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Computing Noncommutative Deformations of Presheaves and Sheaves of Modules

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Abstract. We describe a noncommutative deformation theory for presheaves and sheaves of modules that generalizes the commutative deformation theory of these global algebraic structures and the noncommutative deformation theory of modules over algebras due to Laudal.

In the first part of the paper, we describe a noncommutative deformation functor for presheaves of modules on a small category and an obstruction theory for this functor in terms of global Hochschild cohomology. An important feature of this obstruction theory is that it can be computed in concrete terms in many interesting cases.

In the last part of the paper, we describe a noncommutative deformation functor for quasi-coherent sheaves of modules on a ringed space $(X, \mathcal{A})$. We show that for any good $\mathcal{A}$-affine open cover U of X, the forgetful functor $\mathsf{QCoh}\,\mathcal{A} \to \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ induces an isomorphism of noncommutative deformation functors.

Applications. We consider noncommutative deformations of quasi-coherent $\mathcal{A}$-modules on X when $(X, \mathcal{A}) = (X, \mathcal{O}_X)$ is a scheme or $(X, \mathcal{A}) = (X, \mathcal{D})$ is a D-scheme in the sense of Beilinson and Bernstein. In these cases, we may use any open affine cover of X closed under finite intersections to compute noncommutative deformations in concrete terms using presheaf methods. We compute the noncommutative deformations of the left $\mathcal{D}_X$-module $\mathcal{D}_X$ when X is an elliptic curve as an example.

Introduction

Deformation theory was formalized by Grothendieck in the language of schemes in the 1950s, and is described in the series of Bourbaki seminar expositions *Fondements de la géométrie algébrique* [8]; see in particular Grothendieck [10, 11]. The general philosophy is best described by the following quotation:

> La méthode générale consiste toujours à faire des constructions formelles, ce qui consiste essentiellement à faire de la géométrie algébrique sur un anneau artinien, et à en tirer des conclusions de nature "algébrique" en utilisant les trois théorèmes fondamentaux (Grothendieck [10, p. 11]).

We shall follow Grothendieck's philosophy closely, and we are therefore led to the study of functors of (noncommutative) Artin rings.

Let k be an algebraically closed field, and let l denote the category of local Artinian commutative k-algebras with residue field k, with local homomorphisms. A *functor of Artin rings* is a covariant functor $\mathrm{D}\colon \mathsf{l} \to \mathsf{Sets}$ such that $\mathrm{D}(k)$ only contains one

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element. In Schlessinger [17], criteria were given for functors of Artin rings to have a pro-representable hull, respectively to be pro-representable.

Let A be an Abelian k-category, and let X be an object of A. The flat deformation functor $\mathsf{Def}_X \colon \mathsf{l} \to \mathsf{Sets}$ of X in A is a functor of Artin rings. In many cases, it has a pro-representing hull $\mathsf{H}(\mathsf{Def}_X)$ (see [13, 17]), and there are constructive methods for finding $\mathsf{H}(\mathsf{Def}_X)$ (see Laudal [13, 14]). In fact, if there exists an obstruction theory for Def_X with finite dimensional cohomology $\mathsf{H}^n(X)$ for $n = 1, 2$, then Def_X has a pro-representing hull, given algorithmically in terms of the vector spaces $\mathsf{H}^n(X)$ for $n = 1, 2$ and certain generalized symmetric Massey products on them.

When $\mathsf{A} = \mathsf{Mod}(A)$, the category of left modules over an associative k-algebra A, Laudal introduced a generalization of the deformation functor $\mathsf{Def}_M \colon \mathsf{l} \to \mathsf{Sets}$ for a left A-module M in [15]. He considered the category a_p of p-pointed Artinian rings for any integer $p \geq 1$ and constructed a noncommutative deformation functor $\mathsf{Def}_{\mathcal{M}} \colon \mathsf{a}_p \to \mathsf{Sets}$ for any finite family $\mathcal{M} = \{M_1, \dots, M_p\}$ of left A-modules. This deformation functor has an obstruction theory with cohomology $(\mathrm{Ext}^n_A(M_j, M_i))$ and a pro-representing hull $\mathsf{H}(\mathsf{Def}_{\mathcal{M}})$ given algorithmically in terms of the vector spaces $(\mathrm{Ext}^n_A(M_j, M_i))$ and certain generalized Massey products on them.

The objects in category a_p are Artinian rings R, together with ring homomorphisms $k^p \to R \to k^p$ such that the composition is the identity and such that R is I-adic complete for $I = \ker(R \to k^p)$. The morphisms are the natural commutative diagrams. In Section 1, we give a systematic introduction to functors $\mathsf{D} \colon \mathsf{a}_p \to \mathsf{Sets}$ of noncommutative Artin rings. This notion generalizes the notion of noncommutative deformation functors $\mathsf{Def}_{\mathcal{M}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of families of modules introduced in Laudal [15].

The idea is that there is a noncommutative deformation functor $\mathsf{Def}_X \colon \mathsf{a}_p \to \mathsf{Sets}$ for any finite family $X = \{X_1, \dots, X_p\}$ of algebraic or algebro-geometric objects. The restrictions of Def_X along the p natural full embeddings of categories $\mathsf{a}_1 \subseteq \mathsf{a}_p$ are the noncommutative deformation functors $\mathsf{Def}_{X_i} \colon \mathsf{a}_1 \to \mathsf{Sets}$, and the restriction of Def_{X_i} to $\mathsf{l} \subseteq \mathsf{a}_1$ is the commutative deformation functor $\mathsf{Def}^c_{X_i} \colon \mathsf{l} \to \mathsf{Sets}$. When these deformation functors have pro-representing hulls, we show that

$$\mathsf{H}(\mathsf{Def}_X)^{\mathrm{comm}} \cong \bigoplus_{1 \leq i \leq p} \mathsf{H}(\mathsf{Def}_{X_i})^{\mathrm{comm}} \cong \bigoplus_{1 \leq i \leq p} \mathsf{H}(\mathsf{Def}^c_{X_i}).$$

We remark that the hull $\mathsf{H}(\mathsf{Def}_X)$ *is not* isomorphic to $\bigoplus_i \mathsf{H}(\mathsf{Def}_{X_i})$ in general. We also remark that the noncommutative deformation functor $\mathsf{Def}_X \colon \mathsf{a}_p \to \mathsf{Sets}$ of a family $X = \{X_1, \dots, X_p\}$ *is not* the same as the noncommutative deformation functor $\mathsf{Def}_X \colon \mathsf{a}_1 \to \mathsf{Sets}$ of the direct sum $X = X_1 \oplus \cdots \oplus X_p$.

Let $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$ be the category of presheaves of left $\mathcal{A}$-modules on c, where c is a small category and $\mathcal{A}$ is a k-algebra of presheaves on c. We consider an Abelian k-category A such that $\mathsf{A} \subseteq \mathsf{PreSh}(\mathsf{c}, \mathcal{A})$ is a full subcategory, and in Section 2 we construct a noncommutative deformation functor $\mathsf{Def}^{\mathsf{A}}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ for any finite family $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ of objects in A. We remark that we use a notion of matric freeness, introduced by Laudal, to replace flatness in the definition of the deformation functor. It is not clear if these notions are equivalent if $p \geq 2$.

In Sections 3–5, we consider deformations in the category $\mathsf{A} = \mathsf{PreSh}(\mathsf{c}, \mathcal{A})$ of presheaves. We describe the noncommutative deformation functor $\mathsf{Def}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$

of any finite family $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ of presheaves of left $\mathcal{A}$-modules on c in concrete terms in Section 3, and use this to develop an obstruction theory for $\mathsf{Def}_{\mathcal{F}}$ with cohomology $(\mathsf{HH}^n(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)))$ in Section 5. The global Hochschild cohomology $\mathsf{HH}^n(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i))$ of $\mathcal{A}$ with values in $\mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)$ on c is described in detail in Section 4.

Theorem *Let c be a small category, $\mathcal{A}$ a presheaf of k-algebras on c, and $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ a finite family of presheaves of left $\mathcal{A}$-modules on c. If*

$$\dim_k \mathsf{HH}^n(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) < \infty \text{ for } 1 \le i, j \le p, \ n = 1, 2,$$

then the noncommutative deformation functor $\mathsf{Def}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ in $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$ has a pro-representing hull $\mathsf{H}(\mathsf{Def}_{\mathcal{F}})$, completely determined by the k-linear spaces

$$\mathsf{HH}^n(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i))$$

for $1 \le i, j \le p$, $n = 1, 2$, together with some generalized Massey products on them.

In Section 6, we consider deformations in the category $\mathsf{A} = \mathsf{QCoh}(\mathcal{A})$ of quasi-coherent sheaves of left $\mathcal{A}$-modules on X, where $(X, \mathcal{A})$ is a ringed space over k. We show that if U is a good $\mathcal{A}$-affine open cover of X, then the natural forgetful functor $\pi \colon \mathsf{QCoh}(\mathcal{A}) \to \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ identifies $\mathsf{QCoh}(\mathcal{A})$ with an exact Abelian subcategory of $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ that is closed under extensions, and that π induces an isomorphism $\mathsf{Def}^{qc}_{\mathcal{F}} \to \mathsf{Def}^{\mathsf{U}}_{\mathcal{F}}$ of deformation functors for any finite family $\mathcal{F}$ in $\mathsf{QCoh}(\mathcal{A})$.

Theorem *Let $(X, \mathcal{A})$ be a ringed space over k, U a good $\mathcal{A}$-affine open cover of X, and $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ a finite family of quasi-coherent left $\mathcal{A}$-modules on X. If*

$$\dim_k \mathsf{HH}^n(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) < \infty \text{ for } 1 \le i, j \le p, \ n = 1, 2,$$

then the noncommutative deformation functor $\mathsf{Def}^{qc}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ in $\mathsf{QCoh}(\mathcal{A})$ has a pro-representing hull $\mathsf{H}(\mathsf{Def}^{qc}_{\mathcal{F}})$, completely determined by the k-linear spaces $\mathsf{HH}^n(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i))$ for $1 \le i, j \le p$, $n = 1, 2$, together with some generalized Massey products on them.

We give examples of ringed spaces $(X, \mathcal{A})$ that admit good $\mathcal{A}$-affine open covers in Section 7. The main commutative examples are schemes $(X, \mathcal{O}_X)$ over k. The main noncommutative examples are D-schemes $(X, \mathcal{D})$ over k in the sense of Beilinson–Bernstein [2]. In particular, important examples of D-schemes over an algebraically closed field k of characteristic 0 include $(X, \mathcal{D}_X)$, where X is a locally Noetherian scheme over k and $\mathcal{D}_X$ is the sheaf of k-linear differential operators on X, and $(X, \mathrm{U}(\mathbf{g}))$, where X is a separated scheme of finite type over k and $\mathrm{U}(\mathbf{g})$ is the universal enveloping D-algebra of a Lie algebroid $\mathbf{g}$ on X/k.

Let us consider one of the Abelian categories $\mathsf{A} = \mathsf{PreSh}(\mathsf{c}, \mathcal{A})$, or if X has a good $\mathcal{A}$-affine open cover, $\mathsf{A} = \mathsf{QCoh}(\mathcal{A})$. We expect that the noncommutative deformation functor $\mathsf{Def}^{\mathsf{A}}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ in A is controlled by $(\mathrm{Ext}^n_{\mathsf{A}}(\mathcal{F}_j, \mathcal{F}_i))$ in both these cases. In fact, we show that $t(\mathsf{Def}^{\mathsf{A}}_{\mathcal{F}})_{ij} \cong \mathrm{Ext}^1_{\mathsf{A}}(\mathcal{F}_j, \mathcal{F}_i)$, (see Proposition

5.2 and Proposition 6.3). It might be possible to develop an obstruction theory for $\mathrm{Def}^{\mathsf{A}}_{\mathcal{F}}$ with cohomology $(\mathrm{Ext}^n_{\mathsf{A}}(\mathcal{F}_j, \mathcal{F}_i))$ in both these cases.

However, notice that when $\mathsf{A} = \mathsf{QCoh}(\mathcal{A})$ for a noncommutative sheaf of algebras $\mathcal{A}$, it is often very hard to compute $\mathrm{Ext}^n_{\mathsf{A}}(\mathcal{F}_j, \mathcal{F}_i)$. For instance, localization of injectives can behave badly, even when $(X, \mathcal{A})$ is a D-scheme. On the other hand, the global Hochschild cohomology groups $\mathsf{HH}^n(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i))$ can be computed in concrete terms in many cases of interest.

In Section 8, we give an example of this. Let X be any elliptic curve over an algebraically closed field k of characteristic 0, and consider $\mathcal{O}_X$ as a left $\mathcal{D}_X$-module on X. We show that $\mathsf{HH}^0(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k$, $\mathsf{HH}^1(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k^2$, and $\mathsf{HH}^2(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k$ for an open affine cover U of X closed under intersections. Using these results and the obstruction calculus, we compute the pro-representing hull $\mathsf{H}(\mathrm{Def}_{\mathcal{O}_X}) \cong k\langle\langle t_1, t_2\rangle\rangle/(t_1t_2 - t_2t_1)$ of the noncommutative deformation functor $\mathrm{Def}_{\mathcal{O}_X}$ and we also compute its versal family. In this example, it seems hard to compute $\mathrm{Ext}^n_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)$ for $n = 1, 2$ in other ways.

Noncommutative deformation theory has applications to representation theory. In Laudal [15], it was shown that noncommutative deformations of modules are closely related to iterated extensions in module categories, and we used this result to study finite length categories of modules in Eriksen [4]. These methods work in any Abelian k-category with a reasonable noncommutative deformation theory.

1 Functors of Noncommutative Artin Rings

Let k be an algebraically closed field. We shall define the category a_p of *p-pointed noncommutative Artin rings* for any integer $p \geq 1$. For expository purposes, we first define A_p, the category of p-pointed algebras. An object of A_p is an associative ring R, together with structural ring homomorphisms $f\colon k^p \to R$ and $g\colon R \to k^p$ such that $g \circ f = \mathrm{id}$, and a morphism $u\colon (R, f, g) \to (R', f', g')$ in A_p is ring homomorphism $u\colon R \to R'$ such that $u \circ f = f'$ and $g' \circ u = g$.

We denote by $I = I(R)$ the ideal $I = \ker(g)$ for any $R \in \mathsf{A}_p$, and call it the radical ideal of R. The category a_p is the full subcategory of A_p consisting of objects $R \in \mathsf{A}_p$ such that R is Artinian and (separated) complete in the I-adic topology. For any integer $n \geq 1$, $\mathsf{a}_p(n)$ is the full subcategory of a_p consisting of objects $R \in \mathsf{a}_p$ such that $I^n = 0$. The pro-category $\hat{\mathsf{a}}_p$ is the full subcategory of A_p consisting of objects $R \in \mathsf{A}_p$ such that $R_n = R/I^n$ is Artinian for all $n \geq 1$ and R is (separated) complete in the I-adic topology. It follows that $\mathsf{a}_p \subseteq \hat{\mathsf{a}}_p$.

For any $R \in \mathsf{A}_p$, $R \in \mathsf{a}_p$ if and only if $\dim_k R$ is finite and $I = I(R)$ is nilpotent. If this is the case, then I is the Jacobson radical of R, and there are p isomorphism classes of simple left R-modules, all of dimension 1 over k.

For any object $R \in \mathsf{A}_p$, we write $e_1, \dots, e_p$ for the indecomposable idempotents in k^p and $R_{ij} = e_i R e_j$. Note that R is a matrix ring in the sense that there is a k-linear isomorphism $R \cong (R_{ij}) = \bigoplus R_{ij}$, and multiplication in R corresponds to matric multiplication in (R_{ij}). In what follows, we shall denote the direct sum of any family $\{V_{ij} : 1 \leq i, j \leq p\}$ of k-linear vector spaces by (V_{ij}).

We define a *functor of (p-pointed) noncommutative Artin rings* to be a covariant

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functor $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ such that $\mathsf{D}(k^p) = \{*\}$ is reduced to one element. It follows that there is a distinguished element $*_R \in \mathsf{D}(R)$ given by $\mathsf{D}(k^p \to R)(*)$ for any $R \in \mathsf{a}_p$. We see that $*_R \in \mathsf{D}(R)$ is a lifting of $* \in \mathsf{D}(k^p)$ to R, and call it the *trivial lifting*.

There is a natural extension of $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ to the pro-category $\hat{\mathsf{a}}_p$, which we denote by $\mathsf{D}\colon \hat{\mathsf{a}}_p \to \mathsf{Sets}$. For any $R \in \hat{\mathsf{a}}_p$, it is given by $\mathsf{D}(R) = \varprojlim \mathsf{D}(R_n)$. A pro-couple for $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ is a pair (H, ξ) with $H \in \hat{\mathsf{a}}_p$ and $\xi \in \mathsf{D}(H)$, and a morphism $u\colon (H,\xi) \to (H',\xi')$ of pro-couples is a morphism $u\colon H \to H'$ in $\hat{\mathsf{a}}_p$ such that $\mathsf{D}(u)(\xi) = \xi'$. By Yoneda's lemma, $\xi \in \mathsf{D}(H)$ corresponds to a morphism $\phi\colon \operatorname{Mor}(H,-) \to \mathsf{D}$ of functors on a_p. We say that (H,ξ) *pro-represents* D if ϕ is an isomorphism of functors on a_p, and that (H,ξ) is a *pro-representing hull* of D if $\phi\colon \operatorname{Mor}(H,-) \to \mathsf{D}$ is a smooth morphism of functors on a_p that induces an isomorphism of functors on $\mathsf{a}_p(2)$ by restriction.

Lemma 1.1 *Let $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ be a functor of noncommutative Artin rings. If D has a pro-representing hull, then it is unique up to a (non-canonical) isomorphism of pro-couples.*

Proof Let (H,ξ) and (H',ξ') be pro-representing hulls of D, and let ϕ, ϕ' be the corresponding morphisms of functors on a_p. By the smoothness of ϕ, ϕ', it follows that $\phi_{H'}$ and ϕ'_H are surjective. Hence there are morphisms $u\colon (H,\xi) \to (H',\xi')$ and $v\colon (H',\xi') \to (H,\xi)$ of pro-couples. Restriction to $\mathsf{a}_p(2)$ gives morphisms $u_2\colon (H_2,\xi_2) \to (H'_2,\xi'_2)$ and $v_2\colon (H'_2,\xi'_2) \to (H_2,\xi_2)$. But both (H_2,ξ_2) and (H'_2,ξ'_2) represent the restriction of D to $\mathsf{a}_p(2)$, so u_2 and v_2 are mutual inverses. Let us write $\operatorname{gr}_n(R) = I(R)^n/I(R)^{n+1}$ for all $R \in \hat{\mathsf{a}}_p$ and all $n \geq 1$. By the above argument, it follows that $\operatorname{gr}_1(u)$ and $\operatorname{gr}_1(v)$ are mutual inverses. In particular, $\operatorname{gr}_1(u \circ v) = \operatorname{gr}_1(u) \circ \operatorname{gr}_1(v)$ is surjective. This implies that $\operatorname{gr}_n(u \circ v)$ is a surjective endomorphism of the finite dimensional vector space $\operatorname{gr}_n(H')$ for all $n \geq 1$, and hence an automorphism of $\operatorname{gr}_n(H')$ for all $n \geq 1$. So $u \circ v$ is an automorphism, and by a symmetric argument, $v \circ u$ is an automorphism as well. It follows that u and v are isomorphisms of pro-couples. ∎

For $1 \leq i,j \leq p$, let $k^p[\epsilon_{ij}]$ be the object in $\mathsf{a}_p(2)$ defined by $k^p[\epsilon_{ij}] = k^p + k \cdot \epsilon_{ij}$, with $\epsilon_{ij} = e_i\, \epsilon_{ij}\, e_j$ and $\epsilon_{ij}^2 = 0$. We define the *tangent space* of $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ to be $t(\mathsf{D}) = (t(\mathsf{D})_{ij})$ with $t(\mathsf{D})_{ij} = \mathsf{D}(k^p[\epsilon_{ij}])$ for $1 \leq i,j \leq p$. Note that if (H,ξ) is a pro-representing hull for D, then ξ induces a bijection $t(\mathsf{D})_{ij} \cong I(H)_{ij}/I(H)^2_{ij}$. In particular, $t(\mathsf{D})_{ij}$ has a canonical k-linear structure in this case.

A *small surjection* in a_p is a surjective morphism $u\colon R \to S$ in a_p such that $KI = IK = 0$, where $I = I(R)$ and $K = \ker(u)$. Given a functor $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ of noncommutative Artin rings, a small lifting situation for D is defined by a small surjection $u\colon R \to S$ in a_p and an element $\xi_S \in \mathsf{D}(S)$. In order to study the existence of, and ultimately construct, a pro-representing hull H for D, we are led to consider the possible liftings of ξ_S to R in small lifting situations.

Let $\{\mathsf{H}^n_{ij} : 1 \leq i,j \leq p\}$ be a family of vector spaces over k for $n = 1,2$. We say that a functor $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ of noncommutative Artin rings has an *obstruction theory* with cohomology (H^n_{ij}) if the following conditions hold:

(i) For any small lifting situation, given by a small surjection $u\colon R \to S$ in a_p with kernel $K = \ker(u)$ and an element $\xi_S \in \mathsf{D}(S)$, we have:

(a) There exists a canonical obstruction $o(u, \xi_S) \in (\mathsf{H}^2_{ij} \otimes_k K_{ij})$ such that $o(u, \xi_S) = 0$ if and only if there exists a lifting of ξ_S to R,

(b) If $o(u, \xi_S) = 0$, there is an transitive and effective action of $(\mathsf{H}^1_{ij} \otimes_k K_{ij})$ on the set of liftings of ξ_S to R.

(ii) Let $u_i\colon R_i \to S_i$ be a small surjection with kernel $K_i = \ker(u_i)$ and let $\xi_i \in \mathsf{D}(S_i)$ for $i = 1, 2$. If $\alpha\colon R_1 \to R_2$ and $\beta\colon S_1 \to S_2$ are morphisms in a_p such that $u_2 \circ \alpha = \beta \circ u_1$ and $\mathsf{D}(\beta)(\xi_1) = \xi_2$, then $\alpha^*(o(u_1, \xi_1)) = o(u_2, \xi_2)$, where $\alpha^*\colon (\mathsf{H}^2_{ij} \otimes_k K_{1,ij}) \to (\mathsf{H}^2_{ij} \otimes_k K_{2,ij})$ is the natural map induced by α.

Moreover, if H^n_{ij} has finite k-dimension for $1 \le i, j \le p$, $n = 1, 2$, then we say that D has an obstruction theory with *finite dimensional cohomology* (H^n_{ij}).

In the rest of this section, we shall assume that $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ is a functor of noncommutative Artin rings that has an obstruction theory with finite dimensional cohomology (H^n_{ij}). Note that for any object $R \in \mathsf{a}_p(2)$, the morphism $R \to k^p$ is a small surjection. This implies that there is a canonical set-theoretical bijection

$$(\mathsf{H}^1_{ij} \otimes_k I(R)_{ij}) \cong \mathsf{D}(R),$$

given by the trivial lifting $*_R \in \mathsf{D}(R)$. In particular, there is a set-theoretical bijection between H^1_{ij} and $t(\mathsf{D})_{ij} = \mathsf{D}(k^p[\epsilon_{ij}])$ for $1 \le i, j \le p$.

We define T^n to be the free, formal matrix ring in $\hat{\mathsf{a}}_p$ generated by the k-linear vector spaces $\{(\mathsf{H}^n_{ij})^* : 1 \le i, j \le p\}$ for $n = 1, 2$, where $(\mathsf{H}^n_{ij})^* = \mathrm{Hom}_k(\mathsf{H}^n_{ij}, k)$. For any $R \in \mathsf{a}_p(2)$, we have natural isomorphisms

$$\mathsf{D}(R) \cong (\mathsf{H}^1_{ij} \otimes_k I(R)_{ij}) \cong (\mathrm{Hom}_k((\mathsf{H}^1_{ij})^*, I(R)_{ij})) \cong \mathrm{Mor}(T^1_2, R),$$

where $T^1_2 = T^1/I(T^1)^2$. It follows that there is an isomorphism $\mathrm{Mor}(T^1_2, -) \to \mathsf{D}$ of functors on $\mathsf{a}_p(2)$, *i.e.,* the restriction of D to $\mathsf{a}_p(2)$ is represented by (T^1_2, ξ_2) for some $\xi_2 \in \mathsf{D}(T^1_2)$.

Theorem 1.2 *Let* $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ *be a functor of noncommutative Artin rings. If* D *has an obstruction theory with finite dimensional cohomology, then there is an obstruction morphism* $o\colon T^2 \to T^1$ *in* $\hat{\mathsf{a}}_p$ *such that* $H = T^1 \widehat{\otimes}_{T^2} k^p$ *is a pro-representing hull of* D.

Proof Let us write $I = I(T^1)$, $T^1_n = T^1/I^n$ and $t_n\colon T^1_{n+1} \to T^1_n$ for the natural morphism for all $n \ge 1$. Let $a_2 = I^2$ and $H_2 = T^1/a_2 = T^1_2$. Then the restriction of D to $\mathsf{a}_p(2)$ is represented by (H_2, ξ_2) and $H_2 \cong T^1_2 \otimes_{T^2} k^p$. Using o_2 and ξ_2 as a starting point, we shall construct o_{n+1} and ξ_{n+1} for $n \ge 2$ inductively. So let $n \ge 2$, and assume that the morphism $o_n\colon T^2 \to T^1_n$ and the deformation $\xi_n \in \mathsf{D}(H_n)$ are given, with $H_n = T^1_n \otimes_{T^2} k^p$. We may assume that $t_{n-1} \circ o_n = o_{n-1}$ and that ξ_n is a lifting of ξ_{n-1}.

Let us first construct the morphism $o_{n+1}\colon T^2 \to T^1_{n+1}$. We define a'_n to be the ideal in T^1_n generated by $o_n(I(T^2))$. Then $a'_n = a_n/I^n$ for an ideal $a_n \subseteq T^1$ with $I^n \subseteq a_n$, and $H_n \cong T^1/a_n$. Let $b_n = Ia_n + a_nI$, then we obtain the following commutative

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diagram:

$$
\begin{array}{ccccc}
T^2 & & T^1_{n+1} & \longrightarrow & T^1/b_n \\
 & \searrow^{o_n} & \downarrow & & \downarrow \\
 & & T^1_n & \longrightarrow & H_n = T^1/a_n,
\end{array}
$$

There is an obstruction $o'_{n+1} = o(T^1/b_n \to H_n, \xi_n)$ for lifting ξ_n to T^1/b_n since $T^1/b_n \to T^1/a_n$ is a small surjection, hence a morphism $o'_{n+1} \colon T^2 \to T^1/b_n$. Let a''_{n+1} be the ideal in T^1/b_n generated by $o'_{n+1}(I(T^2))$. Then $a''_{n+1} = a_{n+1}/b_n$ for an ideal $a_{n+1} \subseteq T^1$ with $b_n \subseteq a_{n+1} \subseteq a_n$. Let $H_{n+1} = T^1/a_{n+1}$, then we obtain the following commutative diagram:

$$
\begin{array}{ccccccc}
T^2 & \overset{o'_{n+1}}{\curvearrowright} & T^1_{n+1} & \longrightarrow & T^1/b_n & \longrightarrow & H_{n+1} = T^1/a_{n+1} \\
 & \searrow^{o_n} & \downarrow & & \downarrow & \swarrow & \\
 & & T^1_n & \longrightarrow & H_n = T^1/a_n & &
\end{array}
$$

By the choice of a_{n+1}, the obstruction for lifting ξ_n to H_{n+1} is zero. We can therefore find a lifting $\xi_{n+1} \in \mathrm{D}(H_{n+1})$ of ξ_n to H_{n+1}.

We claim that there is a morphism $o_{n+1} \colon T^2 \to T^1_{n+1}$ that commutes with o'_{n+1} and o_n. Note that $a_{n-1} = I^{n-1} + a_n$ since $t_{n-1} \circ o_n = o_{n-1}$. For simplicity, we write $O(K) = (\mathrm{Hom}_k(\mathrm{gr}_1(T^2)_{ij}, K_{ij}))$ for any family $K = (K_{ij})$ of vector spaces over k. The following diagram of k-vector spaces is commutative with exact columns:

$$
\begin{array}{ccc}
0 & & 0 \\
\downarrow & & \downarrow \\
O(b_n/I^{n+1}) & \xrightarrow{j_n} & O(b_{n-1}/I^n) \\
\downarrow & & \downarrow \\
O(a_n/I^{n+1}) & \xrightarrow{k_n} & O(a_{n-1}/I^n) \\
\downarrow{\scriptstyle r_{n+1}} & & \downarrow{\scriptstyle r_n} \\
O(a_n/b_n) & \xrightarrow{l_n} & O(a_{n-1}/b_{n-1}) \\
\downarrow & & \downarrow \\
0 & & 0
\end{array}
$$

We may consider o_n as an element in $O(a_{n-1}/I^n)$, and $o'_{n+1} \in O(a_n/b_n)$. Since o'_n commutes with o'_{n+1} and o_n, we get $l_n(o'_{n+1}) = r_n(o_n)$. To prove the claim, it is enough

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to find an element $o_{n+1} \in O(a_n/I^{n+1})$ such that $r_{n+1}(o_{n+1}) = o'_{n+1}$ and $k_n(o_{n+1}) = o_n$. Since $o_n(I(T^2)) \subseteq a_n$, there is an element $\overline{o}_{n+1} \in O(a_n/I^{n+1})$ such that $k_n(\overline{o}_{n+1}) = o_n$. But $a_{n-1} = a_n + I^{n-1}$ implies that j_n is surjective, so the claim follows from the snake lemma. In particular, $T^1_{n+1} \otimes_{T^2} k^p \cong H_{n+1}$ when the tensor product is taken over o_{n+1}.

By induction, we find a morphism $o_n\colon T^2 \to T^1_n$ and an element $\xi_n \in \mathsf{D}(H_n)$ for all integers $n \geq 1$, with $H_n = T^1_n \otimes_{T^2} k^p$. Using the universal property of the projective limit, we obtain a morphism $o\colon T^2 \to T^1$ in $\hat{\mathsf{a}}_p$ and an element $\xi \in \mathsf{D}(H)$, with $H = T^1 \hat{\otimes}_{T^2} k^p$. We claim that (H, ξ) is a pro-representable hull for D.

Clearly, it is enough to prove that (H_n, ξ_n) is a pro-representing hull for the restriction of D to $\mathsf{a}_p(n)$ for all $n \geq 3$. So let $\phi_n\colon \operatorname{Mor}(H_n, -) \to \mathsf{D}$ be the morphism of functors on $\mathsf{a}_p(n)$ corresponding to ξ_n for some $n \geq 3$. We shall prove that ϕ_n is a smooth morphism. Let $u\colon R \to S$ be a small surjection in $\mathsf{a}_p(n)$ with kernel K, let $E_R \in \mathsf{D}(R)$ and $v \in \operatorname{Mor}(H_n, S)$ be elements such that $\mathsf{D}(u)(E_R) = \mathsf{D}(v)(\xi_n) = E_S$, and consider the following commutative diagram:

$$\begin{array}{ccccc} T^1 & \longrightarrow & H_{n+1} & & R \\ & \searrow & \downarrow & & \downarrow{\scriptstyle u} \\ & & H_n & \xrightarrow[v]{} & S \end{array}$$

We can find a morphism $v'\colon T^1 \to R$ that makes the diagram commutative. This implies that $v'(a_n) \subseteq K$, and since u is small, that $v'(b_n) = 0$. But the induced map $T^1/b_n \to R$ maps the obstruction o'_{n+1} to $o(u, E_S) = 0$. It follows that $v'(a_{n+1}) = 0$, hence v' induces a morphism $v'\colon H_{n+1} \to R$ making the diagram commutative. Since $v'(I(H_{n+1})^n) = 0$, we may consider v' as a map from $H_{n+1}/I(H_{n+1})^n \cong H_n$. This proves that there is a morphism $v'\colon H_n \to R$ such that $u \circ v' = v$.

Let $E'_R = \mathsf{D}(v')(\xi_n)$. Then E'_R is a lifting of E_S to R, and the difference between E_R and E'_R is given by an element $d \in (\mathsf{H}^1_{ij} \otimes_k K_{ij}) = (\operatorname{Hom}_k(\operatorname{gr}_1(T^1)_{ij}, K_{ij}))$. Let $v''\colon T^1 \to R$ be the morphism given by $v''(x_{ij}(l)) = v'(x_{ij}(l)) + d(\overline{x_{ij}(l)})$, where $\{x_{ij}(l) : 1 \leq l \leq d_{ij}\}$ is a basis for H^1_{ij} for $1 \leq i, j \leq p$. Since $a_{n+1} \subseteq I^2$ and u is small, $v''(a_{n+1}) \subseteq v'(a_{n+1}) + I(R)K + KI(R) + K^2 = v'(a_{n+1}) = 0$. This implies that v'' induces a morphism $v''\colon H_n \to R$. By construction, $u \circ v'' = u \circ v' = v$ and $\mathsf{D}(v'')(\xi_n) = E_R$, and this proves that ϕ_n is smooth. ■

We remark that a more general version of Theorem 1.2 can be proved if D has an obstruction theory with cohomology (H^n_{ij}) and H^1_{ij} has a countable k-basis for $1 \leq i, j \leq p$, using the methods of Laudal [13].

Corollary 1.3 *Let* $\mathsf{D}\colon \mathsf{a}_p \to$ Sets *be a functor of noncommutative Artin rings. If* D *has an obstruction theory with finite dimensional cohomology* (H^n_{ij})*, then there is a k-linear isomorphism* $\mathsf{H}^1_{ij} \cong t(\mathsf{D})_{ij}$ *for* $1 \leq i, j \leq p$*.*

Corollary 1.4 *Let* $\mathsf{D}\colon \mathsf{a}_p \to$ Sets *be a functor of noncommutative Artin rings. If* D *has an obstruction theory with finite dimensional cohomology* (H^n_{ij})*, then the pro-representing hull* $\mathsf{H}(\mathsf{D})$ *is completely determined by the k-linear spaces* H^n_{ij} *for* $1 \leq i, j \leq p$*,* $n = 1, 2$*, together with some generalized Massey products on them.*

These results are natural generalizations of similar results for functors of commutative Artin rings. As in the commutative case, the generalized Massey product structure on H^n_{ij} can be considered as the k-linear dual of the obstruction morphism $o\colon T^2 \to T^1$, see Laudal [13, 14]. If D is obstructed, *i.e.*, $o(I(T^2)) \neq 0$, then it is a non-trivial task to compute $\mathsf{H}(\mathsf{D})$ using generalized Massey products.

Let $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ be a functor of noncommutative Artin rings. For $1 \leq i \leq p$, we write $\mathsf{D}_i\colon \mathsf{a}_1 \to \mathsf{Sets}$ for the restriction of D to a_1 using the i-th natural inclusion of categories $\mathsf{a}_1 \hookrightarrow \mathsf{a}_p$, and $\mathsf{D}^c_i\colon \mathsf{l} \to \mathsf{Sets}$ for the restriction of D_i to l, where l is the full subcategory of a_1 consisting of commutative algebras. For any associative ring R, we define the commutativization of R to be the quotient ring $R^c = R/I^c(R)$, where $I^c(R)$ is the ideal in R generated by the set of commutators $\{ab - ba : a, b \in R\}$.

Proposition 1.5 *Let* $\mathsf{D}\colon \mathsf{a}_p \to \mathsf{Sets}$ *be a functor of noncommutative Artin rings. If* D *has an obstruction theory with finite dimensional cohomology* (H^n_{ij})*, then* D^c_i *has a commutative hull* $\mathsf{H}(\mathsf{D}^c_i) \cong \mathsf{H}(\mathsf{D}_i)^c$ *for* $1 \leq i \leq p$*, and* $\mathsf{H}(\mathsf{D})^c \cong \bigoplus \mathsf{H}(\mathsf{D}^c_i)$.

Proof Clearly, the functor D_i has an obstruction theory with finite dimensional cohomology H^n_{ii} for $1 \leq i \leq p$, and therefore a pro-representing hull $\mathsf{H}(\mathsf{D}_i)$ that is determined by an obstruction morphism $o_i\colon T^2 \to T^1$. Similarly, the functor D^c_i has an obstruction theory with finite dimensional cohomology H^n_{ii} for $1 \leq i \leq p$, and therefore a pro-representing hull $\mathsf{H}(\mathsf{D}^c_i)$ that is determined by an obstruction morphism $o^c_i\colon (T^2)^c \to (T^1)^c$. These morphisms are defined by obstructions in small lifting situations, so it follows from the functorial nature of the obstructions that o_i and o^c_i are compatible. Hence $\mathsf{H}(\mathsf{D}^c_i) \cong \mathsf{H}(\mathsf{D}_i)^c$ for $1 \leq i \leq p$. For the second part, note that $\mathsf{H}(\mathsf{D})^c_{ij} = 0$ whenever $i \neq j$. In fact, for any $x_{ij} \in \mathsf{H}(\mathsf{D})_{ij}$ with $i \neq j$, the commutator $[e_i, x_{ij}] = e_i x_{ij} - x_{ij} e_i = x_{ij}$ is zero in $\mathsf{H}(\mathsf{D})^c$. This implies that $\mathsf{H}(\mathsf{D})^c = \bigoplus \mathsf{H}(\mathsf{D})^c_{ii} \cong \bigoplus \mathsf{H}(\mathsf{D}_i)^c \cong \bigoplus \mathsf{H}(\mathsf{D}^c_i)$. ■

2 Noncommutative Deformation Functors

Let k be an algebraically closed field, and let A be any Abelian k-category. For any object $X \in \mathsf{A}$, we recall the definition of $\mathsf{Def}^{\mathsf{A}}_X\colon \mathsf{l} \to \mathsf{Sets}$, the commutative deformation functor of X in A, which is a functor of Artin rings, and discuss how to generalize this definition to noncommutative deformation functors $\mathsf{Def}^{\mathsf{A}}_X\colon \mathsf{a}_p \to \mathsf{Sets}$ of a family $X = \{X_1, \dots, X_p\}$ in the category A.

Let R be any object in a_p. We consider the category A_R of R-objects in A, *i.e.*, the category with objects (X, ϕ), where X is an object of A and $\phi\colon R \to \mathrm{Mor}_{\mathsf{A}}(X, X)$ is a k-algebra homomorphism, and with morphisms $f\colon (X, \phi) \to (X', \phi')$, where $f\colon X \to X'$ is a morphism in A such that $f \circ \phi(r) = \phi'(r) \circ f$ for all $r \in R$.

Let R be any object of a_p, A any Abelian k-category, and $\mathsf{mod}(R)$ the category of finitely generated left R-modules. For each object $Y \in \mathsf{A}_R$, there is a unique finite colimit preserving functor $Y \otimes_R -\colon \mathsf{mod}(R) \to \mathsf{A}$ that maps R to Y, given in the following way. If $M = \mathrm{coker}(f)$, where $f\colon R^m \to R^n$ is a homomorphism of left R-modules, then $Y \otimes_R M = \mathrm{coker}(F)$, where $F\colon Y^m \to Y^n$ is the morphism in B induced by f and the R-linear structure on B. We say that an object $Y \in \mathsf{A}_R$ is *R-flat* if $Y \otimes_R -$ is exact. It is clear that any morphism $u\colon R \to S$ in a_p induces a functor

$- \otimes_R S \colon \mathsf{A}_R \to \mathsf{A}_S$.

Given an object $X \in \mathsf{A}$, the flat deformation functor $\mathrm{Def}^{\mathsf{A}}_{X}\colon \mathsf{l} \to \mathsf{Sets}$ is given in the following way. For any object $R \in \mathsf{l}$, we define a lifting of X to R to be an object $X_R \in \mathsf{A}_R$ which is R-flat, together with an isomorphism $\eta \colon X_R \otimes_R k \to X$ in A, and we say that two liftings (X_R, η) and (X'_R, η') are equivalent if there is an isomorphism $\tau \colon X_R \to X'_R$ in A_R such that $\eta' \circ (\tau \otimes_R \mathrm{id}) = \eta$. Let $\mathrm{Def}^{\mathsf{A}}_X(R)$ be the set of equivalence classes of liftings of X to R. Then $\mathrm{Def}^{\mathsf{A}}_X \colon \mathsf{l} \to \mathsf{Sets}$ is a functor of Artin rings, called the commutative deformation functor of X in A.

When $\mathsf{A} = \mathsf{Mod}(A)$, the category of left modules over an associative k-algebra A, we remark that the category A_R is the category of A-R bimodules on which k acts centrally, and the tensor product defined above is the usual tensor product over R. It follows that the usual deformation functor Def_M of a left A-module M coincides with the deformation functor defined above.

Given a finite family $X = \{X_1, \dots, X_p\}$ of objects in A, we would like to define a noncommutative deformation functor $\mathrm{Def}^{\mathsf{A}}_X \colon \mathsf{a}_p \to \mathsf{Sets}$ of the family X in A. When $\mathsf{A} = \mathsf{Mod}(A)$, the category of left modules over an associative k-algebra A, such a deformation functor was defined in Laudal [15]. The idea is to replace the condition that M_R is a flat right R-module with the *matric freeness* condition that

$$M_R \cong (M_i \otimes_k R_{ij}) \tag{2.1}$$

as right R-modules. This is reasonable, since an R-module is flat if and only if (2.1) holds when $R \in \mathsf{l}$ or $R \in \mathsf{a}_1$, see Bourbaki [3, Corollary II.3.2]. However, it is not clear whether an R-module is flat if and only if (2.1) holds when $R \in \mathsf{a}_p$ for $p \geq 2$.

We choose to define noncommutative deformation functors using Laudal's matric freeness condition rather than flatness. However, it is not completely clear how to do this for an arbitrary Abelian k-category. We shall therefore restrict our attention to categories of sheaves and presheaves of modules.

Let c be a small category, let $\mathcal{A}$ be a presheaf of k-algebras on c, and assume that A is an exact Abelian subcategory of the category $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$ of presheaves of left $\mathcal{A}$-modules on c. Then there is a forgetful functor $\pi_c \colon \mathsf{A} \to \mathsf{Mod}(k)$ for each object $c \in \mathsf{c}$ and an induced forgetful functor $\pi^R_c \colon \mathsf{A}_R \to \mathsf{Mod}(R)$ for each object $c \in \mathsf{c}$ and each $R \in \mathsf{a}_p$. We say that an object $X_R \in \mathsf{A}_R$ is *R-free* if $\pi^R_c(X_R) \cong (\pi_c(X_i) \otimes_k R_{ij})$ in $\mathsf{Mod}(R)$ for all objects $c \in \mathsf{c}$.

Let A be an Abelian k-category, and let $X = \{X_1, \dots, X_p\}$ be a finite family of objects in A. If A is an exact Abelian subcategory of $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$, we define the noncommutative deformation functor $\mathrm{Def}^{\mathsf{A}}_X \colon \mathsf{a}_p \to \mathsf{Sets}$ in the following way. A lifting of X to R is an object X_R in A_R that is R-free, together with isomorphisms $\eta_i \colon X_R \otimes_R k_i \to X_i$ in A for $1 \leq i \leq p$, and two liftings (X_R, η_i) and (X'_R, η'_i) are equivalent if there is an isomorphism $\tau \colon X_R \to X'_R$ in A_R such that $\eta'_i \circ (\tau \otimes_R \mathrm{id}) = \eta_i$ for $1 \leq i \leq p$. Let $\mathrm{Def}^{\mathsf{A}}_X(R)$ be the set of equivalence classes of liftings of X to R. Then $\mathrm{Def}^{\mathsf{A}}_X$ is a functor of noncommutative Artin rings, the noncommutative deformation functor of the family X in A. When the category A is understood from the context, we often write Def_X for $\mathrm{Def}^{\mathsf{A}}_X$.

3 Deformations of Presheaves of Modules

Let k be an algebraically closed field. The category $\mathrm{PreSh}(\mathsf{c}, \mathcal{A})$ of presheaves of left $\mathcal{A}$-modules on c is an Abelian k-category for any small category c and any presheaf $\mathcal{A}$ of associative k-algebras on c, and we shall consider deformations in this category. To fix notations, a presheaf on c is always covariant in this paper.

For any finite family $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ of presheaves of left $\mathcal{A}$-modules on c, we consider the noncommutative deformation functor $\mathrm{Def}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$, defined by $\mathrm{Def}_{\mathcal{F}} = \mathrm{Def}^{\mathsf{A}}_{\mathcal{F}}$ with $\mathsf{A} = \mathrm{PreSh}(\mathsf{c}, \mathcal{A})$. We shall describe this functor in concrete terms.

Let $R \in \mathsf{a}_p$, and consider a lifting $\mathcal{F}_R$ of the family $\mathcal{F}$ to R. Without loss of generality, we may assume that $\mathcal{F}_R(c) = (\mathcal{F}_i(c) \otimes_k R_{ij})$ with the natural right R-module structure for all $c \in \mathsf{c}$. To describe the lifting completely, we must specify the left action of $\mathcal{A}(c)$ on $(\mathcal{F}_i(c) \otimes_k R_{ij})$ for any object $c \in \mathsf{c}$, and the restriction map $\mathcal{F}_R(\phi) \colon (\mathcal{F}_i(c) \otimes_k R_{ij}) \to (\mathcal{F}_i(c') \otimes_k R_{ij})$ for any morphism $\phi \colon c \to c'$ in c. It is enough to specify the action of $a \in \mathcal{A}(c)$ on elements of the form $f_j \otimes e_j$ in $\mathcal{F}_j(c) \otimes_k R_{jj}$, and we must have

$$a(f_j \otimes e_j) = (af_j) \otimes e_j + \sum f_i' \otimes r_{ij} \tag{3.1}$$

with $f_i' \in \mathcal{F}_i(c)$, $r_{ij} \in I(R)_{ij}$ for all objects $c \in \mathsf{c}$. Similarly, it is enough to specify the restriction map $\mathcal{F}_R(\phi)$ on elements of the form $f_j \otimes e_j$ in $\mathcal{F}_j(c) \otimes_k R_{jj}$, and we must have

$$\mathcal{F}_R(\phi)(f_j \otimes e_j) = \mathcal{F}_j(\phi)(f_j) \otimes e_j + \sum f_i' \otimes r_{ij} \tag{3.2}$$

with $f_i' \in \mathcal{F}_i(c'),\ r_{ij} \in I(R)_{ij}$ for all morphisms $\phi \colon c \to c'$ in c.

Let $Q^R(c, c') = (\mathrm{Hom}_k(\mathcal{F}_j(c), \mathcal{F}_i(c') \otimes_k R_{ij}))$ for all objects $c, c' \in \mathsf{c}$, and write $Q^R(c) = Q^R(c, c)$. There is a natural product $Q^R(c', c'') \otimes_k Q^R(c, c') \to Q^R(c, c'')$ for all objects $c, c', c'' \in \mathsf{c}$, given by composition of maps and multiplication in R, such that $Q^R(c)$ is an associative k-algebra and $Q^R(c, c')$ is a $Q^R(c')$-$Q^R(c)$ bimodule in a natural way.

Lemma 3.1 *For any $R \in \mathsf{a}_p$, there is a bijective correspondence between the following data, up to equivalence, and* $\mathrm{Def}_{\mathcal{F}}(R)$*:*

(i) *For any $c \in \mathsf{c}$, a k-algebra homomorphism $L(c) \colon \mathcal{A}(c) \to Q^R(c)$ that satisfies equation* (3.1)*,*
(ii) *For any morphism $\phi \colon c \to c'$ in c, an element $L(\phi) \in Q^R(c, c')$ that satisfies equation* (3.2) *and $L(\phi)L(c) = L(c')L(\phi)$,*
(iii) *We have $L(\mathrm{id}) = \mathrm{id}$ and $L(\phi')L(\phi) = L(\phi' \circ \phi)$ for all morphisms $\phi \colon c \to c'$ and $\phi' \colon c' \to c''$ in c.*

For any $R \in \mathsf{a}_p$, we may, for any $c \in \mathsf{c}$, consider $L(c) \colon \mathcal{A}(c) \to Q^R(c)$ given by $L(c)(a)(f_j) = af_j \otimes e_j$ for all $a \in \mathcal{A}(c)4$, $f_j \in \mathcal{F}_j(c)$, and for any morphism $\phi \colon c \to c'$ in c consider $L(\phi) \in Q^R(c, c')$ given by $L(\phi)(f_j) = \mathcal{F}_j(\phi)(f_j) \otimes e_j$ for all $f_j \in \mathcal{F}_j(c)$. These data correspond to the trivial deformation $*_R \in \mathrm{Def}_{\mathcal{F}}(R)$.

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Lemma 3.2 *There is a bijection $\phi_{ij}\colon t(\mathsf{Def}_{\mathcal{F}})_{ij} \to \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{F}_j, \mathcal{F}_i)$ for $1 \le i, j \le p$ that maps trivial deformations to split extensions.*

Proof Let $R = k^p[\epsilon_{ij}]$, and let $\mathcal{F}_R$ be a lifting of $\mathcal{F}$ to R. We consider the j-th column $\mathcal{F}_R^j$ of $\mathcal{F}_R$, given by $c \mapsto \mathcal{F}_R(c)e_j$, which is a sub-presheaf of $\mathcal{F}_R$ of left $\mathcal{A}$-modules on c, since $\mathcal{F}_R^j(c)$ is invariant under $L(c)$ and $L(\phi)$ for any $c \in \mathsf{c}$ and any $\phi\colon c \to c'$ in c. Moreover, there is a natural exact sequence $0 \to \mathcal{F}_i \to \mathcal{F}_R^j \to \mathcal{F}_j \to 0$ in $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$, since $\mathcal{F}_R^j(c) = \mathcal{F}_j(c) \oplus \mathcal{F}_i(c)\epsilon_{ij}$ for all $c \in \mathsf{c}$. Clearly, equivalent liftings of $\mathcal{F}$ to R give equivalent extensions in $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$, so $\mathcal{F}_R \mapsto \mathcal{F}_R^j$ defines a map $\phi_{ij}\colon t(\mathsf{Def}_{\mathcal{F}})_{ij} \to \operatorname{Ext}^1_{\mathcal{A}}(\mathcal{F}_j, \mathcal{F}_i)$ that maps trivial deformations to split extensions. To construct an inverse of ϕ_{ij}, we consider an extension $\mathcal{E}$ of $\mathcal{F}_j$ by $\mathcal{F}_i$ in $\mathsf{PreSh}(\mathsf{c}, \mathcal{A})$, and let $\mathcal{F}_R = \mathcal{E} \oplus \mathcal{F}_1 \cdots \oplus \widehat{\mathcal{F}_i} \oplus \cdots \oplus \widehat{\mathcal{F}_j} \oplus \cdots \oplus \mathcal{F}_p$. Then $\mathcal{F}_R$ is a presheaf of left $\mathcal{A}$-modules on c, and it is easy to see that it defines a lifting of $\mathcal{F}$ to R since $\mathcal{E}(c) \cong \mathcal{F}_i(c) \oplus \mathcal{F}_j(c)$ as k-linear vector spaces for any $c \in \mathsf{c}$. It follows that the assignment $\mathcal{E} \mapsto \mathcal{F}_R$ defines an inverse of ϕ_{ij}. ∎

4 Global Hochschild Cohomology

Let k be an algebraically closed field, let c be a small category, and let $\mathcal{A}$ be a presheaf of associative k-algebras on c. For any presheaves $\mathcal{F}, \mathcal{G}$ of left $\mathcal{A}$-modules on c, we may consider the Hochschild complex $\mathsf{HC}^*(\mathcal{A}(c), \operatorname{Hom}_k(\mathcal{F}(c), \mathcal{G}(c)))$ of $\mathcal{A}(c)$ with values in the bimodule $\operatorname{Hom}_k(\mathcal{F}(c), \mathcal{G}(c))$ for any object $c \in \mathsf{c}$. Unfortunately, this construction is not functorial in c. In this section, we consider a variation of this construction that is functorial, and use this to define a global Hochschild cohomology theory.

Let $\mathsf{Mor}\,\mathsf{c}$ denote the category of morphisms in c defined in the following way. An object in $\mathsf{Mor}\,\mathsf{c}$ is a morphism in c, and given objects $f\colon c \to c'$ and $g\colon d \to d'$ in $\mathsf{Mor}\,\mathsf{c}$, a morphism $(\alpha, \beta)\colon f \to g$ in $\mathsf{Mor}\,\mathsf{c}$ is a couple of morphisms $\alpha\colon d \to c$ and $\beta\colon c' \to d'$ in c such that $\beta f \alpha = g$. Clearly, $\mathsf{Mor}\,\mathsf{c}$ is a small category.

Let $\mathcal{F}, \mathcal{G}$ be presheaves of left $\mathcal{A}$-modules on c, and let $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ be the presheaf in $\mathsf{PreSh}(\mathsf{Mor}\,\mathsf{c}, \underline{k})$ given by $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})(\phi) = \operatorname{Hom}_k(\mathcal{F}(c), \mathcal{G}(c'))$ for any morphism $\phi\colon c \to c'$ in c. We define the *Hochschild complex* of $\mathcal{A}$ with values in $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ to be the functor $\mathsf{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))\colon \mathsf{Mor}\,\mathsf{c} \to \mathsf{Compl}(k)$, given by

$$\mathsf{HC}^p(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi) = \operatorname{Hom}_k\Big(\bigotimes_k^p \mathcal{A}(c), \operatorname{Hom}_k(\mathcal{F}(c), \mathcal{G}(c'))\Big)$$

for any morphism $\phi\colon c \to c'$ in c and any integer $p \ge 0$, with differential given by

$$\begin{aligned} d^p(\phi)(f)(a_1 \otimes \cdots \otimes a_{p+1}) = a_1\, f(a_2 \otimes \cdots \otimes a_{p+1}) \\ + \sum_{i=1}^{p} (-1)^i\, f(a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{p+1}) \\ + (-1)^{p+1}\, f(a_1 \otimes \cdots \otimes a_p)\, a_{p+1} \end{aligned}$$

for any $f \in \mathrm{HC}^p(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi)$ and any $a_1, \dots, a_{p+1} \in \mathcal{A}(c)$. We see that $\mathrm{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi)$ is the Hochschild complex of $\mathcal{A}(c)$ with values in the bimodule $\mathrm{Hom}_k(\mathcal{F}(c), \mathcal{G}(c'))$ for any morphism $\phi\colon c \to c'$ in c, and the definition of $\mathsf{Mor}\,\mathsf{c}$ ensures that $\phi \mapsto \mathrm{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi)$ is functorial.

Let us write $\mathcal{E}xt^*_{\mathcal{A}}(\mathcal{F}, \mathcal{G})\colon \mathsf{Mor}\,\mathsf{c} \to \mathsf{Mod}(k)$ for the composition of functors given by $\phi \mapsto \mathrm{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi) \mapsto \mathrm{H}^*(\mathrm{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))(\phi))$. We see that $\mathcal{E}xt^*_{\mathcal{A}}(\mathcal{F}, \mathcal{G})(\phi) \cong \mathrm{Ext}^*_{\mathcal{A}(c)}(\mathcal{F}(c), \mathcal{G}(c'))$ for any morphism $\phi\colon c \to c'$ in c.

For any functor $G\colon \mathsf{Mor}\,\mathsf{c} \to \mathsf{Mod}(k)$, we may consider the resolving complex $\mathrm{D}^*(\mathsf{c}, G)$ in $\mathsf{Mod}(k)$ of the projective limit functor of G, see Laudal [13]. We recall that for any integer $p \geq 0$, $\mathrm{D}^p(\mathsf{c}, G)$ is given by

$$\mathrm{D}^p(\mathsf{c}, G) = \prod_{c_0 \to \cdots \to c_p} G(\phi_p \circ \cdots \circ \phi_1),$$

where the product is taken over all p-tuples $(\phi_1, \dots, \phi_p)$ of composable morphisms $\phi_i\colon c_{i-1} \to c_i$ in c, and the differential $d^p\colon \mathrm{D}^p(\mathsf{c}, G) \to \mathrm{D}^{p+1}(\mathsf{c}, G)$ is given by

$$\begin{aligned}(d^p g)(\phi_1, \dots, \phi_{p+1}) = \; & G(\phi_1, \mathrm{id})(g(\phi_2, \dots, \phi_{p+1})) \\ & + \sum_{i=1}^{p} (-1)^i \, g(\phi_1, \dots, \phi_{i+1} \circ \phi_i, \dots, \phi_{p+1}) \\ & + (-1)^{p+1} \, G(\mathrm{id}, \phi_{p+1})(g(\phi_1, \dots, \phi_p))\end{aligned}$$

for all $g \in \mathrm{D}^p(\mathsf{c}, G)$ and for all $(p+1)$-tuples $(\phi_1, \dots, \phi_{p+1})$ of composable morphisms $\phi_i\colon c_{i-1} \to c_i$ in c. We denote the cohomology of $\mathrm{D}^*(\mathsf{c}, G)$ by $\mathrm{H}^*(\mathsf{c}, G)$, and recall the following standard result.

Proposition 4.1 *Let c be a small category. The resolving complex $\mathrm{D}^*(\mathsf{c}, G)$ has the following properties:*

(i) $\mathrm{D}^*(\mathsf{c}, -)\colon \mathsf{PreSh}(\mathsf{Mor}\,\mathsf{c}, \underline{k}) \to \mathsf{Compl}(k)$ *is exact,*

(ii) $\mathrm{H}^p(\mathsf{c}, G) \cong \varprojlim^{(p)} G$ *for all* $G \in \mathsf{PreSh}(\mathsf{Mor}\,\mathsf{c}, \underline{k})$ *and for all* $p \geq 0$.

In particular, $\mathrm{H}^*(\mathsf{c}, -)\colon \mathsf{PreSh}(\mathsf{Mor}\,\mathsf{c}, \underline{k}) \to \mathsf{Mod}(k)$ *is an exact* δ*-functor.*

For any functor $\mathrm{C}^*\colon \mathsf{Mor}\,\mathsf{c} \to \mathsf{Compl}(k)$, we may consider the double complex $\mathrm{D}^{**} = \mathrm{D}^*(\mathsf{c}, \mathrm{C}^*)$ of vector spaces over k. Explicitly, have that $\mathrm{D}^{pq} = \mathrm{D}^p(\mathsf{c}, \mathrm{C}^q)$ for all integers p, q with $p \geq 0$, that $d_I^{pq}\colon \mathrm{D}^{pq} \to \mathrm{D}^{p+1,q}$ is the differential d^p in $\mathrm{D}^*(\mathsf{c}, \mathrm{C}^q)$, and that $d_{II}^{pq}\colon \mathrm{D}^{pq} \to \mathrm{D}^{p,q+1}$ is the differential given by $d_{II}^{pq} = (-1)^p \, \mathrm{D}^p(\mathsf{c}, d^q)$, where $d^q\colon \mathrm{C}^q \to \mathrm{C}^{q+1}$ is the differential in C^*. Note that if $\mathrm{C}^q = 0$ for all $q < 0$, then D^{**} lies in the first quadrant.

We define the *global Hochschild complex* of $\mathcal{A}$ with values in $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ on c to be the total complex of the double complex $\mathrm{D}^{**} = \mathrm{D}^*(\mathsf{c}, \mathrm{HC}^*(\mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G})))$, and denote it by $\mathrm{HC}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$. Moreover, we define *global Hochschild cohomology* $\mathrm{HH}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$ of $\mathcal{A}$ with values in $\mathcal{H}om_k(\mathcal{F}, \mathcal{G})$ on c to be the cohomology of the global Hochschild complex $\mathrm{HC}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$.

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Proposition 4.2 *There is a spectral sequence for which* $E_2^{pq} = \mathsf{H}^p(\mathsf{c}, \mathcal{E}xt^q_{\mathcal{A}}(\mathcal{F}, \mathcal{G}))$*, such that* $E_\infty = \operatorname{gr} \mathsf{HH}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$*, the associated graded vector space over* k *with respect to a suitable filtration of* $\mathsf{HH}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$.

We remark that $\mathsf{HH}^*(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}, \mathcal{G}))$ can be calculated in concrete terms in many situations, using the above spectral sequence. We shall give an example of such a computation in last section of this paper.

5 Obstruction Theory for Presheaves of Modules

Let k be an algebraically closed field. For any small category c and any presheaf $\mathcal{A}$ of associative k-algebras on c, we shall construct an obstruction theory for the noncommutative deformation functor $\operatorname{Def}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ with cohomology

$$(\mathsf{HH}^n(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)))$$

for any finite family $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ of presheaves of left $\mathcal{A}$-modules on c.

Proposition 5.1 *Let* $u \colon R \to S$ *be a small surjection in* a_p *with kernel* K*, and let* $\mathcal{F}_S \in \operatorname{Def}_{\mathcal{F}}(S)$ *be a deformation. Then there exists a canonical obstruction*

$$o(u, \mathcal{F}_S) \in (\mathsf{HH}^2(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) \otimes_k K_{ij})$$

such that $o(u, \mathcal{F}_S) = 0$ *if and only if there exists a deformation* $\mathcal{F}_R \in \operatorname{Def}_{\mathcal{F}}(R)$ *lifting* $\mathcal{F}_S$ *to* R*. Moreover, if* $o(u, \mathcal{F}_S) = 0$*, then there is a transitive and effective action of* $(\mathsf{HH}^1(\mathsf{c}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) \otimes_k K_{ij})$ *on the set of liftings of* $\mathcal{F}_S$ *to* R.

Proof Let $\mathcal{F}_S \in \operatorname{Def}_{\mathcal{F}}(S)$ be given. By Lemma 3.1, this deformation corresponds to the following data. A k-algebra homomorphism $L^S(c) \colon \mathcal{A}(c) \to Q^S(c)$ for each object $c \in \mathsf{c}$ and an element $L^S(\phi) \in Q^S(c, c')$ for each morphism $\phi \colon c \to c'$ in c, such that the conditions of Lemma 3.1 are satisfied. Moreover, to lift $\mathcal{F}_S$ to R is the same as to lift these data to R.

Choose a k-linear section $\sigma \colon S \to R$ such that $\sigma(I(S)_{ij}) \subseteq I(R)_{ij}$ and $\sigma(e_i) = e_i$ for $1 \le i, j \le p$. Clearly, σ induces a k-linear map $Q^S(c, c') \to Q^R(c, c')$ for all c, c' in c, which we shall denote by $Q(\sigma, c, c')$. We define $L^R(c) = Q(\sigma, c, c) \circ L^S(c)$ for all objects $c \in \mathsf{c}$ and $L^R(\phi) = Q(\sigma, c, c')(L^S(c, c'))$ for all morphisms $\phi \colon c \to c'$ in c. Then $L^R(c) \colon \mathcal{A}(c) \to Q^R(c)$ is a k-linear map which lifts $L^S(c)$ to R for all $c \in \mathsf{c}$, and we define the obstruction

$$o(0,2)(c) \colon \mathcal{A}(c) \otimes_k \mathcal{A}(c) \to Q^K(c)$$

by $o(0,2)(c)(a \otimes b) = L^R(c)(ab) - L^R(c)(a)\, L^R(c)(b)$ for all $a, b \in \mathcal{A}(c)$. It is clear that $L^R(c)$ is a k-algebra homomorphism if and only if $o(0,2)(c) = 0$. Moreover, $L^R(\phi)$ lifts $L^S(\phi)$ to R for all morphisms $\phi \colon c \to c'$ in c, and we define the obstruction

$$o(1,1)(\phi) \colon \mathcal{A}(c) \to Q^K(c, c')$$

by $o(1,1)(\phi)(a) = L^R(\phi) \circ L^R(c)(a) - L^R(c')(\mathcal{A}(\phi)(a)) \circ L^R(\phi)$ for all $a \in \mathcal{A}(c)$. It is clear that $L^R(\phi)$ is $\mathcal{A}(\phi)$-linear if and only if $o(1,1)(\phi) = 0$. Finally, we define the obstruction $o(2,0)(\phi,\phi') \in Q^K(c,c'')$ by $o(2,0)(\phi,\phi') = L^R(\phi')\,L^R(\phi) - L^R(\phi'\circ\phi)$ for all morphisms $\phi\colon c \to c'$ and $\phi'\colon c' \to c''$ in c. It is clear that L^R satisfies the cocycle condition if and only if $o(2,0) = 0$.

We see that $o = (o(0,2), o(1,1), o(2,0))$ is a 2-cochain in global Hochschild complex $(\mathsf{HC}^n(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i)) \otimes_k K_{ij})$. A calculation shows that o is a 2-cocycle, and that its cohomology class $o(u,\mathcal{F}_S) \in (\mathsf{HH}^2(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i)) \otimes_k K_{ij})$ is independent of the choice of $L^R(c)$ and $L^R(\phi)$. It is clear that if there is a lifting of $\mathcal{F}_S$ to R, we may choose $L^R(c)$ and $L^R(\phi)$ such that $o = 0$, hence $o(u,\mathcal{F}_S) = 0$. Conversely, assume that $o(u,\mathcal{F}_S) = 0$. Then there exists a 1-cochain of the form (ϵ,Δ) with $\epsilon \in \mathsf{D}^{01}$ and $\Delta \in \mathsf{D}^{10}$ such that $d(\epsilon,\Delta) = o$. Let $L'(c) = L^R(c) + \epsilon(c)$ and $L'(\phi) = L^R(\phi) + \Delta(\phi)$. Then $L'(c)$ is another lifting of $L^S(c)$ to R, $L'(\phi)$ is another lifting of $L^S(\phi)$ to R, and essentially the same calculation as above shows that the corresponding 2-cocycle $o' = 0$. Hence there is a lifting of $\mathcal{F}_S$ to R, and this proves the first part of the proposition.

For the second part, assume that $\mathcal{F}_R$ is a lifting of $\mathcal{F}$ to R. Then $\mathcal{F}_R$ is defined by liftings $L^R(c)$ and $L^R(\phi)$ to R such that the corresponding 2-cocycle $o = 0$. Let us consider a 1-cochain (ϵ,Δ), and consider the new liftings $L'(c) = L^R(c) + \epsilon(c)$ and $L'(\phi) = L^R(\phi) + \Delta(\phi)$. From the previous calculations, it is clear that the new 2-cocycle $o' = 0$ if and only if (ϵ,Δ) is a 1-cocycle. Moreover, if this is the case, the lifting $\mathcal{F}'_R$ defined by $L'(c)$ and $L'(\phi)$ is equivalent to $\mathcal{F}_R$ if and only if (ϵ,Δ) is a 1-coboundary, since an equivalence between $\mathcal{F}_R$ and $\mathcal{F}'_R$ must have the form $\mathrm{id} + \pi$ for some 0-cochain π with $d(\pi) = (\epsilon,\Delta)$. ∎

We see that the obstruction $o(u,\mathcal{F}_S)$ is functorial, so it defines an obstruction theory for the noncommutative deformation functor $\mathsf{Def}_{\mathcal{F}}\colon \mathsf{a}_p \to \mathsf{Sets}$ by definition. If the condition

$$(5.1) \qquad \dim_k \mathsf{HH}^n(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i)) < \infty \text{ for } 1 \le i,j \le p,\ n = 1,2,$$

holds, it follows that $\mathsf{Def}_{\mathcal{F}}$ has an obstruction theory with finite dimensional cohomology $(\mathsf{HH}^n(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i)))$.

Proposition 5.2 *Let c be a small category, let $\mathcal{A}$ be a presheaf of k-algebras on c, and let $\mathcal{F} = \{\mathcal{F}_1,\dots,\mathcal{F}_p\}$ be a finite family of presheaves of left $\mathcal{A}$-modules on c. Then $\mathsf{HH}^1(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i)) \cong t(\mathsf{Def}_{\mathcal{F}})_{ij} \cong \mathrm{Ext}^1_{\mathcal{A}}(\mathcal{F}_j,\mathcal{F}_i)$ for $1 \le i,j \le p$.*

Theorem 5.3 *Let c be a small category, let $\mathcal{A}$ be a presheaf of k-algebras on c, and let $\mathcal{F} = \{\mathcal{F}_1,\dots,\mathcal{F}_p\}$ be a finite family of presheaves of left $\mathcal{A}$-modules on c. If condition* (5.1) *holds, then the noncommutative deformation functor $\mathsf{Def}_{\mathcal{F}}\colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ in $\mathsf{PreSh}(\mathsf{c},\mathcal{A})$ has a pro-representing hull $\mathsf{H}(\mathsf{Def}_{\mathcal{F}})$, completely determined by the k-linear spaces $\mathsf{HH}^n(\mathsf{c},\mathcal{A},\mathcal{H}om_k(\mathcal{F}_j,\mathcal{F}_i))$ for $1 \le i,j \le p$, $n = 1,2$, together with some generalized Massey products on them.*

6 Deformations of Quasi-Coherent Sheaves of Modules

Let k be an algebraically closed field, and let $(X,\mathcal{A})$ be a ringed space over k. We recall that a sheaf $\mathcal{F} \in \mathsf{Sh}(X,\mathcal{A})$ of left $\mathcal{A}$-modules on X is *quasi-coherent* if for every point $x \in X$, there exists an open neighbourhood $U \subseteq X$ of x, free sheaves $\mathcal{L}_0, \mathcal{L}_1$ of left $\mathcal{A}|_U$-modules on U, and an exact sequence $0 \leftarrow \mathcal{F}|_U \leftarrow \mathcal{L}_0 \leftarrow \mathcal{L}_1$ of sheaves of left $\mathcal{A}|_U$-modules on U. The category $\mathsf{QCoh}(\mathcal{A})$ of *quasi-coherent sheaves* of left $\mathcal{A}$-modules on X is the full subcategory of $\mathsf{Sh}(X, \mathcal{A})$ consisting of quasi-coherent sheaves.

The full subcategory $\mathsf{QCoh}(\mathcal{A}) \subseteq \mathsf{Sh}(X, \mathcal{A})$ is closed under finite direct sums, but it is not clear if $\mathsf{QCoh}(\mathcal{A})$ is closed under kernels and cokernels in general. Hence $\mathsf{QCoh}(\mathcal{A})$ is an additive but not necessarily an Abelian k-category. In this section, we give sufficient conditions for $\mathsf{QCoh}(\mathcal{A})$ to be an exact Abelian subcategory of $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ for an open cover U of X, and consider deformations in the category $\mathsf{QCoh}(\mathcal{A})$ in these cases.

Let us consider the global sections functor $\Gamma(X, -) \colon \mathsf{Sh}(X, \mathcal{A}) \to \mathsf{Mod}(A)$, where we write $A = \Gamma(X, \mathcal{A})$. This functor is left exact, and we denote its right derived functors by $\mathsf{H}^*(X, -) = R^*\Gamma(X, -)$. We say that X is $\mathcal{A}$*-affine* if the following conditions hold:

(i) $\Gamma(X, -)$ induces an equivalence of categories $\mathsf{QCoh}(\mathcal{A}) \to \mathsf{Mod}(A)$,
(ii) $\mathsf{H}^n(X, \mathcal{F}) = 0$ for all $\mathcal{F} \in \mathsf{QCoh}(\mathcal{A})$ and for all integers $n \geq 1$.

Moreover, we say that an open subset $U \subseteq X$ is $\mathcal{A}$-affine if U is $\mathcal{A}|_U$-affine, and that an open cover U of X is $\mathcal{A}$-affine if U is $\mathcal{A}$-affine for any $U \in \mathsf{U}$.

An open cover U of X is *good* if any finite intersection $V = U_1 \cap U_2 \cap \cdots \cap U_r$ with $U_i \in \mathsf{U}$ for $1 \leq i \leq r$ can be covered by open subsets $W \subseteq V$ with $W \in \mathsf{U}$. In particular, any open cover closed under finite intersections is good.

Proposition 6.1 *If* U *is a good* $\mathcal{A}$*-affine open cover of* X*, then the natural forgetful functor* $\pi \colon \mathsf{QCoh}(\mathcal{A}) \to \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ *is a full embedding, and* π *identifies* $\mathsf{QCoh}(\mathcal{A})$ *with an exact Abelian subcategory of* $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ *that is closed under extensions.*

Proof It is clear that π is a full embedding, and π is exact since $\mathsf{H}^1(U, \mathcal{F}) = 0$ for all $U \in \mathsf{U}, \mathcal{F} \in \mathsf{QCoh}(\mathcal{A})$. It is therefore enough to show that for any exact sequence

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

in $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ with $\mathcal{F}' = \pi(\mathcal{G}'), \mathcal{F}'' = \pi(\mathcal{G}'')$ for $\mathcal{G}', \mathcal{G}'' \in \mathsf{QCoh}(\mathcal{A})$, there is a quasi-coherent sheaf $\mathcal{G} \in \mathsf{QCoh}(\mathcal{A})$ such that $\pi(\mathcal{G}) = \mathcal{F}$. Since U is an $\mathcal{A}$-affine open cover, we can find $\mathcal{G}^U \in \mathsf{QCoh}(\mathcal{A}|_U)$ with $\Gamma(U, \mathcal{G}^U) = \mathcal{F}(U)$ for all $U \in \mathsf{U}$. For any inclusion $U \supseteq V$ in U, there is a natural isomorphism $\mathcal{G}^U|_V \to \mathcal{G}^V$ of quasi-coherent sheaves on V since $\mathcal{G}', \mathcal{G}'' \in \mathsf{QCoh}(\mathcal{A})$. Hence it follows from the fact that U is a good cover of X that the quasi-coherent sheaves $\{\mathcal{G}^U \in \mathsf{QCoh}(\mathcal{A}|_U) : U \in \mathsf{U}\}$ can be glued to a quasi-coherent sheaf $\mathcal{G} \in \mathsf{QCoh}(\mathcal{A})$ with $\pi(\mathcal{G}) = \mathcal{F}$. ■

Let U be a good $\mathcal{A}$-affine open cover of X, and let $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ be a finite family of quasi-coherent sheaves of left $\mathcal{A}$-modules on X. Since $\mathsf{QCoh}(\mathcal{A})$ is an exact Abelian subcategory of $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ by Proposition 6.1, we may consider the noncommutative deformation functor $\mathsf{Def}^{qc}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ as a family of quasi-coherent sheaves, defined by $\mathsf{Def}^{qc}_{\mathcal{F}} = \mathsf{Def}^{\mathsf{A}}_{\mathcal{F}}$ with $\mathsf{A} = \mathsf{QCoh}(\mathcal{A})$. We may also consider the noncommutative deformation functor $\mathsf{Def}^{\mathsf{U}}_{\mathcal{F}} \colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ as a family of presheaves on U, defined by $\mathsf{Def}^{\mathsf{U}}_{\mathcal{F}} = \mathsf{Def}^{\mathsf{A}}_{\mathcal{F}}$ with $\mathsf{A} = \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$.

Proposition 6.2 *If U is a good $\mathcal{A}$-affine open cover of X, then the forgetful functor $\pi\colon \mathsf{QCoh}(\mathcal{A}) \to \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ induces an isomorphism $\mathsf{Def}^{qc}_{\mathcal{F}} \to \mathsf{Def}^{\mathsf{U}}_{\mathcal{F}}$ of deformation functors for any finite family $\mathcal{F}$ of quasi-coherent left $\mathcal{A}$-modules on X.*

Proof Clearly, π induces a morphism of noncommutative deformation functors, and it is enough to show that the induced map of sets $\pi^*_R\colon \mathsf{Def}^{qc}_{\mathcal{F}}(R) \to \mathsf{Def}^{\mathsf{U}}_{\mathcal{F}}(R)$ is a bijection for any $R \in \mathsf{a}_p$. If X is $\mathcal{A}$-affine and $\mathsf{U} = \{X\}$, then $\mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ is naturally equivalent to $\mathsf{Mod}(A)$, so $\pi\colon \mathsf{QCoh}(\mathcal{A}) \to \mathsf{PreSh}(\mathsf{U}, \mathcal{A})$ is an equivalence of categories, and this implies that π_R is a bijection for any $R \in \mathsf{a}_p$. In the general case, let $\mathcal{F}_R \in \mathsf{Def}^{\mathsf{U}}_{\mathcal{F}}(R)$. Then $\mathcal{F}_R(U)$ is a deformation of the family $\{\mathcal{F}_1(U), \dots, \mathcal{F}_p(U)\}$ in $\mathsf{Mod}(\mathcal{A}(U))$ to R for any $U \in \mathsf{U}$. By the result in the $\mathcal{A}$-affine case, we can find a deformation $\mathcal{F}^U_R$ of the family $\{\mathcal{F}_1|_U, \dots, \mathcal{F}_p|_U\}$ in $\mathsf{QCoh}(\mathcal{A}|_U)$ to R that is compatible with $\mathcal{F}_R(U)$. We remark that if $V \subseteq U$ is an inclusion in U, then there is a natural isomorphism $\mathcal{F}^U_R|_V \to \mathcal{F}^V_R$ of sheaves of left $\mathcal{A}$-modules on V, since $\mathcal{F}$ is a family of quasi-coherent sheaves of $\mathcal{A}$-modules on X. We must glue the local deformations $\mathcal{F}^U_R$ to a deformation $\mathcal{F}_R$ of the family $\mathcal{F}$ to R in $\mathsf{QCoh}(\mathcal{A})$, and this is clearly possible since U is a good open cover of X. ■

Proposition 6.3 *Let $(X, \mathcal{A})$ be a ringed space over k, and let U be a good $\mathcal{A}$-affine open cover of X. Then*

$$t(\mathsf{Def}^{qc}_{\mathcal{F}})_{ij} \cong \mathsf{HH}^1(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) \cong \mathrm{Ext}^1_{\mathsf{QCoh}(\mathcal{A})}(\mathcal{F}_j, \mathcal{F}_i)$$

for any finite family $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ of quasi-coherent left $\mathcal{A}$-modules on X.

Proof Since $\mathsf{Def}^{qc}_{\mathcal{F}} \cong \mathsf{Def}^{\mathsf{U}}_{\mathcal{F}}$ by Proposition 6.2, we have $t(\mathsf{Def}^{qc}_{\mathcal{F}})_{ij} \cong t(\mathsf{Def}^{\mathsf{U}}_{\mathcal{F}})_{ij}$, and $\mathrm{Ext}^1_{\mathsf{QCoh}(\mathcal{A})}(\mathcal{F}_j, \mathcal{F}_i) \cong \mathrm{Ext}^1_{\mathsf{PreSh}(\mathsf{U},\mathcal{A})}(\mathcal{F}_j, \mathcal{F}_i)$ by Proposition 6.1 and Oort [16, Lemma 3.2]. Hence the result follows from Proposition 5.2. ■

Theorem 6.4 *Let $(X, \mathcal{A})$ be a ringed space over k, let U be a good $\mathcal{A}$-affine open cover of X, and let $\mathcal{F} = \{\mathcal{F}_1, \dots, \mathcal{F}_p\}$ be a finite family of quasi-coherent left $\mathcal{A}$-modules on X. If $\dim_k \mathsf{HH}^n(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i)) < \infty$ for $1 \le i, j \le p$, $n = 1, 2$, then the noncommutative deformation functor $\mathsf{Def}^{qc}_{\mathcal{F}}\colon \mathsf{a}_p \to \mathsf{Sets}$ of $\mathcal{F}$ in $\mathsf{QCoh}(\mathcal{A})$ has a pro-representing hull $\mathsf{H}(\mathsf{Def}^{qc}_{\mathcal{F}})$, completely determined by the k-linear spaces $\mathsf{HH}^n(\mathsf{U}, \mathcal{A}, \mathcal{H}om_k(\mathcal{F}_j, \mathcal{F}_i))$ for $1 \le i, j \le p$, $n = 1, 2$, together with some generalized Massey products on them.*

7 Quasi-Coherent Ringed Schemes

Let k be an algebraically closed field. We shall consider some important examples of ringed spaces $(X, \mathcal{A})$ over k such that X has a good $\mathcal{A}$-affine open cover U. It is often possible to choose U to be a finite cover, and this is important for effective computations of noncommutative deformations using presheaf methods.

Example 1 Let $(X, \mathcal{O}_X)$ be a scheme over k. If $U \subseteq X$ is an open affine subscheme of X, then U is $\mathcal{O}_X$-affine by Hartshorne [12, Corollary II.5.5] and Grothendieck [7, Theorem 1.3.1]. Hence any open affine cover of X is an $\mathcal{O}_X$-affine open cover.

If X is separated over k, then any finite intersection of open affine subschemes of X is affine. Hence if X is quasi-compact and separated over k, then there is a finite $\mathcal{O}_X$-affine open cover of X closed under intersections.

Example 2 A *ringed scheme* over k is a ringed space $(X, \mathcal{A})$ over k defined by a scheme $(X, \mathcal{O}_X)$ over k and a morphism $i\colon \mathcal{O}_X \to \mathcal{A}$ of sheaves of associative k-algebras on X. A *quasi-coherent ringed scheme* over k is a ringed scheme $(X, \mathcal{A})$ over k such that $\mathcal{A}$ is quasi-coherent as a left and right $\mathcal{O}_X$-module. The notion of quasi-coherent ringed schemes was introduced in Yekutieli and Zhang [20], and the following result follows from [20, Corollary 5.13].

Lemma 7.1 *A ringed scheme $(X, \mathcal{A})$ over k is quasi-coherent if and only if the morphism $\mathcal{A}(U) \to \mathcal{A}(D(f))$ is a ring of fractions with respect to $S = \{f^n : n \geq 0\}$ for any open affine subscheme $U \subseteq X$ and any $f \in \mathcal{O}_X(U)$.*

For any quasi-coherent ringed scheme $(X, \mathcal{A})$ over k, a left $\mathcal{A}$-module $\mathcal{F}$ is quasi-coherent if and only if it is quasi-coherent as a left $\mathcal{O}_X$-module. This follows from Grothendieck [6, Proposition 9.6.1], when $\mathcal{A}$ is a sheaf of commutative rings on X, and the proof can easily be extended to the noncommutative case.

Lemma 7.2 *Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme over k. If $U \subseteq X$ is an open affine subscheme of X, then U is $\mathcal{A}$-affine.*

Proof Write $A = \mathcal{A}(U)$, and consider $\Gamma(U, -)\colon \mathsf{QCoh}(\mathcal{A}|_U) \to \mathsf{Mod}(A)$. We claim that $\Gamma(U, -)$ is an equivalence of categories. By the comments preceding the lemma, $\mathsf{QCoh}(\mathcal{A}|_U)$ can be considered as a subcategory of $\mathsf{QCoh}(\mathcal{O}_U)$, and $\Gamma(U, -)\colon \mathsf{QCoh}(\mathcal{O}_U) \to \mathsf{Mod}(O)$ is an equivalence of categories with $O = \mathcal{O}_X(U)$. So the claim follows from Lemma 7.1. Finally, $\mathsf{H}^n(U, \mathcal{F}) = 0$ for any integer $n \geq 1$ and any $\mathcal{F} \in \mathsf{QCoh}(\mathcal{A}|_U)$ by the above comments. ∎

Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme over k. Then any open affine cover of X is an $\mathcal{A}$-affine open cover of X. If X is quasi-compact and separated over k, then there is a finite $\mathcal{A}$-affine open cover of X closed under intersections.

Example 3 Let $(X, \mathcal{A})$ be a quasi-coherent ringed scheme over k, and assume that $\operatorname{char}(k) = 0$. We say that $\mathcal{A}$ is a *D-algebra*, and that $(X, \mathcal{A})$ is a *D-scheme*, if the following condition holds. For any open subset $U \subseteq X$ and for any section $a \in \mathcal{A}(U)$, there exists an integer $n \geq 0$ (depending on a) such that

$$[\cdots[[a, f_1], f_2], \dots, f_n] = 0$$

for all sections $f_1, \dots, f_n \in \mathcal{O}_X(U)$, where $[a, f] = af - fa$ is the usual commutator for all $a \in \mathcal{A}(U)$, $f \in \mathcal{O}_X(U)$. The notion of D-schemes was considered in Beilinson and Bernstein [2], and most quasi-coherent ringed schemes that appear naturally are D-schemes. We give some important examples of D-schemes below.

Example 4 Let $(X, \mathcal{O}_X)$ be a scheme over k, and assume that $\operatorname{char}(k) = 0$. For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, we denote by $\operatorname{Diff}(\mathcal{F})$ the sheaf of k-linear *differential operators* on $\mathcal{F}$, see Grothendieck [9, §16.8]. By definition, $\operatorname{Diff}(\mathcal{F})$ is a sheaf of associative

k-algebras on X, equipped with a morphism $i\colon \mathcal{O}_X \to \mathrm{Diff}(\mathcal{F})$ of sheaves of rings, and $\mathrm{Diff}(\mathcal{F})$ is clearly a D-algebra on X if and only if $\mathrm{Diff}(\mathcal{F})$ is quasi-coherent as a left and right $\mathcal{O}_X$-module. By Beilinson–Bernstein [2, Example 1.1.6], this is the case if $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module.

Let X be locally Noetherian, and consider the sheaf $\mathcal{D}_X = \mathrm{Diff}(\mathcal{O}_X)$ of k-linear *differential operators* on X. Since $\mathcal{O}_X$ is a coherent sheaf of rings, it follows that $\mathcal{D}_X$ is a D-algebra on X.

We remark that there are some examples of schemes over k that are $\mathcal{D}_X$-affine but not affine. For instance, this holds for the projective space $X = \mathbf{P}^n$ for all integers $n \geq 1$ (see [1]). It also holds for the weighted projective space $X = \mathbf{P}(a_1, \dots, a_n)$ (see [19]).

Example 5 Let $(X, \mathcal{O}_X)$ be a separated scheme of finite type over k, and assume that $\mathrm{char}(k) = 0$. A *Lie algebroid* of X is a quasi-coherent $\mathcal{O}_X$-module $\mathbf{g}$ with a k-Lie algebra structure, together with a morphism $\tau\colon \mathbf{g} \to \mathcal{D}er_k(\mathcal{O}_X)$ of sheaves of $\mathcal{O}_X$-modules and of k-Lie algebras, such that

$$[g, f \cdot h] = f[g, h] + \tau_g(f) \cdot h$$

for any open subset $U \subseteq X$ and any sections $g, h \in \mathbf{g}(U)$, $f \in \mathcal{O}_X(U)$. The notion of Lie algebroids in algebraic geometry was considered in Beilinson–Bernstein [2].

For any sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules, an integrable $\mathbf{g}$-*connection* on $\mathcal{F}$ is a morphism $\nabla\colon \mathbf{g} \to \mathcal{E}nd_k(\mathcal{F})$ of sheaves of $\mathcal{O}_X$-modules and of k-Lie algebras such that

$$\nabla_U(g)(fm) = f\nabla_U(g)(m) + g(f)m$$

for any open subset $U \subseteq X$ and any sections $f \in \mathcal{O}_X(U)$, $g \in \mathbf{g}(U)$, $m \in \mathcal{F}(U)$. The quasi-coherent sheaves of $\mathcal{O}_X$-modules with integrable $\mathbf{g}$-connections form an Abelian k-category, and there is a universal enveloping D-algebra $\mathrm{U}(\mathbf{g})$ of $\mathbf{g}$ such that this category is equivalent to $\mathsf{QCoh}(\mathrm{U}(\mathbf{g}))$ (see [2]). In particular, $(X, \mathrm{U}(\mathbf{g}))$ is a D-scheme.

The tangent sheaf $\theta_X = \mathcal{D}er_k(\mathcal{O}_X)$ of X is a Lie algebroid of X in a natural way, and $\mathrm{U}(\theta_X)$ is a subsheaf of the sheaf $\mathcal{D}_X$ of k-linear differential operators on X. If X is a smooth irreducible quasi-projective variety over k, then $\mathrm{U}(\theta_X) = \mathcal{D}_X$.

8 Calculations for D-modules on Elliptic Curves

Let k be an algebraically closed field of characteristic 0, and let X be a smooth irreducible variety over k of dimension d. Then the sheaf $\mathcal{D}_X$ of k-linear differential operators on X is a D-algebra on X. We consider the noncommutative deformations of $\mathcal{O}_X$ as a quasi-coherent left $\mathcal{D}_X$-module via the natural left action of $\mathcal{D}_X$ on $\mathcal{O}_X$. As an example, we compute the pro-representing hull of $\mathcal{O}_X$ as a quasi-coherent left $\mathcal{D}_X$-module when X is an elliptic curve (see also [5]).

Let U be an open affine cover of X. Then $U \subseteq X$ is a smooth, irreducible affine variety over k of dimension d for all $U \in \mathsf{U}$. It is well known that $\mathcal{D}_X(U)$ is a simple

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Noetherian ring of global dimension d and that $\mathcal{O}_X(U)$ is a simple left $\mathcal{D}_X(U)$-module (see [18]). Hence $\mathcal{E}xt^q_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) = 0$ for $q \geq d + 1$ and $\mathcal{E}nd_{\mathcal{D}_X}(\mathcal{O}_X) = \underline{k}$. If X is a curve, then the spectral sequence in Proposition 4.2 degenerates, and

$$\mathsf{HH}^n(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong \mathsf{H}^{n-1}(\mathsf{U}, \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)) \text{ for } n \geq 1,$$

$$\mathsf{HH}^0(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k.$$

Let $X \subseteq \mathbf{P}^2$ be the irreducible projective plane curve given by the homogeneous equation $f = 0$, where $f = y^2z - x^3 - axz^2 - bz^3$ for fixed parameters $(a, b) \in k^2$. We assume that $\Delta = 4a^3 + 27b^2 \neq 0$, so that X is smooth and therefore an elliptic curve over k. We choose an open affine cover $\mathsf{U} = \{U_1, U_2, U_3\}$ of X closed under intersections, given by $U_1 = D_+(y)$, $U_2 = D_+(z)$, and $U_3 = U_1 \cap U_2$. We shall compute $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)$ and $\mathsf{H}^{n-1}(\mathsf{U}, \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X))$ for $n = 1, 2$.

Let $A_i = \mathcal{O}_X(U_i)$ and $D_i = \mathcal{D}_X(U_i)$ for $i = 1, 2, 3$. We see that $A_1 \cong k[x, z]/(f_1)$ and $A_2 \cong k[x, y]/(f_2)$, where $f_1 = z - x^3 - axz^2 - bz^3$ and $f_2 = y^2 - x^3 - ax - b$. Moreover, we have that $\operatorname{Der}_k(A_i) = A_i\partial_i$ and $D_i = A_i\langle\partial_i\rangle$ for $i = 1, 2$, where

$$\partial_1 = (1 - 2axz - 3bz^2)\, \partial/\partial x + (3x^2 + az^2)\partial/\partial z,$$

$$\partial_2 = -2y\partial/\partial x - (3x^2 + a)\partial/\partial y.$$

On the intersection $U_3 = U_1 \cap U_2$, we choose an isomorphism $A_3 \cong k[x, y, y^{-1}]/(f_3)$ with $f_3 = f_2$, and see that $\operatorname{Der}_k(A_3) = A_3\partial_3$ and $D_3 = A_3\langle\partial_3\rangle$ for $\partial_3 = \partial_2$. The restriction maps of $\mathcal{O}_X$ and $\mathcal{D}_X$, considered as presheaves on U, are given by

$$x \mapsto xy^{-1}, \quad z \mapsto y^{-1}, \quad \partial_1 \mapsto \partial_2$$

for the inclusion $U_1 \supseteq U_3$, and the natural localization map for $U_2 \supseteq U_3$. Finally, we find a free resolution of A_i as a left D_i-module for $i = 1, 2, 3$, given by

$$0 \leftarrow A_i \leftarrow D_i \xleftarrow{\cdot\partial_i} D_i \leftarrow 0$$

and use this to compute $\operatorname{Ext}^1_{D_i}(A_i, A_j) \cong \operatorname{coker}(\partial_i|_{U_j} : A_j \to A_j)$ for all $U_i \supseteq U_j$ in U. We see that $\operatorname{Ext}^1_{D_i}(A_i, A_3) \cong \operatorname{coker}(\partial_3 : A_3 \to A_3)$ is independent of i, and find the following k-linear bases for $\operatorname{Ext}^1_{D_i}(A_i, A_j)$:

	$a \neq 0$	$a = 0$
$U_1 \supseteq U_1$	$1, z, z^2, z^3$	$1, z, x, xz$
$U_2 \supseteq U_2$	$1, y^2$	$1, x$
$U_3 \supseteq U_3$	$x^2y^{-1}, 1, y^{-1}, y^{-2}, y^{-3}$	$x^2y^{-1}, 1, y^{-1}, x, xy^{-1}$

The functor $\mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X) \colon \mathsf{Mor}\,\mathsf{U} \to \mathsf{Mod}(k)$ defines the following diagram in $\mathsf{Mod}(k)$, where the maps are induced by the restriction maps on $\mathcal{O}_X$:

$$\begin{array}{ccccc} \operatorname{Ext}^1_{D_1}(A_1, A_1) & & & & \operatorname{Ext}^1_{D_2}(A_2, A_2) \\ \downarrow & & & & \downarrow \\ \operatorname{Ext}^1_{D_1}(A_1, A_3) & = & \operatorname{Ext}^1_{D_3}(A_3, A_3) & = & \operatorname{Ext}^1_{D_2}(A_2, A_3) \end{array}$$

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We use that $15y^2 = \Delta\ y^{-2}$ in $\mathrm{Ext}^1_{D_3}(A_3, A_3)$ when $a \neq 0$ and that $-3b\ xy^{-2} = x$ in $\mathrm{Ext}^1_{D_3}(A_3, A_3)$ when $a = 0$ to describe these maps in the given bases, and compute $\mathsf{H}^{n-1}(\mathsf{U}, \mathrm{Ext}^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X))$ for $n = 1, 2$ using the resolving complex $\mathsf{D}^*(\mathsf{U}, -)$. We find the following k-linear bases:

	$a \neq 0$	$a = 0$
$n = 1$	$\xi_1 = (1, 1, 1)$, $\xi_2 = (\Delta z^2, 15y^2, \Delta y^{-2})$	$\xi_1 = (1, 1, 1)$, $\xi_2 = (-3b\ xz, x, x)$
$n = 2$	$\omega = (0, 0, 0, 0, 6ax^2y^{-1})$	$\omega = (0, 0, 0, 0, x^2y^{-1})$

We recall that ξ_1, ξ_2, and ω are represented by cocycles of degree $p = 0$ and $p = 1$ in the resolving complex $\mathsf{D}^*(\mathsf{U}, \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X))$, where

$$\mathsf{D}^p(\mathsf{U}, \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)) = \prod_{U_0 \supseteq \cdots \supseteq U_p} \mathcal{E}xt^1_{\mathcal{D}_X}(\mathcal{O}_X, \mathcal{O}_X)(U_0 \supseteq U_p)$$

and the product is indexed by $\{U_1 \supseteq U_1, U_2 \supseteq U_2, U_3 \supseteq U_3\}$ when $p = 0$, and $\{U_1 \supseteq U_1, U_2 \supseteq U_2, U_3 \supseteq U_3, U_1 \supseteq U_3, U_2 \supseteq U_3\}$ when $p = 1$.

This proves that the noncommutative deformation functor $\mathrm{Def}_{\mathcal{O}_X} \colon \mathsf{a}_1 \to \mathsf{Sets}$ of the left $\mathcal{D}_X$-module $\mathcal{O}_X$ has tangent space $\mathsf{HH}^1(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k^2$ and obstruction space $\mathsf{HH}^2(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X)) \cong k$ for any elliptic curve X over k, and a pro-representing hull $H = k\langle\langle t_1, t_2\rangle\rangle/(F)$ for some noncommutative power series $F \in k\langle\langle t_1, t_2\rangle\rangle$.

We shall compute the noncommutative power series F and the versal family $\mathcal{F}_H \in \mathrm{Def}_{\mathcal{O}_X}(H)$ using the obstruction calculus. We choose base vectors t_1^*, t_2^* in $\mathsf{HH}^1(\mathsf{U}, \mathcal{A}, \mathcal{E}nd_k(\mathcal{O}_X))$, and representatives $(\psi_l, \tau_l) \in \mathsf{D}^{01} \oplus \mathsf{D}^{10}$ of t_l^* for $l = 1, 2$, where $\mathsf{D}^{pq} = \mathsf{D}^p(\mathsf{U}, \mathsf{HC}^q(\mathcal{A}, \mathcal{E}nd_k(\mathcal{O}_X)))$. We may choose $\psi_l(U_i)$ to be the derivation defined by

$$\psi_l(U_i)(P_i) = \begin{cases} 0 & \text{if } P_i \in A_i, \\ \xi_l(U_i) \cdot \mathrm{id}_{A_i} & \text{if } P_i = \partial_i, \end{cases}$$

for $l = 1, 2$ and $i = 1, 2, 3$, and $\tau_l(U_i \supseteq U_j)$ to be the multiplication operator in $\mathrm{Hom}_{A_i}(A_i, A_j) \cong A_j$ given by $\tau_1 = 0$, $\tau_2(U_i \supseteq U_i) = 0$ for $i = 1, 2, 3$ and

$a \neq 0$	$a = 0$
$\tau_2(U_1 \supseteq U_3) = 0$	$\tau_2(U_1 \supseteq U_3) = x^2y^{-1}$
$\tau_2(U_2 \supseteq U_3) = -4a^2y^{-1} - 3xy + 9bxy^{-1} - 6ax^2y^{-1}$	$\tau_2(U_2 \supseteq U_3) = 0$

The restriction of $\mathrm{Def}_{\mathcal{O}_X} \colon \mathsf{a}_1 \to \mathsf{Sets}$ to $\mathsf{a}_1(2)$ is represented by $(H_2, \mathcal{F}_{H_2})$, where

$$H_2 = k\langle t_1, t_2\rangle/(t_1, t_2)^2$$

and the deformation $\mathcal{F}_{H_2} \in \mathrm{Def}_{\mathcal{O}_X}(H_2)$ is defined by $\mathcal{F}_{H_2}(U_i) = A_i \otimes_k H_2$ as a right H_2-module for $i = 1, 2, 3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

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for $i = 1, 2, 3$ and for all $P_i \in D_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j)\; m_i|_{U_j} \otimes t_2$$

for $i = 1, 2$, $j = 3$ and for all $m_i \in A_i$.

Let us attempt to lift the family $\mathcal{F}_{H_2} \in \mathrm{Def}_{\mathcal{O}_X}(H_2)$ to $R = k\langle\langle t_1, t_2\rangle\rangle/(t_1, t_2)^3$. We let $\mathcal{F}_R(U_i) = A_i \otimes_k R$ as a right R-module for $i = 1, 2, 3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

for $i = 1, 2, 3$ and for all $P_i \in D_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto m_i|_{U_j} \otimes 1 + \tau_2(U_i \supseteq U_j)\; m_i|_{U_j} \otimes t_2 + \frac{\tau_2(U_i \supseteq U_j)^2}{2}\; m_i|_{U_j} \otimes t_2^2$$

for $i = 1, 2$, $j = 3$ and for all $m_i \in A_i$. We see that $\mathcal{F}_R(U_i)$ is a left $\mathcal{D}_X(U_i)$-module for $i = 1, 2, 3$, and that $t_1 t_2 - t_2 t_1 = 0$ is a necessary and sufficient condition for $\mathcal{D}_X$-linearity of the restriction maps for the inclusions $U_1 \supseteq U_3$ and $U_2 \supseteq U_3$. This implies that $\mathcal{F}_R$ is not a lifting of $\mathcal{F}_{H_2}$ to R. But if we consider the quotient $H_3 = R/(t_1 t_2 - t_2 t_1)$, we see that the family $\mathcal{F}_{H_3} \in \mathrm{Def}_{\mathcal{O}_X}(H_3)$ induced by $\mathcal{F}_R$ is a lifting of $\mathcal{F}_{H_2}$ to H_3.

In fact, we claim that the restriction of $\mathrm{Def}_{\mathcal{O}_X} \colon \mathsf{a}_1 \to \mathsf{Sets}$ to $\mathsf{a}_1(3)$ is represented by $(H_3, \mathcal{F}_{H_3})$. One way to prove this is to show that it is not possible to find any lifting $\mathcal{F}'_R \in \mathrm{Def}_{\mathcal{O}_X}(R)$ of $\mathcal{F}_{H_2}$ to R. Another approach is to calculate the cup products $\langle t_i^*, t_j^*\rangle$ in global Hochschild cohomology for $i, j = 1, 2$, and this gives

$$\begin{aligned} &\langle t_1^*, t_2^*\rangle = o^*, \quad \langle t_2^*, t_1^*\rangle = -o^* && \text{for } a \neq 0,\\ &\langle t_1^*, t_2^*\rangle = o^*, \quad \langle t_2^*, t_1^*\rangle = -o^* && \text{for } a = 0, \end{aligned}$$

where $o^* \in \mathrm{HH}^2(\mathsf{U}, \mathcal{D}_X, \mathcal{E}nd_k(\mathcal{O}_X))$ is the base vector corresponding to ω. Since all other cup products vanish, this implies that $F = t_1 t_2 - t_2 t_1 \mod (t_1, t_2)^3$.

Let $H = k\langle\langle t_1, t_2\rangle\rangle/(t_1 t_2 - t_2 t_1)$. We shall show that it is possible to find a lifting $\mathcal{F}_H \in \mathrm{Def}_{\mathcal{O}_X}(H)$ of $\mathcal{F}_{H_3}$ to H. We let $\mathcal{F}_{\mathrm{H}}(U_i) = A_i \widehat{\otimes}_k H$ as a right H-module for $i = 1, 2, 3$, with left D_i-module structure given by

$$P_i(m_i \otimes 1) = P_i(m_i) \otimes 1 + \psi_1(U_i)(P_i)(m_i) \otimes t_1 + \psi_2(U_i)(P_i)(m_i) \otimes t_2$$

for $i = 1, 2, 3$ and for all $P_i \in \mathcal{D}_i$, $m_i \in A_i$, and with restriction map for the inclusion $U_i \supseteq U_j$ given by

$$m_i \otimes 1 \mapsto \sum_{n=0}^{\infty} \frac{\tau_2(U_i \supseteq U_j)^n}{n!}\; m_1|_{U_j} \otimes t_2^n = \exp(\tau_2(U_i \supseteq U_j) \otimes t_2) \cdot (m_1|_{U_j} \otimes 1)$$

for $i = 1, 2,\ \ j = 3$ and for all $m_i \in A_i$. This implies that $(H, \mathcal{F}_H)$ is the pro-representing hull of $\mathrm{Def}_{\mathcal{O}_X}$, and that $F = t_1t_2 - t_2t_1$. We remark that the versal family $\mathcal{F}_H$ does not admit an algebraization, *i.e.*, an algebra H_{alg} of finite type over k such that H is a completion of H_{alg}, together with a deformation in $\mathrm{Def}_{\mathcal{O}_X}(H_{\mathrm{alg}})$ that induces the versal family $\mathcal{F}_H \in \mathrm{Def}_{\mathcal{O}_X}(H)$.

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