $R(n)$ if and only if (1) p is a path from $(0, 0, 0)$ to (n, n, n), (2) except for the point $(0, 0, 0)$ and points of the line $x = y = n$, p contains as terms only points on the $(n, n, 0)$ side of the plane $y = x$, and (3) each step of p is either (a) x-increasing only, (b) y-increasing only, (c) z-increasing only, or (d) a cube diagonal, then the number $|R(n)|$ of paths belonging to $R(n)$ is

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$$f(n-1, n, n) - \left\{ 2 \sum_{t=0}^{n} f(n-2, n, n-t) + \sum_{t=1}^{n} f(n-2, n-1, n-t) \right\} .$$

In the plane an α-path is defined to be a path which contains only points on or below the main diagonal [1]. We will describe paths in three dimensional space which are analogous to α-paths.

Let $Q'(n)$ denote the set of lattice paths such that p belongs to it if and only if (1) p is a path from $(0, 0, 0)$ to (n, n, n), (2) no term of p is a point of the non-$(n, 0, 0)$ side of the plane $y = x$, and (3) each step of p is either (a) x-increasing only, (b) y-increasing only, (c) z-increasing only, or (d) a cube diagonal. The number $|Q'(n)|$ is the same as the number of paths from $(0, 0, 0)$ to $(n+1, n+1, n)$, which, except for the end points, contain only points on the $(n, 0, 0)$ side of the plane $y = x$. The steps of the paths are restricted to be of types 1, 2, 3, and 7.

We have shown that

$$|Q(n)| = f(n, n, n-1) - 2f(n, n, n-2) - f(n-1, n-1, n) - f(n-1, n-1, n-2),$$

or

$$|Q(n)| = f(n-1, n, n) - f(n-2, n-1, n-1) - f(n-1, n, n-1) - 2f(n-2, n, n).$$

From this and the above statement concerning the number $|Q'(n)|$ we see that

$$|Q'(n)| = f(n, n+1, n) - f(n-1, n, n-1) - f(n, n+1, n-1) - 2f(n-1, n+1, n),$$

or

$$|Q'(n)| = f(n, n, n+1) - f(n, n-1, n-1) - 3f(n, n-1, n+1).$$

Using the definition of $f(m, n, k)$ we see that

$$|Q'(n)| = \sum_{r=0}^{n} \frac{(3n-2r+1)!}{(n-r)!\,(n-r)!\,(n-r+1)!\,r!} - \sum_{r=0}^{n-1} \frac{(3n-2r-2)!}{(n-r)!\,((n-r-1)!)^2\,r!}$$

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$$- 3 \sum_{r=0}^{n-1} \frac{(3n-2r)!}{(n-r)!(n-r-1)!(n-r+1)!r!}$$

Simplification of the above yields the

THEOREM: If $|Q'(n)|$ is the number of paths in the set $Q'(n)$, then

$$|Q'(n)| = (n+1) + \sum_{r=0}^{n-1} \frac{(3n-2r)!(r+1)}{(n-r-1)!(n-r)!(n-r+1)!r!(n-r)}$$

$$- \sum_{r=0}^{n-1} \frac{(3n-2r-2)!}{((n-r-1)!)^3 r!(n-r)}$$

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This work was supported in part by a grant of the Coordinating Board, Texas College and University System.

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https://doi.org/10.4153/CMB-1967-064-2 Published online by Cambridge University Press