Bull. Aust. Math. Soc. **106** (2022), 177–184
doi:10.1017/S0004972721001271

ABELIAN DIFFERENCE SETS AS LATTICE COVERINGS AND LATTICE TILINGS

MLADEN KOVAČEVIĆ

(Received 17 November 2021; accepted 26 November 2021; first published online 24 January 2022)

Abstract

We demonstrate that every difference set in a finite Abelian group is equivalent to a certain 'regular' covering of the lattice $A_n = \{\boldsymbol{x} \in \mathbb{Z}^{n+1} : \sum_i x_i = 0\}$ with balls of radius 2 under the ℓ_1 metric (or, equivalently, a covering of the integer lattice $\mathbb{Z}^n$ with balls of radius 1 under a slightly different metric). For planar difference sets, the covering is also a packing, and therefore a tiling, of A_n. This observation leads to a geometric reformulation of the prime power conjecture and of other statements involving Abelian difference sets.

2020 *Mathematics subject classification*: primary 05B40; secondary 05B10, 05B45, 11B13, 11B75, 11H31, 52C17, 52C22.

Keywords and phrases: difference set, prime power conjecture, group splitting, lattice packing, lattice covering, tiling, Manhattan metric.

1. Introduction

Let G be a finite Abelian group of order $|G| = v$, written additively. A subset $D \subseteq G$ of cardinality k is said to be a (v, k, λ)-*difference set* [2] if every nonzero element of G can be expressed as a difference $d_i - d_j$ of two elements from D in exactly λ ways. The parameters v, k, λ then necessarily satisfy the identity $\lambda(v-1) = k(k-1)$. Difference sets with parameter $\lambda = 1$ are called *planar*. These objects appeared first in the work of Singer [8] and have attracted the interest of mathematicians ever since. The ensuing research has produced numerous beautiful results at the crossroads of algebra, combinatorics and geometry [2, 7], and has also found several applications, for example in coding theory [4]. The purpose of this note is to contribute to this line of work by providing another geometric and combinatorial interpretation of difference sets, more precisely, by showing that difference sets can be represented in a simple and natural way as sublattices of $\mathbb{Z}^n$ having certain packing/covering/tiling properties.

This work was supported by the European Union's Horizon 2020 research and innovation programme under Grant Agreement number 856967, and by the Ministry of Education, Science and Technological Development of the Republic of Serbia through the project number 451-03-68/2020-14/200156.

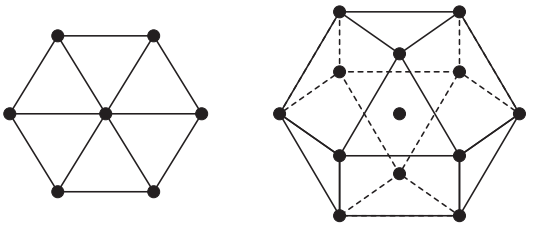


FIGURE 1. Ball of radius 1 in (A_2, d) (hexagon) and in (A_3, d) (cuboctahedron).

Consequently, many statements involving difference sets, in particular those dealing with existence questions, can be reformulated in purely geometric terms.

The A_n lattice under the ℓ_1 metric. A lattice in $\mathbb{R}^n$ is a discrete subgroup of $(\mathbb{R}^n, +)$. The A_n lattice is

$$A_n = \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{Z}^{n+1} \ : \ \sum_{i=0}^{n} x_i = 0\right\},$$

where $\mathbb{Z}$ denotes the integers, as usual. In particular, A_1 is equivalent to $\mathbb{Z}$, A_2 to the hexagonal lattice and A_3 to the face-centred cubic lattice [3].

The metric on A_n that we consider is essentially the ℓ_1 (also termed Manhattan or taxi) distance,

$$d(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}\|\boldsymbol{x} - \boldsymbol{y}\|_1 = \frac{1}{2}\sum_{i=0}^{n} |x_i - y_i|,$$

where $\boldsymbol{x} = (x_0, x_1, \ldots, x_n)$, $\boldsymbol{y} = (y_0, y_1, \ldots, y_n)$; the constant $1/2$ is adopted for convenience because $\|\boldsymbol{x} - \boldsymbol{y}\|_1$ is always even for $\boldsymbol{x}, \boldsymbol{y} \in A_n$. The metric d also represents the graph distance in A_n. If $\Gamma(A_n)$ is a graph with the vertex set A_n and with edges joining neighbouring points (points at distance 1 under d), then $d(\boldsymbol{x}, \boldsymbol{y})$ is the length of the shortest path between $\boldsymbol{x}$ and $\boldsymbol{y}$ in $\Gamma(A_n)$. The ball of radius 1 around $\boldsymbol{x} \in A_n$ contains $2\binom{n+1}{2} + 1 = n^2 + n + 1$ points of the form $\boldsymbol{x} + \boldsymbol{f}_{i,j}$, where $\boldsymbol{f}_{i,j}$ is the vector having 1 at the ith coordinate, -1 at the jth coordinate and zeros elsewhere (by convention, $\boldsymbol{f}_{i,i} = \boldsymbol{0}$) (see Figure 1). The convex interior of the points in this ball forms a highly symmetrical polytope with the property that the distance between any vertex and the centre is equal to the distance between any two neighbouring vertices.

REMARK 1.1. For the purpose of studying packing and covering problems, it is sometimes more convenient to visualise $\mathbb{Z}^n$ instead of an arbitrary lattice. In our case, there is a simple mapping that makes the transition to $\mathbb{Z}^n$ and back very easy, namely $\boldsymbol{x} = (x_0, x_1, \ldots, x_n) \mapsto \boldsymbol{x}' = (x_1, \ldots, x_n)$. If we define the following metric on $\mathbb{Z}^n$:

$$d^+(\boldsymbol{x}', \boldsymbol{y}') = \max\left\{\sum_{\substack{i=1 \\ x_i > y_i}}^{n} (x_i - y_i), \sum_{\substack{i=1 \\ x_i < y_i}}^{n} (y_i - x_i)\right\},$$

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press

then it is not difficult to show that the above mapping is an isometry between (A_n, d) and $(\mathbb{Z}^n, d^+)$ [6, Theorem 4]. Consequently, packing and similar problems in (A_n, d) are equivalent to those in $(\mathbb{Z}^n, d^+)$.

2. (v, k, λ)-difference sets and coverings of A_n

In the following, when using notions from graph theory in our setting, we have in mind the graph representation $\Gamma(A_n)$ of A_n, as introduced in Section 1. An (r, i, j)-cover in a graph $\Gamma = (V, E)$ [1] is a set of its vertices $S \subseteq V$ having the property that every element of S, respectively $V \setminus S$, is covered by exactly i, respectively j, balls of radius r centred at elements of S. Special cases of such sets, namely $(1, i, j)$ covers, have also been studied in the context of domination in graphs [10]. Note also that an $(r, 1, 1)$-cover is a *tiling* of V with balls of radius r (in coding theory, this is known as an r-perfect code). An independent set in a graph $\Gamma = (V, E)$ is a subset of its vertices $I \subseteq V$, no two of which are adjacent in Γ.

We now state our main result. The proof is a generalisation of the connection between lattice packing/tiling and so-called group splitting [9].

THEOREM 2.1. *There exists an Abelian $(v, n+1, \lambda)$-difference set if and only if the lattice A_n contains a $(1, 1, \lambda)$-covering sublattice.*

PROOF. Suppose that $D = \{d_0, d_1, \ldots, d_n\}$ is a $(v, n+1, \lambda)$-difference set in an Abelian group G and consider the sublattice

$$\mathcal{L}_D = \Big\{ \boldsymbol{x} \in A_n : \sum_{i=0}^{n} x_i d_i = 0 \Big\}, \tag{2.1}$$

where $x_i d_i$ denotes the sum in G of $|x_i|$ copies of d_i, respectively $-d_i$, if $x_i > 0$, respectively $x_i < 0$. Let us show that $\mathcal{L}_D$ is a $(1, 1, \lambda)$-cover of A_n. Consider a point $\boldsymbol{y} = (y_0, y_1, \ldots, y_n) \notin \mathcal{L}_D$, meaning that $\sum_{i=0}^{n} y_i d_i = a \in G$, $a \neq 0$. The neighbours of $\boldsymbol{y}$ are of the form $\boldsymbol{y} + \boldsymbol{f}_{i,j}$, $i \neq j$ (recall that $\boldsymbol{f}_{i,j}$ denotes the vector having 1 at the ith coordinate, -1 at the jth coordinate and zeros elsewhere). Because D is a difference set, $-a \in G$ can be written as a difference of two elements from D in exactly λ ways, meaning that there are λ different pairs (s, t) for which $d_s - d_t = -a$, $d_s, d_t \in D$. For every such pair, consider the point $\boldsymbol{z}_{s,t} = \boldsymbol{y} + \boldsymbol{f}_{s,t}$. Note that $\boldsymbol{z}_{s,t} \in \mathcal{L}_D$ because $\sum_{i=0}^{n} z_i d_i = \sum_{i=0}^{n} y_i d_i + d_s - d_t = a - a = 0$. Therefore, there are exactly λ points in the lattice $\mathcal{L}_D$ that are adjacent to $\boldsymbol{y}$, that is, such that balls of radius 1 around them cover $\boldsymbol{y}$. To show that the elements of $\mathcal{L}_D$ are covered only by the balls around themselves (that is, $\mathcal{L}_D$ is an independent set in $\Gamma(A_n)$), note that if there were two points at distance 1 in $\mathcal{L}_D$, then, by the same argument as above, we would obtain $d_s - d_t = 0$, that is, $d_s = d_t$ for some $s \neq t$, which is not possible if $|D| = n + 1$.

For the other direction, assume that $\mathcal{L}$ is a $(1, 1, \lambda)$-covering sublattice of A_n. Consider the quotient group $G = A_n / \mathcal{L}$ and take $D_{\mathcal{L}} = \{d_0, d_1, \ldots, d_n\} \subseteq G$, where $d_i = [\boldsymbol{f}_{i,0}] \equiv \boldsymbol{f}_{i,0} + \mathcal{L}$ are cosets (elements of G). Let us first check that all the d_i terms are distinct. Suppose that $d_s = d_t$ for some $s \neq t$. This implies that $d_s - d_t = [\boldsymbol{f}_{s,t}] = [\boldsymbol{0}]$,

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press

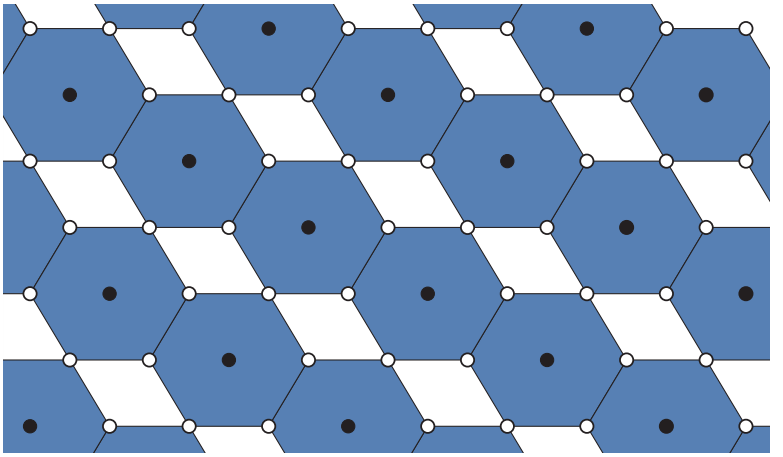


FIGURE 2. A $(1, 1, 2)$-covering sublattice of A_2, representing the difference set $D = \{0, 1, 2\} \subset \mathbb{Z}_4$.

which means that $\boldsymbol{f}_{s,t} \in \mathcal{L}$. However, because $\mathbf{0} \in \mathcal{L}$ and $\mathbf{0}$ and $\boldsymbol{f}_{s,t}$ are at distance 1, this would contradict the fact that $\mathcal{L}$ is an independent set in $\Gamma(A_n)$. Hence, $|D_{\mathcal{L}}| = n + 1$. Now take any nonzero element of G, say $[\boldsymbol{y}]$, $\boldsymbol{y} \notin \mathcal{L}$. By assumption, $\boldsymbol{y}$ is covered by exactly λ elements of $\mathcal{L}$, that is, $\boldsymbol{y} + \boldsymbol{f}_{s,t} \in \mathcal{L}$ for exactly λ vectors $\boldsymbol{f}_{s,t}$. Because $\boldsymbol{f}_{s,t} = \boldsymbol{f}_{s,0} - \boldsymbol{f}_{t,0}$, this means that $d_t - d_s = [\boldsymbol{f}_{t,0}] - [\boldsymbol{f}_{s,0}] = [\boldsymbol{y}]$ for exactly λ pairs (s, t). Therefore, $D_{\mathcal{L}}$ is a $(v, n + 1, \lambda)$-difference set. □

Note that we have not specified the order of the elements of D when defining the corresponding lattice $\mathcal{L}_D$ in (2.1) because it would only affect it in an insignificant way. Note also that if we write $d'_i = zd_i + g$ instead of d_i in (2.1), where z is a fixed integer coprime with v and g is a fixed element of G, the same lattice is obtained because

$$\sum_{i=0}^{n} x_i d_i = 0 \quad \Leftrightarrow \quad \sum_{i=0}^{n} x_i d'_i = 0,$$

which follows from $\sum_{i=0}^{n} x_i = 0$ and $\gcd(z, v) = 1$. (Recall that two difference sets D and D' in an Abelian group G are said to be *equivalent* [2, Remark 1.11, page 302] if $D' = \{zd + g : d \in D\}$, for some $z \in \mathbb{Z}$ coprime with $v = |G|$ and some $g \in G$.)

Geometrically, the theorem states that balls of radius 1 around the points of the sublattice $\mathcal{L}_D$ overlap in such a way that every point that does not belong to $\mathcal{L}_D$ is covered by exactly λ balls. (The points in $\mathcal{L}_D$ (centres of the balls) are covered by one ball only, and hence this notion is different from multitiling [5].) Note that increasing λ increases the density of the lattice $\mathcal{L}_D$ in A_n. The densest such lattice is therefore obtained for $\lambda = n + 1$ (which is the maximum value because $\lambda(v - 1) = n(n + 1)$ and $n \leq v - 1$). It corresponds to the trivial (v, v, v)-difference set $D = G$ in an arbitrary Abelian group G.

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press

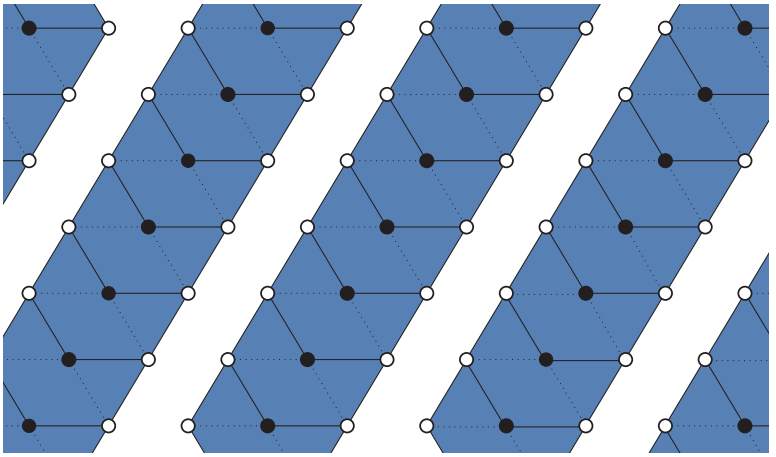


FIGURE 3. A $(1, 3, 2)$-covering sublattice of A_2.

EXAMPLE 2.2. $D = \{0, 1, 2\}$ is a $(4, 3, 2)$-difference set in the cyclic group $\mathbb{Z}_4$. A $(1, 1, 2)$-covering sublattice $\mathcal{L}_D \subset A_2$ corresponding to this difference set (see (2.1)) is illustrated in Figure 2. Points in $\mathcal{L}_D$ are depicted as black and those in $A_2 \setminus \mathcal{L}_D$ as white dots. For illustration, Figure 3 shows an example of a $(1, \mathbf{3}, 2)$-covering sublattice of A_2, which does not represent any difference set.

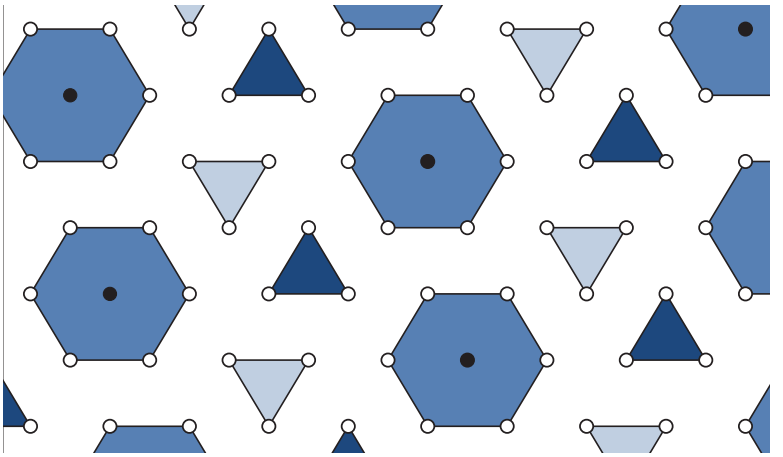
3. Planar difference sets and tilings of A_n

A $(v, k, 1)$-difference set $D \subseteq G$ is called *planar* (or *simple*). The condition $\lambda = 1$ means that every nonzero element of the group G can be expressed as a difference of two elements from D in a unique way. In this case, we necessarily have $v = |G| = k^2 - k + 1$. The *order* of a planar difference set D of cardinality k is defined as $k - 1$. These objects are very well-studied, and a large body of literature is devoted to their constructions and investigation of their properties [2]. One of the most well-known problems in the area concerning the existence of planar difference sets for specific sets of parameters is the so-called *prime power conjecture* [2, Conjecture 7.5, page 346], which states that a planar difference set of order n exists if and only if n is a prime power (counting $n = 1$ as a prime power). Existence of such sets for $n = p^m$, p prime, $m \in \mathbb{N}$, was demonstrated by Singer [8], but the necessity of this condition remains an open problem for over eight decades.

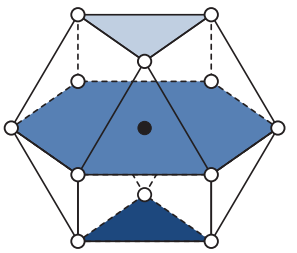
As we noted earlier, a $(1, 1, 1)$-cover of A_n is in fact a tiling of (A_n, d) with balls of radius 1, meaning that every point in A_n is covered by exactly one ball. If the centres of the balls form a sublattice of A_n, then this is said to be a lattice tiling. We can now state what Theorem 2.1 reduces to in the special case $\lambda = 1$.

COROLLARY 3.1. *There exists an Abelian planar difference set of order n if and only if the space* (A_n, d) *admits a lattice tiling with balls of radius* 1.

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press



(a) The lattice $\mathcal{L}_D$ viewed in the plane $x_0 = 0$.



(b) Intersections of a ball in (A_3, d) with the planes $x_0 = \text{const}$.

FIGURE 4. Lattice tiling of (A_3, d) corresponding to the difference set $D = \{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$.

Existence of such tilings when n is a prime power follows from the existence of the corresponding planar difference sets [8], but the necessity of this condition is open and is equivalent to the prime power conjecture.

CONJECTURE 3.2 (Prime power conjecture). The space (A_n, d) admits a lattice tiling with balls of radius 1 if and only if the dimension n is a prime power.

A stronger conjecture would claim the above even for nonlattice tilings.

EXAMPLE 3.3. Consider a planar difference set $D = \{0, 1, 3, 9\} \subset \mathbb{Z}_{13}$. The corresponding lattice tiling of (A_3, d) is illustrated in Figure 4(a). The figure shows the intersection of A_3 with the plane $x_0 = 0$; the intersections of a ball of radius 1 in (A_3, d) with the planes $x_0 = \text{const.}$ are shown in Figure 4(b) as a clarification.

Corollary 3.1 and Example 3.3 were also stated in [6] by using coding theoretic terminology.

Another important unsolved problem in the field is the following: all Abelian planar difference sets live in cyclic groups [2, Conjecture 7.7, page 346]. Because the group

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press

G containing a difference set D which defines a lattice $\mathcal{L}_D$ is isomorphic to $A_n/\mathcal{L}_D$, the statement that G is cyclic, that is, that it has a generator, is equivalent to the following statement.

CONJECTURE 3.4 (All Abelian planar difference sets are cyclic). Suppose a lattice $\mathcal{L} \subset A_n$ defines a lattice tiling of (A_n, d) with balls of radius 1. Then the period of $\mathcal{L}$ in A_n along the direction $\boldsymbol{f}_{i,j}$ is equal to $n^2 + n + 1$ for at least one vector $\boldsymbol{f}_{i,j}$, $(i,j) \in \{0, 1, \ldots, n\}^2$.

The cyclic case. To conclude the paper, let us consider briefly the case of cyclic planar difference sets of order n, where it is assumed that the group we are working with is $\mathbb{Z}_v$, $v = n^2 + n + 1$. As mentioned above, the restriction to cyclic groups might not be a restriction at all. So let $D = \{d_0, d_1, \ldots, d_n\} \subset \mathbb{Z}_v$ be a difference set and assume that $d_0 = 0$, $d_1 = 1$. (This is not a loss in generality because if D is a difference set, then there exist two elements, say $d_0, d_1 \in D$, such that $d_1 - d_0 = 1$, so one can instead consider the equivalent difference set $D' = \{d_i - d_0 : d_i \in D\}$ which obviously contains 0 and 1.) Let $\mathcal{L}_D \subset \mathbb{Z}^n$ be the lattice defined as in (2.1), but with the 0-coordinate of all vectors left out (the latter is done for convenience, because $d_0 = 0$). Leaving out the 0-coordinate essentially transforms the space (A_n, d) to $(\mathbb{Z}^n, d^+)$ (see Remark 1.1). The generator matrix of the lattice $\mathcal{L}_D$ can then be written in the following simple and explicit form:

$$B(\mathcal{L}_D) = \begin{pmatrix} v & 0 & 0 & \cdots & 0 \\ -d_2 & 1 & 0 & \cdots & 0 \\ -d_3 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -d_n & 0 & 0 & \cdots & 1 \end{pmatrix},$$

that is, the elements of the lattice are the vectors $\boldsymbol{x} = \boldsymbol{\xi} \cdot B(\mathcal{L}_D)$, $\boldsymbol{\xi} \in \mathbb{Z}^n$. The generator matrix of the dual lattice $\mathcal{L}_D^*$ is

$$B(\mathcal{L}_D^*) = B(\mathcal{L}_D)^{-T} = \begin{pmatrix} 1/v & d_2/v & d_3/v & \cdots & d_n/v \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}.$$

References

[1] M. A. Axenovich, 'On multiple coverings of the infinite rectangular grid with balls of constant radius,' *Discrete Math.* **268** (2003), 31–48.

[2] T. Beth, D. Jungnickel and H. Lenz, *Design Theory*, 2nd edn (Cambridge University Press, Cambridge, 1999).

[3] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd edn (Springer, New York, 1999).

[4] C. Ding, *Codes from Difference Sets* (World Scientific, Singapore, 2015).

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press

[5] N. Gravin, S. Robins and D. Shiryaev, 'Translational tilings by a polytope, with multiplicity', *Combinatorica* **32** (2012), 629–648.
[6] M. Kovačević and V. Y. F. Tan, 'Codes in the space of multisets—coding for permutation channels with impairments,' *IEEE Trans. Inform. Theory* **64** (2018), 5156–5169.
[7] E. H. Moore and H. S. Pollatsek, *Difference Sets: Connecting Algebra, Combinatorics, and Geometry* (American Mathematical Society, Providence, RI, 2013).
[8] J. Singer, 'A theorem in finite projective geometry and some applications to number theory,' *Trans. Amer. Math. Soc.* **43** (1938), 377–385.
[9] S. Stein and S. Szabó, *Algebra and Tiling: Homomorphisms in the Service of Geometry*, Carus Mathematical Monographs, 25 (The Mathematical Association of America, Washington, DC, 1994).
[10] J. A. Telle, 'Complexity of domination-type problems in graphs,' *Nordic J. Comput.* **1** (1994), 157–171.

MLADEN KOVAČEVIĆ, Faculty of Technical Sciences,
University of Novi Sad, Serbia
e-mail: kmladen@uns.ac.rs

https://doi.org/10.1017/S0004972721001271 Published online by Cambridge University Press