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MAXIMAL OPERATORS ALONG FLAT CURVES WITH ONE VARIABLE VECTOR FIELD

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Abstract We study a maximal average along a family of curves $\{(t, m(x_1)\gamma(t)) : t \in [-r, r]\}$, where $\gamma|_{[0,\infty)}$ is a convex function and m is a measurable function. Under the assumption of the doubling property of γ' and $1 \leqslant m(x_1) \leqslant 2$, we prove the $L^p(\mathbb{R}^2)$ boundedness of the maximal average. As a corollary, we obtain the pointwise convergence of the average in $r > 0$ without any size assumption for a measurable m.

Keywords: Maximal functions along curves; pseudo-differential operators

1. Introduction

In this study, we analyse a maximal operator defined by a convex function $\gamma|_{[0,\infty)}$ and a measurable function $m : \mathbb{R} \to \mathbb{R}$. Specifically, our focus lies on the operator:

$$\mathcal{M}_\gamma^m f(x_1, x_2) := \sup_{r>0} \frac{1}{2r} \int_{-r}^{r} |f(x_1 - t, x_2 - m(x_1)\gamma(t))| \mathrm{d}t,$$

where $\gamma : \mathbb{R} \to \mathbb{R}$ is an extension of $\gamma|_{[0,\infty)}$, which is a even or odd function. Recently, Guo, Hickman, Lie and Roos [13] proved the L^p boundedness of maximal operators $\mathcal{M}_\gamma^m$ for the homogeneous curve $\gamma(t) = t^n$, with $n \geqslant 2$, assuming that m is measurable. However, the L^p boundedness of $\mathcal{M}_\gamma^m$ for the case $n = 1$ remains an open problem. So, we focus on flat convex curves, including piecewise linear curves. Given a convex extension $\gamma : \mathbb{R} \to \mathbb{R}$, we define the bounded doubling property for a derivative γ' as follows:

$$\text{there exists a constant } \omega > 1 \text{ such that } \gamma'(\omega|t|) \geq 2\gamma'(|t|) \text{ for all } t \in \mathbb{R}. \tag{1.1}$$

Now, we state the main theorem:

Main Theorem 1. *Let $m : \mathbb{R} \to \mathbb{R}$ be a measurable function such that $1 \leqslant m(x) \leqslant 2$ for all $x \in \mathbb{R}$. Suppose that an extension γ of a convex function $\gamma|_{[0,\infty)}$ satisfies the* *bounded doubling property of γ' in (1.1), with $\gamma(0)=0$. Then, there exists a constant C_ω such that $\|\mathcal{M}_\gamma^m\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leqslant C_{\omega,p}$ holds for $1<p\leqslant\infty$.*

Remark 1.1.

- The theorem can be extended to certain types of piecewise linear curves. Refer to Section 7 in [7] or Remark 5 in [14] for more details. Additionally, the condition (1.1) admits flat convex curves, such as $\gamma(t)=\mathrm{e}^{-\frac{1}{|t|}}$ and $\mathrm{e}^{-\mathrm{e}^{\frac{1}{|t|}}}$, which are flat at the origin.
- By using the dilation technique, we can extend our results to $\|\mathcal{M}_\gamma^m\|_{L^p\to L^p} \leqslant C\log_2(\frac{b}{a})$ under the assumption $0<a\leqslant m(x)\leqslant b$.

In the view of pointwise convergence, we can drop the assumption $1\leqslant m(x_1)\leqslant 2$.

Corollary 1.1. *For a measurable function $m:\mathbb{R}\to\mathbb{R}$ and a convex extension γ on $\mathbb{R}^1$ passing through the origin with its derivative γ' satisfying property (1.1), we have*

$$\lim_{r\to 0}\frac{1}{2r}\int_{-r}^{r} f(x_1-t, x_2-m(x_1)\gamma(t))\,dt = f(x_1,x_2) \qquad a.e.$$

for $f\in L^p(\mathbb{R}^2)$.

The study of maximal operators along flat convex curves has a rich history in Harmonic analysis by itself. In the 1970s, Stein and Wainger [24] asked the general class of curves $(t,\gamma(t))$ for which there are L^p results for $\mathcal{M}_\gamma^1$. In the 1980s, Carlsson *et al.* [11] proved that $\mathcal{M}_\gamma^1$ is bounded on $L^p(\mathbb{R}^2)$ under the bounded doubling condition (1.1). In the 1990s, the study of maximal operators was extended to the curves with a variable coefficient, as demonstrated in [4, 9, 10, 15, 23]. Carbery, Wainger and Wright [9] established the L^p boundedness of $\mathcal{M}_\gamma^{x_1}$ along plane curves γ whose derivative satisfies the infinitesimal doubling property. Under the same assumption, Bennett [4] extended the L^2 results for $\mathcal{M}_\gamma^P$, where P is a polynomial. As a corollary of our main theorem, we derive the L^p boundedness of $\mathcal{M}_\gamma^P$ under much weaker assumptions on γ.

Corollary 1.2. *For a polynomial $P:\mathbb{R}\to\mathbb{R}$ with degree d and a convex extension γ on $\mathbb{R}^1$ passing through the origin with its derivative γ' satisfying property (1.1), there exists a constant $C_{\omega,d}$ independent of the coefficients of P such that $\|\mathcal{M}_\gamma^P\|_{L^p(\mathbb{R}^2)\to L^p(\mathbb{R}^2)} \leqslant C_{\omega,d,p}$ for $1<p\leqslant\infty$.*

Note that the infinitesimal doubling property implies the bounded doubling property. For more details, refer to [4].

1.1. Historical background

Zygmund conjecture is a long-standing open problem in harmonic analysis. This question inquires whether the Lipschitz regularity of u is sufficient to guarantee any non-trivial L^p bounds for the maximal operator:

$$\mathcal{M}_\gamma^u(f)(x_1,x_2) := \sup_{r>0} \frac{1}{2r}\int_{-r}^{r} |f(x_1-t, x_2-u(x_1,x_2)\gamma(t))|\mathrm{d}t,$$

where $\gamma(t) = t$. Since the discovery of the Besicovitch set in the 1920s, it has been shown that the conjecture is false when the function u is only Hölder continuous C^α with $\alpha < 1$. However, the problem remains open under the Lipschitz assumption for u. In the 1970s, Stein and Wainger [24] proposed an analogous conjecture for the Hilbert transform. Regarding the Hilbert transforms along vector fields, Lacey and Li [18] made a significant progress regarding the regularity of u in 2006, using time–frequency analysis tools. Later, Bateman and Thiele [2] obtained the L^p estimates for the Hilbert transform along a one-variable vector field. Their proof relied on the commutation relation between the Hilbert transform and Littlewood–Paley projection operators, which cannot be directly applied to the maximal operator $\mathcal{M}_\gamma^m$ due to its sub-linearity. Therefore, the problem for maximal operators remains open. For additional discussion on Stein's conjecture, we recommend references [1, 2, 17]. In the study of maximal operators, Bourgain [5] demonstrated the L^2 boundedness of $\mathcal{M}_t^u$ for real analytic functions u. In 1999, Carbery, Seeger, Wainger and Wright [8] examined the maximal operators $\mathcal{M}_t^m$ along one variable vector field. One of the authors in this paper further extended this result in [16].

Recently, in [13], Guo *et al.* investigated the L^p boundedness of $\mathcal{M}_\gamma^u$ under the Lipschitz assumption for u and homogeneous curve $\gamma(t) = t^n$ for $n > 1$. Later, Liu, Song and Yu [20] extended the results to more general curves with the condition $\left|\frac{t\gamma''(t)}{\gamma'(t)}\right| \sim 1$. A crucial tool used in the proofs of both papers was the local smoothing estimate, which was established in [3, 21]. For more history, we recommend the study [19] by Victor Lie, which presents a unified approach and includes a more general view of this topic as well as problems related to the concept of non-zero curvature.

1.2. Notation

Let $\psi : \mathbb{R} \to \mathbb{R}$ be a non-negative C^∞ function supported on $[-2,2]$ such that $\psi \equiv 1$ on $[-1,1]$. Define $\varphi(t) = \psi(t) - \psi(2t)$ and $\varphi_l(t) = \frac{1}{2^l}\varphi(\frac{t}{2^l})$. Also, define $\psi^c(t) = 1 - \psi(t)$. Note that $\sum_{l\in\mathbb{Z}} \varphi\left(\frac{t}{2^l}\right) = 1$ for $t \neq 0$ and $\mathrm{supp}(\varphi) \subset \left\{\frac{1}{2} \leqslant |x| \leqslant 2\right\}$. We define the Littlewood–Paley projection $\mathcal{L}_s f$ as $\widehat{\mathcal{L}_s f}(\xi) := \hat{f}(\xi)\varphi\left(\frac{\xi_1}{2^s}\right)$. We shall use the notation $A \lesssim_d B$ when $A \leqslant C_d B$ with a constant $C_d > 0$ depending on the parameter d. Moreover, we write $A \sim_d B$, if $A \lesssim_d B$ and $B \lesssim_d A$. Let M_{HL} be the Hardy–Littlewood maximal operator and M^{str} be the strong maximal operator. Let χ_A be a characteristic function, which is equal to 1 on A and otherwise 0. Denote the dyadic pieces of intervals by

$$I_i = [2^{i-1}, 2^{i+1}] \cup [-2^{i+1}, -2^{i-1}],$$
$$\tilde{I}_i = [2^{i-2}, 2^{i+2}] \cup [-2^{i+2}, -2^{i-2}],$$

and the corresponding strips by $S_i = I_i \times \mathbb{R}$, $\tilde{S}_i = \tilde{I}_i \times \mathbb{R}$.

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2. Reduction

In this section, we present three propositions that have broad applicability. Let $\Gamma : \mathbb{R}^2 \to \mathbb{R}$ be a measurable function and define a general class of operators

$$T_j f(x_1, x_2) := \int f(x_1 - t, x_2 - \Gamma(x_1, t))\varphi_j(t)\mathrm{d}t.$$

Proposition 2.1. *Define* $T_j^{glo} f(x_1, x_2) := \psi_{j+4}^c(x_1) T_j f(x_1, x_2)$*. Under the measurability assumption of* Γ*, we have*

$$\| \sup_j |T_j - T_j^{glo}| \|_p \leqslant C_p,$$

for $1 < p \leqslant \infty$*.*

Proof. Denote that $\tilde{\varphi}(\frac{x}{2^j}) = \sum_{k=-3}^{4} \varphi(\frac{x}{2^{j+k}})$, which has a localized support $|x| \sim 2^j$. Let T_j^{loc} and T_j^{mid} be operator, defined by

$$T_j^{\mathrm{loc}} f(x_1, x_2) := \psi_{j-4}(x_1) T_j f(x_1, x_2),$$
$$T_j^{\mathrm{mid}} f(x_1, x_2) := \tilde{\varphi}\left(\frac{x_1}{2^j}\right) T_j f(x_1, x_2).$$

Then, we can decompose $T_j - T_j^{\mathrm{glo}}$ into $T_j^{\mathrm{mid}} + T_j^{\mathrm{loc}}$. For the operator T_j^{mid}, replace the sup as ℓ^p sum. Then, we have

$$\left\| \sup_{j \in \mathbb{Z}} |T_j^{\mathrm{mid}} f| \right\|_{L^p(\mathbb{R}^2)} \leqslant \left(\sum_{j \in \mathbb{Z}} \|T_j^{\mathrm{mid}} f\|_{L^p(\mathbb{R}^2)}^p \right)^{\frac{1}{p}}.$$

Denote $F(x_1) = \|f(x_1, \cdot)\|_{L^p(dx_2)}$. By applying Minkowski's integral inequality and a change of variables, we get the pointwise inequality:

$$\begin{aligned} \|T_j^{\mathrm{mid}} f(x_1, \cdot)\|_{L^p(dx_2)} &\leqslant \int \left(\int |f(x_1 - t, x_2 - \Gamma(x_1, t))|^p dx_2 \right)^{\frac{1}{p}} \varphi_j(t)\mathrm{d}t \\ &\leqslant \int F(x_1 - t)\varphi_j(t) dt \lesssim_\varphi M_{\mathrm{HL}} F(x_1), \end{aligned} \tag{2.1}$$

where the second inequality follows form the fact that $\Gamma(x_1, t)$ is independent of x_2. By (2.1) and the L^p boundedness of M_{HL}, we obtain

$$\left(\sum_{j \in \mathbb{Z}} \|T_j^{\mathrm{mid}} f\|_{L^p(\mathbb{R}^2)}^p \right)^{\frac{1}{p}} \leqslant \left(\sum_j \int \tilde{\varphi}\left(\frac{x_1}{2^j}\right) |M_{\mathrm{HL}} F(x_1)|^p dx_1 \right)^{\frac{1}{p}} \lesssim \|f\|_p.$$

which implies the L^p boundedness of $f \mapsto \sup_j |T_j^{\mathrm{mid}} f|$ for $p > 1$. For the operator $T_j^{\mathrm{loc}} f$, we observe the localization principle:

$$T_j^{\mathrm{loc}} f(x_1, x_2) = T_j^{\mathrm{loc}}(\chi_{S_j} f)(x_1, x_2).$$

By combining this with $\sup_{j\in\mathbb{Z}} \|T_j\|_p \leqslant C$, we get the following estimate:

$$\Big\| \sup_{j\in\mathbb{Z}} |T_j^{\mathrm{loc}} f| \Big\|_p^p = \sum_{j\in\mathbb{Z}} \int |T_j^{\mathrm{loc}} \chi_{S_j} f(x_1, x_2)|^p \mathrm{d}x \leqslant C \sum_{j\in\mathbb{Z}} \int |\chi_{S_j} f(x_1, x_2)|^p \mathrm{d}x \lesssim \|f\|_p^p.$$

Therefore, we prove $\| \sup_j |T_j - T_j^{\mathrm{glo}}| \|_p \leqslant C_p$ for $1 < p \leqslant \infty$. □

By Proposition 2.1, in order to prove Theorem 1, it suffices to consider the maximal operator defined as

$$f \mapsto \sup_j |T_j^{\mathrm{glo}} f| \text{ , where } T_j^{\mathrm{glo}} = \psi_{j+4}^c T_j.$$

Proposition 2.2 (Space Reduction). *Let $T_j^\ell f(x_1, x_2) := \chi_{S_\ell}(x_1, x_2) T_j^{glo} f(x_1, x_2)$. Then, the following inequality holds:*

$$\| \sup_{j\in\mathbb{Z}} |T_j^{glo}| \|_{L^p \to L^p} \lesssim \sup_{\ell\in\mathbb{Z}} \| \sup_{j\in\mathbb{Z}} |T_j^\ell| \|_{L^p \to L^p}. \tag{2.2}$$

Proof. One can obtain (2.2) from the localization $T_j^\ell f(x_1, x_2) = T_j^\ell(\chi_{\tilde{S}_\ell} f)(x_1, x_2)$. □

Combining Proposition 2.1 and Proposition 2.2, we may restrict our attention to the maximal operator defined by $f \mapsto \sup_j |T_j^\ell|$, supported on $|x_1| \sim 2^\ell \gg 2^j$.

Proposition 2.3 (Frequency Reduction). *Suppose $\Gamma : \mathbb{R} \times [0, \infty) \to \mathbb{R}$ is measurable on $\mathbb{R}^2$ with $\Gamma(x_1, 0) = 0$ satisfying the following conditions:*

For every $x_2 \in \mathbb{R}$, $x_1 \mapsto \Gamma(x_1, x_2)$ is measurable function.

For every $x_1 \in \mathbb{R}$, $x_2 \mapsto \Gamma(x_1, x_2)$ is convex increasing function.

Let $\widehat{\mathcal{L}_j^{low} f}(\xi_1, \xi_2) := \hat{f}(\xi_1, \xi_2)\psi(2^j \xi_1)$ for $f \in \mathcal{S}(\mathbb{R}^2)$. Then, there exists a constant C independent of Γ such that

$$\sup_{j\in\mathbb{Z}} |T_j(\mathcal{L}_j^{low} f)(x_1, x_2)| \leqslant C M^2 M^1 f(x_1, x_2),$$

where M^i is the Hardy–Littlewood maximal operator taken in the ith variable.

Proof. For $g \in \mathcal{S}(\mathbb{R}^1)$ and $2^{j-1} \leqslant |t| \leqslant 2^{j+1}$, we have

$$\int g(x_1 - t - s)\frac{1}{2^j}\check{\psi}\left(\frac{s}{2^j}\right)ds \lesssim_\psi M_{\mathrm{HL}}g(x_1),$$
$$\frac{1}{r}\int_0^r g(x_2 - \Gamma(x_1,t))dt \leqslant 2M_{\mathrm{HL}}f(x_2 - \Gamma(x_1,0)) = 2M_{\mathrm{HL}}g(x_2),$$

where the second inequality follows form the convexity of $t \mapsto \Gamma(x_1,t)$. For more details, we refer to Lemma 2 in [12] and [6]. Since $T_j(\mathcal{L}_j^{\mathrm{low}}f)(x_1,x_2)$ is a composition of the above two functions, we obtain the desired pointwise inequality. □

Set $\widehat{\mathcal{L}_j^{\mathrm{high}}f}(\xi_1,\xi_2) = \hat{f}(\xi_1,\xi_2)\psi^c(2^j\xi_1)$. Following Proposition 2.3, it is enough to show the estimate $\|\sup_j |T_j^\ell(\mathcal{L}_j^{\mathrm{high}}f)|\|_p \lesssim \|f\|_p$.

3. Proof of main theorem 1

Following the reduction section, we only consider $\mathcal{T}_j^\ell(\mathcal{L}_j^{\mathrm{high}}f)$, which is given by

$$\mathcal{T}_j^\ell(\mathcal{L}_j^{\mathrm{high}}f)(x_1,x_2) := \psi_{j+4}^c(x_1)\chi_{S_\ell}(x)\int \mathcal{L}_j^{\mathrm{high}}f(x_1 - t, x_2 - m(x_1)\gamma(t))\varphi_j(t)\mathrm{d}t,$$

supported on $|x_1| \sim 2^\ell \gg 2^j$.

3.1. Main difficulty

In a view of pseudo-differential operator, we write

$$\mathcal{T}_j^\ell(\mathcal{L}_j^{\mathrm{high}}f)(x_1,x_2) = \int \mathrm{e}^{2\pi i(x_1\xi_1 + x_2\xi_2)}b_j(x_1,\xi_1,\xi_2)\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2,$$

with the symbol $b_j(x_1,\xi_1,\xi_2)$ given by

$$b_j(x_1,\xi_1,\xi_2) = \chi_{I_\ell}(x_1)\psi^c(2^j\xi_1)\int \mathrm{e}^{-2\pi i(2^j t\xi_1 + m(x_1)\gamma(2^j t)\xi_2)}\varphi(t)\mathrm{d}t.$$

When analysing an oscillatory integral with a phase $t\xi_1 + m(x_1)\gamma(t)\xi_2$, it is usual to decompose each frequency variable ξ_1 and ξ_2 with dyadic scale. Specifically, in the case of a homogeneous curve, we can even estimate the asymptotic behaviour of oscillatory integral. However, under the flat condition (1.1), this usual approach does not work, as there are no comparablity condition $\left|\frac{\gamma'(2t)}{\gamma'(t)}\right| \sim 1$ and a finite type assumption for the curve. To overcome this situation, we will perform an angular decomposition in [11] for a function f and utilize the method in one of the author's paper [15].

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3.2. Angular decomposition

Set

$$A_j(\xi_1,\xi_2) := \psi\left(\frac{\xi_1}{\xi_2\gamma'(2^{j+1})}\right) - \psi\left(\frac{\xi_1}{\xi_2\gamma'(2^{j-1})}\right)$$

and

$$\widehat{\mathcal{A}_j f}(\xi_1,\xi_2) := \hat{f}(\xi_1,\xi_2)A_j(\xi_1,\xi_2),$$
$$\mathcal{A}_j^c f(x_1,x_2) := f(x_1,x_2) - \mathcal{A}_j f(x_1,x_2).$$

Note that we have the following Littlewood–Paley estimate in [11]:

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{A}_j f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \text{ for } 1<p<\infty.$$

We have $\mathcal{A}_j\mathcal{L}_j^{\text{high}} f(x) = \mathcal{A}_j f(x) - \mathcal{L}_j^{\text{low}}\mathcal{A}_j f(x)$. Then, it gives

$$|\mathcal{A}_j\mathcal{L}_j^{\text{high}} f(x_1,x_2)| \lesssim |\mathcal{A}_j f(x_1,x_2)| + |M^1\mathcal{A}_j f(x_1,x_2)|$$

from the pointwise estimate $|\mathcal{L}_j^{\text{low}} f(x_1,x_2)| \lesssim M^1 f(x_1,x_2)$. By the vector valued estimate for Hardy–Littlewood maximal operator, the following estimate holds:

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{A}_j\mathcal{L}_j^{\text{high}} f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)} \text{ for } 1<p<\infty. \tag{3.1}$$

We split $\mathcal{T}_j^\ell(\mathcal{L}_j^{\text{high}} f)$ into two terms:

$$\mathcal{T}_j^\ell(\mathcal{L}_j^{\text{high}} f) = \mathcal{T}_j^\ell(\mathcal{A}_j\mathcal{L}_j^{\text{high}} f) + \mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_j^{\text{high}} f).$$

Then, we shall prove the following:

$$\left\|\sup_{j\in\mathbb{Z}}|\mathcal{T}_j^\ell(\mathcal{A}_j\mathcal{L}_j^{\text{high}} f)|\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}, \tag{3.2}$$

$$\left\|\sup_{j\in\mathbb{Z}}|\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_j^{\text{high}} f)|\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_{L^p(\mathbb{R}^2)}. \tag{3.3}$$

We can obtain the estimate (3.2) for $p=2$ from the following process:

$$\left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{T}_j^\ell(\mathcal{A}_j\mathcal{L}_j^{\text{high}} f)|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^2)} \lesssim \left\|\left(\sum_{j\in\mathbb{Z}}|\mathcal{A}_j\mathcal{L}_j^{\text{high}} f|^2\right)^{\frac{1}{2}}\right\|_{L^p(\mathbb{R}^2)} \lesssim \|f\|_p. \tag{3.4}$$

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Furthermore, the range of p can be extended by a bootstrap argument detailed in Section 3.4. In the following proposition, we focus particularly on the term $\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_j^{\text{high}}f)$ and prove the estimate (3.3). Furthermore, the range of p can be extended by a bootstrap argument detailed in Section 3.4. In the following proposition, we focus particularly on the term $\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_j^{\text{high}}f)$ and prove the estimate (3.3).

Proposition 3.1. *Define the Littlewood–Paley projection* $\widehat{\mathcal{L}_j f}(\xi_1,\xi_2) := \hat{f}(\xi_1,\xi_2)$ $\varphi(\frac{\xi_1}{2^j})$ *so that* $\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_j^{high}f) = \sum_{n=0}^{\infty}\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_{n-j}f)$*. For* $f\in L^p(\mathbb{R}^2)$*, It holds that*

$$\left\|\sup_{j\in\mathbb{Z}}|\mathcal{T}_j^\ell(\mathcal{A}_j^c\mathcal{L}_{n-j}f)|\right\|_{L^p(\mathbb{R}^2)} \le C2^{-\varepsilon_p n}\|f\|_{L^p(\mathbb{R}^2)}, \tag{3.5}$$

for $1<p<\infty$ *and* $n\geqslant 0$.

Note that we need the following:

Lemma 3.1 (Reduction to one variable operator). *Consider the two operators* $\mathcal{R}_1$ *and* $\mathcal{R}_2^\lambda$*, given by*

$$\begin{aligned}\mathcal{R}_1 f(x_1,x_2) &:= \int_{\mathbb{R}^2} e^{2\pi i(x_1\xi_1+x_2\xi_2)}a(x_1,\xi_1,\xi_2)\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2,\\ \mathcal{R}_2^\lambda g(x_1) &:= \int_{\mathbb{R}} e^{2\pi i x_1\xi_1}a(x_1,\xi_1,\lambda)\hat{g}(\xi_1)d\xi_1.\end{aligned}$$

for $f\in\mathcal{S}(\mathbb{R}^2)$ *and* $g\in\mathcal{S}(\mathbb{R})$*. Then,* $\|\mathcal{R}_1\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \leqslant \sup_{\lambda\in\mathbb{R}}\|\mathcal{R}_2^\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})}$.

Proof of Lemma 3.1 Consider a function $f\in\mathcal{S}(\mathbb{R}^2)$ with $\|f\|_{L^2(\mathbb{R}^2)}=1$. Denote $\mathcal{F}_2 f(x_1,\xi_2) = g_{\xi_2}(x_1)$. By Plancheral's theorem with respect to x_2, we get

$$\begin{aligned}\|\mathcal{R}_1 f\|_2^2 &= \int\left|\int_{\mathbb{R}^2} \mathrm{e}^{2\pi i(x_1\xi_1+x_2\xi_2)}a(x_1,\xi_1,\xi_2)\hat{f}(\xi_1,\xi_2)d\xi_1 d\xi_2\right|^2 \mathrm{d}x_1\mathrm{d}x_2\\ &= \int\left|\int_{\mathbb{R}} \mathrm{e}^{2\pi i x_2\lambda}\mathcal{R}_2^\lambda g_\lambda(x_1)\mathrm{d}\lambda\right|^2 \mathrm{d}x_2\mathrm{d}x_1\\ &= \int|\mathcal{R}_2^\lambda g_\lambda(x_1)|^2\mathrm{d}x_1\mathrm{d}\lambda \leqslant \sup_{\lambda\in\mathbb{R}}\|\mathcal{R}_2^\lambda\|^2_{L^2(\mathbb{R})\to L^2(\mathbb{R})}\int|g_\lambda\|^2_{L^2(\mathbb{R})}\mathrm{d}\lambda.\end{aligned}$$

which yields the desired estimate. □

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3.3. Proof of Proposition 3.1

We shall prove $\|\mathcal{T}_j^\ell \mathcal{L}_{n-j}\mathcal{A}_j^c\|_{L^2(\mathbb{R}^2)\to L^2(\mathbb{R}^2)} \lesssim 2^{-\frac{n}{2}}$, which implies

$$\begin{aligned}\Big\|\sup_j |\mathcal{T}_j^\ell(\mathcal{L}_{n-j}\mathcal{A}_j^c f)|\Big\|_2^2 &\lesssim \sum_j \|\mathcal{T}_j^\ell \mathcal{L}_{n-j}\mathcal{A}_j^c(\mathcal{L}_{n-j}f)\|_2^2\\ &\lesssim 2^{-n}\|\sum_j \mathcal{L}_{n-j}f\|_2^2 = 2^{-n}\|f\|_2^2.\end{aligned}$$

We write $\mathcal{T}_j^\ell \mathcal{L}_{n-j}\mathcal{A}_j^c f$ as

$$\mathcal{T}_j^\ell \mathcal{L}_{n-j}\mathcal{A}_j^c f(x_1,x_2) = \int \mathrm{e}^{2\pi i(x_1\xi_1+x_2\xi_2)} a_j(x_1,\xi_1,\xi_2)\hat{f}(\xi_1,\xi_2)\mathrm{d}\xi_1\mathrm{d}\xi_2,$$

with symbol $a_j(x_1,\xi_1,\xi_2)$ given by

$$\chi_{I_\ell}(x_1)\varphi\left(\frac{\xi_1}{2^{n-j}}\right)A_j^c(\xi_1,\xi_2)\int_{\mathbb{R}} \mathrm{e}^{-2\pi i(2^j t\xi_1 + m(x_1)\gamma(2^j t)\xi_2)}\varphi(t)\mathrm{d}t.$$

By Lemma 3.1, to prove (3.5), it suffices to show

$$\|\mathcal{R}_j^\lambda\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant c_1 2^{-c_2 n},$$

where c_1 and c_2 are constants independent of j and λ and $\mathcal{R}_j^\lambda g(x) := \int \mathrm{e}^{2\pi i x\xi} a_j(x,\xi,\lambda)\hat{g}(\xi)\mathrm{d}\xi$ for $g \in \mathcal{S}(\mathbb{R})$. Note that $x \in \mathbb{R}$ and $\xi \in \mathbb{R}$. Hereafter, we omit j and λ in operators for simplicity. Observe that we write $\mathcal{R}$ with kernel K

$$\begin{aligned}\mathcal{R}g(x) &= \int \mathrm{e}^{2\pi i x\xi} a_j(x,\xi,\lambda)\Big(\int \mathrm{e}^{-2\pi i y\xi} g(y)\mathrm{d}y\Big)\mathrm{d}\xi\\ &= \int K(x,y)g(y)\mathrm{d}y,\end{aligned}$$

where

$$K(x,y) := \chi_{I_\ell}(x)\int \mathrm{e}^{-2\pi i\lambda m(x)\gamma(2^j t)}\left(\int \mathrm{e}^{2\pi i(x-2^j t-y)\xi}\varphi\left(\frac{\xi_1}{2^{n-j}}\right)\widehat{A_j^c}(\xi,\lambda)d\xi\right)\varphi(t)\mathrm{d}t.$$

Recall that $|x| \sim 2^\ell \gg 2^j$ and denote

$$\begin{aligned}Q_k &:= \{x \in \mathbb{R} : 2^{\ell-1} + k\cdot 2^j \leqslant |x| < 2^{\ell-1} + (k+1)\cdot 2^j\},\\ Q_k' &:= \{x \in \mathbb{R} : 2^{\ell-1} + (k-4)\cdot 2^j \leqslant |x| < 2^{\ell-1} + (k+5)\cdot 2^j\},\end{aligned}$$

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for each integer k. We define the functions

$$G_k(x,y) := K(x,y)\chi_{Q_k}(x)\chi^c_{Q'_k}(y),$$
$$B_k(x,y) := K(x,y)\chi_{Q_k}(x)\chi_{Q'_k}(y)$$

and use them to split the operator $\mathcal{R}$ as

$$\begin{aligned}\mathcal{R}g(x) &= \sum_{k=0}^{3\cdot 2^{\ell-j-1}-1}\left(\int G_k(x,y)g(y)\mathrm{d}y + \int B_k(x,y)g(y)\mathrm{d}y\right)\\ &:= \sum_{k=0}^{3\cdot 2^{\ell-j-1}-1}\Big(\mathcal{G}_k g(x) + \mathcal{B}_k g(x)\Big).\end{aligned}$$

Then, we shall prove the following:

Lemma 3.2. *There exist constants C_1 and C_2 independent of j,ℓ and λ such that*

$$\left\|\sum_{k=0}^{3\cdot 2^{\ell-j-1}-1}\mathcal{G}_k\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant C_1 2^{-n}, \tag{3.6}$$

$$\left\|\sum_{k=0}^{3\cdot 2^{\ell-j-1}-1}\mathcal{B}_k\right\|_{L^2(\mathbb{R})\to L^2(\mathbb{R})} \leqslant C_2 2^{-\frac{n}{2}}. \tag{3.7}$$

Proof of (3.6). Recall that

$$K(x,y) := \int \mathrm{e}^{-2\pi i\lambda m(x)\gamma(2^j t)}\left(\int \mathrm{e}^{2\pi i(x-2^j t-y)\xi}\varphi\left(\frac{\xi}{2^{n-j}}\right)A^c_j(\xi,\lambda)\mathrm{d}\xi\right)\varphi(t)\mathrm{d}t.$$

We build our proof upon the following observation:

$$|G_k(x,y)| \lesssim \frac{2^j}{2^n|x-y|^2}\chi_{Q_k}(x)\psi^c\left(\frac{|x-y|}{2^j}\right). \tag{3.8}$$

□

Proof of (3.8). Note that $supp(\psi^c)\subset\left\{|x|>\frac{1}{2}\right\}$. We utilize the integration by parts twice with respect to ξ. Then, we get

$$\begin{aligned}\left|\int \mathrm{e}^{2\pi i(x-2^j t-y)\xi}\varphi_{n-j}(\xi)A^c_j(\xi,\lambda)\mathrm{d}\xi\right| &\lesssim \frac{1}{(x-2^j t-y)^2}\int\left|\partial^2_\xi\left[\varphi\left(\frac{\xi}{2^{n-j}}\right)A^c_j(\xi,\lambda)\right]\right|\mathrm{d}\xi\\ &\lesssim \frac{2^j}{2^n}\cdot\frac{1}{(x-2^j t-y)^2}.\end{aligned}$$

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Since $|x-2^jt-y| \gtrsim |x-y|$ on $x \in Q_k$, $y \in \mathbb{R} \setminus Q_k'$ for $\frac{1}{2} \leqslant t \leqslant 2$, we get the desired estimate. □

We shall deduce the following estimate:

$$\sum_{k=0}^{3\cdot 2^{\ell-j-1}-1} \left(\int |G_k(x,y)| \mathrm{d}x + \int |G_k(x,y)| \mathrm{d}y \right) \lesssim 2^{-n}. \tag{3.9}$$

Proof of (3.9). By estimate (3.8) and the disjointness of Q_ks, we have

$$\begin{aligned}\sum_{k=0}^{3\cdot 2^{\ell-j-1}-1} \int |G_k(x,y)| \mathrm{d}x &\lesssim \frac{2^j}{2^n} \sum_k \int_{|x-y|>2^j} \frac{\chi_{Q_k}(x)}{|x-y|^2} \mathrm{d}x \\ &\lesssim \frac{2^j}{2^n} \cdot \int_{|x|>2^j} \frac{1}{|x|^2} \mathrm{d}x = 2^{-n}.\end{aligned}$$

and the second estimate also holds by the similar way. □

By Schur's lemma with the estimate (3.9), we finish the proof of (3.6).

Proof of (3.7). For the operator $\mathcal{B}_k$, denote $g_k(y) = \chi_{Q_k'}(y)g(y)$. By the localization principle, we have

$$\left\| \sum_{k=0}^{3\cdot 2^{\ell-j-1}-1} \mathcal{B}_k \right\|_{L^2 \to L^2} \lesssim \sup_{k\in\mathbb{Z}} \left(\sup_{\|g_k\|_2=1} \|\mathcal{B}_k g_k\|_2 \right). \tag{3.10}$$

To estimate $\|\mathcal{B}_k g_k\|_2$, we write it with the symbol expression again, which is

$$\mathcal{B}_k g_k(x) = \int \mathrm{e}^{2\pi i x\xi} \chi_{Q_k}(x) a_j(x,\xi,\lambda) \widehat{g_k}(\xi) \mathrm{d}\xi,$$

where

$$a_j(x,\xi,\lambda) = \chi_{I_\ell}(x) \varphi\left(\frac{\xi}{2^{n-j}}\right) A_j^c(\xi,\lambda) \int \mathrm{e}^{-2\pi i(2^j t\xi + m(x_1)\gamma(2^j t)\lambda)} \varphi(t) \mathrm{d}t,$$

Observe that

$$|a_j(x,\xi,\lambda)| \lesssim \frac{1}{2^j|\xi|}. \tag{3.11}$$

□

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Proof of (3.11). From the support of $A_j^c(\xi,\lambda)$, we have $|\frac{\xi}{\lambda}| \not\approx |\gamma'(2^j t)|$ for $|t| \sim 1$. This enables us to apply the integration by parts with respect to variable t. Then, we get

$$\begin{aligned}
&\left|\int \mathrm{e}^{-2\pi i(\xi 2^j t+\lambda m(x)\gamma(2^j t))}\varphi(t)\mathrm{d}t\right| \\
&\lesssim \left|\int \mathrm{e}^{-2\pi i(\xi 2^j t+\lambda m(x)\gamma(2^j t))}\partial_t\left(\frac{\varphi(t)}{2^j(\xi+\lambda m(x)\gamma'(2^j t))}\right)\mathrm{d}t\right| \\
&\lesssim \int \frac{|2^j\lambda m(x)2^j\gamma''(2^j t)|}{\{2^j(\xi+\lambda m(x)\gamma'(2^j t))\}^2}\cdot\varphi(t)\mathrm{d}t+\int\frac{|\varphi'(t)|}{|2^j(\xi+\lambda m(x)\gamma'(2^j t))|}\mathrm{d}t \\
&\lesssim \int \frac{|\varphi'(t)|}{|2^j(\xi+\lambda m(x)\gamma'(2^j t))|}dt \lesssim \frac{1}{2^j|\xi|}.
\end{aligned}$$

Then, we get the desired estimate. □

From the observation (3.11), it is easy to check

$$\int |\chi_{Q_k}(x)a_j(x,\xi,\lambda)|\mathrm{d}x \lesssim 2^{-(n-j)},$$
$$\int |a_j(x,\xi,\lambda)|\mathrm{d}\xi \lesssim 2^{-j}.$$

By Schur's lemma with the above estimate and (3.10), we obtain (3.7) in Lemma 3.2.

3.4. A bootstrap argument for the proof of Theorem 1

In the spirit of Nagel, Stein and Wainger [22], we claim that

Lemma 3.3. *If* $\|\sup_j|\mathcal{T}_j^\ell f|\|_{L^p(\mathbb{R}^2)} \leqslant C_1\|f\|_{L^p(\mathbb{R}^2)}$ *and* $\|\mathcal{T}_j^\ell f\|_{L^r(\mathbb{R}^2)} \leqslant C_2\|f\|_{L^r(\mathbb{R}^2)}$ *for* $1<r<\infty$,

$$\left\|\left(\sum_j|\mathcal{T}_j^\ell f_j|^2\right)^{\frac{1}{2}}\right\|_{L^q(\mathbb{R}^2)} \leqslant (C_1C_2)^{\varepsilon q}\left\|\left(\sum_j|f_j|^2\right)^{\frac{1}{2}}\right\|_{L^q(\mathbb{R}^2)} \tag{3.12}$$

holds for all q *with* $\frac{1}{q}<\frac{1}{2}(1+\frac{1}{p})$.

Proof. Consider vector valued functions $\mathfrak{f}=\{f_j\}$ and $\mathfrak{T}\mathfrak{f}=\{\mathcal{T}_j^\ell f_j\}$. Since the operator $\mathcal{A}_j$ is a positive, it follows that $\|\mathfrak{T}\mathfrak{f}\|_{L^p(\mathbb{R}^2,l^\infty)} \lesssim \|\mathfrak{f}\|_{L^p(\mathbb{R}^2,l^\infty)}$ and $\|\mathfrak{T}\mathfrak{f}\|_{L^r(\mathbb{R}^2,l^r)} \lesssim \|\mathfrak{f}\|_{L^r(\mathbb{R}^2,l^r)}$ for r near 1. Applying the Riesz–Thorin interpolation for vector-valued function, we get the conclusion. □

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Combining (3.4), Proposition 2.3 and Proposition 3.1, we obtain the estimate

$$\Big\| \sup_{j\in\mathbb{Z}} |\mathcal{T}_j^{\ell} f| \Big\|_p \leqslant C_p \|f\|_p \tag{3.13}$$

for $p=2$. Moreover, we have

$$\|\mathcal{T}_j^{\ell} f\|_r \leqslant \|f\|_r \tag{3.14}$$

for $r>1$. By using Lemma 3.3 with (3.13) and (3.14), we obtain (3.12) for $\frac{4}{3} < p \leqslant 2$. Then, by setting $\{f_j\}_{j\in\mathbb{Z}} = \{\mathcal{A}_j^c \mathcal{L}_{n-j} f\}_{j\in\mathbb{Z}}$ in (3.12) and applying interpolation with the decay estimate (3.5), we obtain Proposition 3.1 for $\frac{4}{3} < p \leqslant 2$. To treat the bad part in (3.4), set $\{f_j\}_{j\in\mathbb{Z}} = \{\mathcal{A}_j \mathcal{L}_j^{\text{high}} f\}_{j\in\mathbb{Z}}$. Then, we apply Lemma 3.3 again to get the first inequality of (3.4), which implies (3.13) for $\frac{4}{3} < p \leqslant 2$. We can iteratively apply Lemma 3.3 with a wider range of p until we get (3.13) for all $p>1$. With this, we complete the proof of Main Theorem 1.

4. Application

In this section, we shall prove Corollary 1.1 and Corollary 1.2.

4.1. Proof of Corollary 1.1

For a measurable function $m:\mathbb{R}\to\mathbb{R}$, denote that

$$S_r^m f(x_1,x_2) = \frac{1}{2r}\int_{-r}^{r} f(x_1-t, x_2-m(x_1)\gamma(t))\mathrm{d}t,$$
$$\tilde{E}^k = \{(x_1,x_2)\in\mathbb{R}^2 : 2^k \leqslant m(x_1) \leqslant 2^{k+1}\}.$$

By Main Theorem 1 and the second part of Remark 1.1, one can easily check that

$$\| \sup_{r>0} |\chi_{\tilde{E}^k}(\cdot) S_r^m f| \|_p \lesssim \|f\|_p. \tag{4.1}$$

To prove Corollary 1.1, it suffices to show that for each $\alpha>0$ and $k\in\mathbb{Z}$, the set

$$E_\alpha^k = \left\{ (x_1,x_2)\in\tilde{E}^k : \limsup_{r\to 0} |S_r^m f(x_1,x_2) - f(x_1,x_2)| > 2\alpha \right\}$$

has measure zero. Consider a continuous function g_ε of compact support with $\|f - g_\varepsilon\|_p < \varepsilon$. One can see that $\limsup_{r\to 0} |S_r^m f(x_1,x_2) - f(x_1,x_2)| \leqslant \mathcal{M}_\gamma^m (f-g_\varepsilon)(x) + |g_\varepsilon(x) - f(x)|$. For F_α^k and G_α^k, defined by

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$$F_\alpha^k = \{x \in E_\alpha^k : \mathcal{M}_\gamma^m (f - g_\varepsilon)(x) > \alpha\},$$
$$G_\alpha^k = \{x \in E_\alpha^k : |f(x) - g_\varepsilon(x)| > \alpha\},$$

we have $m(E_\alpha^k) \leqslant m(F_\alpha^k) + m(G_\alpha^k)$. Applying estimate (4.1), we get

$$m(F_\alpha^k) + m(G_\alpha^k) \leqslant \frac{2\varepsilon^p}{\alpha^p}.$$

As $\varepsilon \to 0$, we get the conclusion.

4.2. Proof of Corollary 1.2

In order to achieve our goal of removing the dependence of the coefficients of polynomial P on factors other than its degree, we consider the following lemma.

Lemma 4.1. *Given a polynomial P with degree d, we can find a partition $\{s_0, s_1, s_2, \ldots, s_{n(d)}\}$ such that for each interval $[s_i, s_{i+1}]$, there exists a pair (m_i, s_{j_i}) with $1 \leqslant m_i \leqslant d$, satisfying*

$$\sup_{x \in [s_i, s_{i+1}]} \frac{|P(x)|}{|x - s_{j_i}|^{m_i}} \sim_d \inf_{x \in [s_i, s_{i+1}]} \frac{|P(x)|}{|x - s_{j_i}|^{m_i}}. \tag{4.2}$$

Proof of Lemma 4.1. We seek to construct a partition $\mathcal{P} = \{s_1, s_2, ..., s_{n(d)}\}$ of $(-\infty, \infty)$ such that, for each subinterval $[s_i, s_{i+1}]$, there exist non-negative integers m_i and j_i satisfying (4.2). Consider a polynomial $P(x)$ represented by the following expression:

$$P(x) = \prod_{i=1}^{d_1} (x - \alpha_i)^{q_i},$$

where α_i are distinct real numbers. Let $U_i = \{x \in \mathbb{R} : |x - \alpha_i| < |x - \alpha_k| \text{ for all } k = 1, \ldots, d_1\}$. For each i and k, let $\mathcal{U}_i^k(1) = \{x \in U_i : 2|x - \alpha_i| \geqslant |x - \alpha_k|\}$ and $\mathcal{U}_i^k(0) = \{x \in U_i : 2|x - \alpha_i| < |x - \alpha_k|\}$. Then, for any $x \in \mathbb{R}$, there exists an index i such that $x \in U_i$. We define the set-valued function F_i on $\{0,1\}^{d_1}$ by $F_i(a) = \bigcap_{k=1}^{d_1} \mathcal{U}_i^k(a_k)$ for $a = (a_k) \in \{0,1\}^{d_1}$. By using the set-valued function F, we can decompose each set U_i into a finite number of disjoint open intervals, that is,

$$U_i = \mathcal{U}_i^k(0) \cup \mathcal{U}_i^k(1) = \bigcap_{k=1}^{d_1} \left(\mathcal{U}_i^k(0) \cup \mathcal{U}_i^k(1) \right) = \bigcup_{a \in \{0,1\}^{d_1}} F_i(a).$$

For each interval $F_i(a) = [s_i, s_{i+1}]$, we take $m = \sum_{\{k : a_k = 1\}} q_k$ and $s_{j_i} = \alpha_i$. Observe that we have the following inequalities for each fixed i:

$$|x - \alpha_k| \sim |x - \alpha_i| \text{ for all } k \text{ such that } a_k = 1,$$
$$|x - \alpha_k| \sim |\alpha_i - \alpha_k| \text{ for all } k \text{ such that } a_k = 0.$$

By using these observation, we have (4.2) on $[s_i, s_{i+1}]$.

To handle a general polynomial, we can employ a similar approach. First, we can express the polynomial as

$$P(x) = \prod_{i=1}^{d_1}(x-\alpha_i)^{q_i}\prod_{i=1}^{d_2}\{(x-\beta_i)^2+\delta_i^2\}^{r_i}.$$

To treat this, we give one more criterion comparing between $2|x-\alpha_i|$ and $\max\{|x-\beta_k|, |\delta_k|\}$ instead of $|x-\alpha_k|$. Then. the last part can be proved similarly. □

Proof of the Corollory 1.2. Given a polynomial $P(x)$, we obtain a partition $\mathcal{P} = \{s_0, s_1, \ldots, s_{n(d)}\}$ from Lemma 4.1. We then decompose $\mathcal{M}_\gamma^P f(x)$ as

$$\mathcal{M}_\gamma^P f(x) = \sum_{i=0}^{n(d)} \chi_{[s_i,s_{i+1}]}(x)\mathcal{M}_i f(x),$$

where $\mathcal{M}_i f(x) := \chi_{[s_i,s_{i+1}]}(x)\mathcal{M}_\gamma^P f(x)$. To complete the proof, it suffices to demonstrate that

$$\|\mathcal{M}_i f\|_p \leqslant C_d\|f\|_p.$$

By Lemma 4.1, there exists a pair (m_i, s) such that the following holds for $[s_i, s_{i+1}]$:

$$\sup_{x_1\in[s_i,s_{i+1}]}\frac{|P(x_1)|}{|x_1-s|^{m_i}} \sim_d \inf_{x_1\in[s_i,s_{i+1}]}\frac{|P(x_1)|}{|x_1-s|^{m_i}}.$$

Denote that $g_s(x_1,x_2) := f(x_1+s, x_2)$ and consider the estimate

$$\|\mathcal{M}_i f\|_p^p = \int_{s_i-s}^{s_{i+1}-s}\int_{\mathbb{R}}\left(\sup_{r>0}\frac{1}{r}\int_0^r |g_s(x_1-t, x_2-P(x_1+s)\gamma(t))|dt\right)^p \mathrm{d}x_2\mathrm{d}x_1.$$

By applying Proposition 2.2, we can reduce matters to $|x_1| \sim 2^\ell$:

$$\left\|\sup_{j\in\mathbb{Z}}|\mathcal{P}_j^\ell g_s|\right\|_p \leqslant C_d\|f\|_p, \tag{4.3}$$

where $\mathcal{P}_j^\ell g_s(x)$ is defined as

$$\mathcal{P}_j^\ell g_s(x) := \chi_{I_\ell}(x_1)\psi_{j+4}^c(x_1)\int g_s(x_1-t, x_2-P(x_1+s)\gamma(t))\varphi_j(t)\mathrm{d}t,$$

for ℓ such that $[2^{\ell-1}, 2^{\ell+1}]\cap[s_i-s, s_{i+1}-s] \neq \emptyset$. To prove (4.3), it is enough to check the hypothesis of Remark 1.1:

$$\frac{\sup_x|P(x+s)|}{\inf_x|P(x+s)|} \lesssim_d \frac{2^{(\ell+1)m_i}}{2^{(\ell-1)m_i}} \lesssim_d 1 \text{ for } |x|\in[2^{\ell-1}, 2^{\ell+1}],$$

where $1 \leqslant m_i \leqslant d$. This implies the conclusion. □

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