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$\partial\overline{\partial}$-PROBLEM ON WEAKLY 1-COMPLETE KÄHLER MANIFOLDS

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To the memory of Prof. Makoto Suzuki

Abstract. We consider a problem whether Kodaira's $\partial\overline{\partial}$-Lemma holds on weakly 1-complete Kähler manifolds or not. This problem was proposed by S. Nakano. We prove that the Lemma holds for some class of complex quasi-tori $\mathbb{C}^n/\Gamma$, and it does not hold for the other class of them. Every complex quasi-tori is weakly 1-complete and complete Kähler. Then we get a negative answer for the above problem.

§1. Introduction

The following lemma proved by Kodaira is well-known and usually called "$\partial\overline{\partial}$-Lemma" ([9, Proposition 7.1]).

$\partial\overline{\partial}$-LEMMA. *Let X be a compact Kähler manifold and φ a d-exact $(1,1)$-form on X. Then there exists a C^∞-function Ψ on X such that*

$$\varphi = \partial\overline{\partial}\Psi$$

on X.

In [14] many problems concerning function theory of several complex variables are posed. There S. Nakano gives a problem concerning the above $\partial\overline{\partial}$-Lemma as follows.

A complex manifold X is called *weakly* 1*-complete* if there exists a C^∞-plurisubharmonic exhaustive function on X. Easily we can see that a compact complex manifold is weakly 1-complete, a strongly 1-convex manifold is weakly 1-complete and then every Stein manifold is weakly 1-complete.

PROBLEM 1.1. Can one show $\partial\overline{\partial}$-Lemma on weakly 1-complete Kähler manifolds ?

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We can give a very easy counterexample to this problem (Example 4.1); nevertheless, it is very interesting to consider this from the other aspects. We give reformed problems of it in §4.

A connected complex Lie group without global non-constant holomorphic function is called a *toroidal* group. Every complex n-dimensional toroidal group is isomorphic to $\mathbb{C}^n/\Gamma$ for some discrete subgroup Γ ([8]). A complex torus is an example of a toroidal group.

It is shown that every toroidal group is always weakly 1-complete ([4], [11]). From the natural covering structure

$$\mathbb{C}^n \longrightarrow \mathbb{C}^n/\Gamma$$

it follows that every toroidal group $\mathbb{C}^n/\Gamma$ is a complete Kähler manifold.

In this paper we will consider whether $\partial\overline{\partial}$-Lemma holds on toroidal groups or not.

Every toroidal group $\mathbb{C}^n/\Gamma$ satisfies either of the following statements (1) and (2) ([5], [12]):

(1) $H^p(\mathbb{C}^n/\Gamma, \mathcal{O})$ is finite-dimensional for any p;

(2) $H^p(\mathbb{C}^n/\Gamma, \mathcal{O})$ is a non-Hausdorff and then infinite-dimensional locally convex space for any p with $1 \leq p \leq q$,

where $\mathcal{O}$ denotes the structure sheaf of $\mathbb{C}^n/\Gamma$ and $q := \operatorname{rank}\Gamma - n$. From this result we can classify all toroidal groups. We say that a toroidal group is *of cohomologically finite type* if it satisfies the above property (1) and *of non-Hausdorff type* if it satisfies the above property (2), respectively.

We will show that $\partial\overline{\partial}$-Lemma holds for toroidal groups of cohomologically finite type and that it does not hold for toroidal groups of non-Hausdorff type.

This gives the negative answer for the above problem even if we consider it only for toroidal groups.

We wish to thank Prof. Koji Cho who gave a suggestion for us to generalize our former statements of Theorem 3.3.

§2. Toroidal groups

Throughout this section we consider a toroidal group $\mathbb{C}^n/\Gamma$, where Γ is a discrete subgroup of $\mathbb{C}^n$ and of rank $n+q$ generated by $\mathbb{R}$-linearly independent vectors $\{e_1, e_2, \ldots, e_n, v_1 = (v_{11}, \ldots, v_{1n}), v_2 = (v_{21}, \ldots, v_{2n}), \ldots, v_q = (v_{q1}, \ldots, v_{qn})\}$ over $\mathbb{Z}$ and e_i denotes the i-th unit vector of $\mathbb{C}^n$. We take

$\operatorname{Re} v_i, \operatorname{Im} v_i \in \mathbb{R}^n$ with $v_i = \operatorname{Re} v_i + \sqrt{-1}\operatorname{Im} v_i$. Since $e_1, e_2, \dots, e_n, v_1, v_2, \dots, v_q$ are $\mathbb{R}$-linearly independent, $\operatorname{Im} v_1, \operatorname{Im} v_2, \dots, \operatorname{Im} v_q$ are $\mathbb{R}$-linearly independent. Then without loss of generality we may assume $\det[\operatorname{Im} v_{ij}\ ;\ 1 \le i, j \le q] \ne 0$ from now on. We set

$$(2.1)\quad K_{m,i} := \sum_{j=1}^{n} v_{ij} m_j - m_{n+i} \quad\text{and}\quad K_m := \max\{|K_{m,i}|\ ;\ 1 \le i \le q\}$$

for $m = (m_1, m_2, \dots, m_{n+q}) \in \mathbb{Z}^{n+q}$. From the result of [8] it follows that $\mathbb{C}^n/\Gamma$ is toroidal if and only if

$$(2.2)\qquad K_m > 0 \quad\text{for any } m \in \mathbb{Z}^{n+q} \setminus \{0\}.$$

We denote by π_q the projection $\mathbb{C}^n \ni (z_1, \dots, z_n) \mapsto (z_1, \dots, z_q) \in \mathbb{C}^q$. Since $\pi_q(e_i)$, $\pi_q(v_i)$ $(1 \le i \le q)$ are $\mathbb{R}$-linearly independent, π_q induces the $\mathbb{C}^{*n-q}$-principal bundle

$$(2.3)\qquad \pi_q : \mathbb{C}^n/\Gamma \ni z + \Gamma \longmapsto \pi_q(z) + \Gamma^* \in \mathbb{T}^q_C := \mathbb{C}^q/\Gamma^*$$

over the complex q-dimensional torus $\mathbb{T}^q_C$, where $\Gamma^* := \pi_q(\Gamma)$ ([5]). We put

$$\alpha_{ij} := \begin{cases} \operatorname{Re} v_{ij} & (1 \le i \le q,\ 1 \le j \le n) \\ 0 & (q+1 \le i \le n,\ 1 \le j \le n), \end{cases}$$
$$\beta_{ij} := \begin{cases} \operatorname{Im} v_{ij} & (1 \le i \le q,\ 1 \le j \le n) \\ \delta_{ij} & (q+1 \le i \le n,\ 1 \le j \le n), \end{cases}$$

$[\gamma_{ij}\ ;\ 1 \le i, j \le n] := [\beta_{ij}\ ;\ 1 \le i, j \le n]^{-1}$ and $v_i := \sqrt{-1}\, e_i$ for $q+1 \le i \le n$. Since $\{e_1, \dots, e_n, v_1, \dots, v_n\}$ are $\mathbb{R}$-linearly independent, we have an isomorphism

$$\phi : \mathbb{C}^n \ni (z_1, \dots, z_n) \longmapsto (t_1, \dots, t_{2n}) \in \mathbb{R}^{2n}$$

as a real Lie group, where $(z_1, \dots, z_n) = \sum_{i=1}^{n}(t_i e_i + t_{n+i} v_i)$. Then we obtain the relations

$$(2.4)\qquad t_j = x_j - \sum_{i,k=1}^{n} y_k \gamma_{ki} \alpha_{ij} \quad\text{and}\quad t_{n+j} = \sum_{i=1}^{n} y_i \gamma_{ij}$$

for $1 \le j \le n$, where $z_i = x_i + \sqrt{-1}\, y_i$. We put $t = (t', t'')$, $t' = (t_1, \dots, t_{n+q}) \in \mathbb{R}^{n+q}$ and $t'' = (t_{n+q+1}, \dots, t_{2n}) \in \mathbb{R}^{n-q}$. ϕ induces the isomorphism:

$\phi : \mathbb{C}^n/\Gamma \cong \mathbb{T}^{n+q} \times \mathbb{R}^{n-q}$ as a real Lie group, where $\mathbb{T}^{n+q}$ is an $(n+q)$-dimensional real torus. Sometimes we identify $\mathbb{C}^n/\Gamma$ with the real Lie group $\mathbb{T}^{n+q} \times \mathbb{R}^{n-q}$ and use the real coordinate system $(t_1, \dots, t_{2n})$ instead of holomorphic coordinates.

We make the following change of holomorphic coordinates of $\mathbb{C}^n$:

$$\zeta_i = \sum_{j=1}^{n} z_j \gamma_{ji}.$$

Then we can regard $(\zeta_1, \dots, \zeta_n)$ as a local holomorphic coordinate system of $\mathbb{C}^n/\Gamma$ and we have global vector fields and global 1-forms:

$$\frac{\partial}{\partial \overline{\zeta}_i} = \sum_{j=1}^{n} \beta_{ij} \frac{\partial}{\partial \overline{z}_j}, \quad \frac{\partial}{\partial \zeta_i} = \sum_{j=1}^{n} \beta_{ij} \frac{\partial}{\partial z_j},$$

$$d\overline{\zeta}_i = \sum_{j=1}^{n} \gamma_{ij}\, d\overline{z}_j, \ \text{ and } \ d\zeta_i = \sum_{j=1}^{n} \gamma_{ij}\, dz_j$$

$(1 \le i \le n)$ on $\mathbb{C}^n/\Gamma$. It follows from (2.4) that

$$\text{(2.5)} \quad \begin{aligned} \frac{\partial}{\partial \overline{\zeta}_i} &= \frac{1}{2}\left(\sum_{j=1}^{n} \beta_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \sum_{j=1}^{n} \alpha_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right), \\ \frac{\partial}{\partial \zeta_i} &= \frac{1}{2}\left(\sum_{j=1}^{n} \beta_{ij} \frac{\partial}{\partial t_j} + \sqrt{-1} \sum_{j=1}^{n} \alpha_{ij} \frac{\partial}{\partial t_j} - \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right). \end{aligned}$$

Then particularly for $q+1 \le i \le n$ we have

$$\text{(2.6)} \quad \frac{\partial}{\partial \overline{\zeta}_i} = \frac{1}{2}\left(\frac{\partial}{\partial t_i} + \sqrt{-1} \frac{\partial}{\partial t_{n+i}} \right).$$

§3. $\partial\overline{\partial}$-Lemma

Let $\mathcal{A}$ be the sheaf of germs of real analytic functions on $\mathbb{C}^n/\Gamma$ and $\mathcal{H}$ its subsheaf of germs of real analytic functions on $\mathbb{C}^n/\Gamma$ that are holomorphic along each fiber of π_q of (2.3). We may consider $(\zeta_{q+1}, \dots, \zeta_n)$ is a holomorphic coordinate of each fiber of π_q. For $0 \le p \le q$ we denote by $\mathcal{H}^{r,p}$ the sheaf of germs of (r,p)-forms as follows

$$\varphi = \frac{1}{r!p!} \sum_{1 \le j_1, \dots, j_r \le n,\, 1 \le i_1, \dots, i_p \le q} \varphi_{j_1 \cdots j_r, i_1 \cdots i_p} d\zeta_{j_1} \wedge \cdots \wedge d\zeta_{j_r} \wedge d\overline{\zeta}_{i_1} \wedge \cdots \wedge d\overline{\zeta}_{i_p},$$

where $\varphi_{j_1\cdots j_r,i_1\cdots i_p} \in \mathcal{H}$ is skew-symmetric in all indices. Henceforth all differential forms are denoted skew-symmetrically and we use the notations

$$J_r = (j_1, \ldots, j_r), \quad d\zeta_{J_r} = d\zeta_{j_1} \wedge \cdots \wedge d\zeta_{j_r},$$
$$I_p = (i_1, \ldots, i_p), \quad d\overline{\zeta}_{I_p} = d\overline{\zeta}_{i_1} \wedge \cdots \wedge d\overline{\zeta}_{i_p}.$$

Then we write $\varphi = 1/(r!p!) \sum_{J_r,I_p} \varphi_{J_r,I_p}\, d\zeta_{J_r} \wedge d\overline{\zeta}_{I_p}$.

Let Ω^r be the sheaf of germs of holomorphic $(r,0)$-forms on $\mathbb{C}^n/\Gamma$. We have the following lemma.

LEMMA 3.1. *The sequence*

$$0 \longrightarrow \Omega^r \longrightarrow \mathcal{H}^{r,0} \xrightarrow{\overline{\partial}} \mathcal{H}^{r,1} \xrightarrow{\overline{\partial}} \cdots \xrightarrow{\overline{\partial}} \mathcal{H}^{r,q} \longrightarrow 0$$

is exact on $\mathbb{C}^n/\Gamma$ and one obtain a kind of Dolbeault isomorphism

$$H^p(\mathbb{C}^n/\Gamma, \Omega^r) = \frac{\{\varphi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p}) \mid \overline{\partial}\varphi = 0\}}{\overline{\partial} H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p-1})}$$

for $p \geq 1$.

Proof. If $r = 0$, then $\Omega^r = \mathcal{O}$. We obtain the exact sequence:

$$0 \longrightarrow \Omega^0 \longrightarrow \mathcal{H}^{0,0} \longrightarrow \mathcal{H}^{0,1} \longrightarrow \cdots \longrightarrow \mathcal{H}^{0,q} \longrightarrow 0 \tag{3.1}$$

by [5, Proposition 3.4]. We can take a basis

$$\{d\zeta_{J_r} \mid 1 \leq j_1 < \cdots < j_r \leq n\}$$

of $H^0(\mathbb{C}^n/\Gamma, \Omega^r)$. For every points $[z] \in \mathbb{C}^n/\Gamma$ we have the isomorphisms

$$\Omega^r_{[z]} \cong \bigoplus_{J_r} \Omega^0_{[z]}(d\zeta_{J_r})_{[z]},$$
$$\mathcal{H}^{r,p}_{[z]} \cong \bigoplus_{J_r} \mathcal{H}^{0,p}_{[z]}(d\zeta_{J_r})_{[z]}$$

of each stalk of sheaves. Observing coefficients of each $(d\zeta_{J_r})_{[z]}$, we can divide the sheaf complex of the statement of the lemma to $\binom{n}{r}$ complexes so that each complex can be identified with (3.1). This argument shows also the latter half of the lemma. ∎

Now we recall the argument of §4 of [5]. For $\varphi \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{r,p})$. We can write

$$\varphi = \frac{1}{r!p!} \sum_{J_r, I_p} \varphi_{J_r,I_p}\, d\zeta_{J_r} \wedge d\overline{\zeta}_{I_p},$$

where $\varphi_{J_r,I_p} \in H^0(\mathbb{C}^n/\Gamma, \mathcal{H}^{0,0})$. The function φ_{J_r,I_p} has the Fourier expansion on $\mathbb{C}^n/\Gamma$:

$$\varphi_{J_r,I_p} = \sum_{m \in \mathbb{Z}^{n+q}} C^m_{J_r,I_p}(t'') \exp(2\pi\sqrt{-1}\,\langle m, t' \rangle),$$

where $C^m_{J_r,I_p}(t'')$'s are C^∞ functions on t'' and $\langle m, t' \rangle := \sum_{i=1}^{n+q} m_i t_i$. Since the function φ_{J_r,I_p} is holomorphic along the fibers of the map of (2.3), then for $q+1 \le i \le n$

$$\frac{\partial C^m_{J_r,I_p}(t'')}{\partial \overline{\zeta}_i} = 0.$$

From (2.6) we have the following Fourier series:

$$\varphi_{J_r,I_p} = \sum_{m \in \mathbb{Z}^{n+q}} c^m_{J_r,I_p} \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\right) \exp(2\pi\sqrt{-1}\,\langle m, t' \rangle),$$

where $c^m_{J_r,I_p}$'s are constants.

We put

$$\varphi^m_{J_r,I_p} = c^m_{J_r,I_p} \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\right) \exp(2\pi\sqrt{-1}\,\langle m, t' \rangle) \tag{3.2}$$

and

$$\varphi^m = \frac{1}{r!p!} \sum_{J_r, I_p} \varphi^m_{J_r,I_p}\, d\zeta_{J_r} \wedge d\overline{\zeta}_{I_p}.$$

Then $\varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m$. It follows from (2.1), (2.5) and (3.2) that for $1 \le \ell \le q$

$$\frac{\partial \varphi^m_{J_r,I_p}}{\partial \overline{\zeta}_\ell} = \pi K_{m,\ell}\, \varphi^m_{J_r,I_p}, \qquad \frac{\partial \varphi^m_{J_r,I_p}}{\partial \zeta_\ell} = \pi \overline{K}_{m,\ell}\, \varphi^m_{J_r,I_p}. \tag{3.3}$$

Now we suppose φ is $\overline{\partial}$-closed, that is, $\overline{\partial}\varphi^m = 0$ for any $m \in \mathbb{Z}^{n+q}$. The compatiblity condition for φ to be $\overline{\partial}$-closed is expressed by the Fourier coefficients such that

$$\sum_{\ell=1}^{p+1} (-1)^\ell K_{m,i_\ell} c^m_{J_r, i_1 \cdots \hat{i_\ell} \cdots i_{p+1}} = 0 \tag{3.4}$$

for any $J_r, I_{p+1} = (i_1, \dots, i_{p+1})$, and $m \in \mathbb{Z}^{n+q}$. For $m \in \mathbb{Z}^{n+q} \setminus \{0\}$ we put $i(m) := \min\{i \mid |K_{m,i}| = K_m,\ 1 \le i \le q\}$. Replacing $I_{p+1} = (i_1, \dots, i_{p+1})$ of (3.4) by $(i(m), i_1, \dots, i_p)$, then we have

$$K_{m,i(m)} c^m_{J_r, i_1 \cdots i_p} = \sum_{\ell=1}^{p} (-1)^{\ell+1} K_{m,i_\ell} c^m_{J_r, i(m) i_1 \cdots \hat{i_\ell} \cdots i_p} = 0. \tag{3.5}$$

For $m \neq 0$ we have, by (2.2), $K_{m,i(m)} \neq 0$ and then we can put

$$\psi^m := \frac{(-1)^r}{\pi r!(p-1)!} \sum_{J_r, I_{p-1}} \frac{c^m_{J_r, i(m) i_1 \cdots i_{p-1}}}{K_{m,i(m)}} \exp\Big(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\Big) \times \exp(2\pi\sqrt{-1}\,\langle m, t'\rangle)\, d\zeta_{J_r} \wedge d\overline{\zeta}_{I_{p-1}},$$

where $I_{p-1} := (i_1, \dots, i_{p-1})$. Then by (3.3) and (3.5) we obtain

$$\overline{\partial}\psi^m = \varphi^m$$

for $m \neq 0$. This means that any $\overline{\partial}$-closed form $\varphi = \sum_{m \in \mathbb{Z}^{n+q}} \varphi^m$ has a formal solution $\sum_{m \neq 0} \psi^m$ of the $\overline{\partial}$-equation:

$$\overline{\partial} \sum_{m \neq 0} \psi^m = \sum_{m \neq 0} \varphi^m.$$

Hence it is determined by the behavior of the lower limit of the sequence of positive numbers:

$$\{K_m \mid m \in \mathbb{Z}^{n+q}\}$$

whether the formal solution is a real solution or not.

The following theorem characterizes toroidal groups of cohomologically finite type.

THEOREM 3.2. ([5], [13]) *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then the following statements* (1), (2), (3) *and* (4) *are equivalent.*

(1) *$\mathbb{C}^n/\Gamma$ is of cohomologically finite type.*

(2) *There exists $a > 0$ such that*

$$\sup_{m \neq 0} \exp(-a\|m^*\|)/K_m < \infty,$$

where $\|m^*\| = \max\{|m_i|\ ;\ 1 \le i \le n\}$.

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(3)

$$\dim H^p(\mathbb{C}^n/\Gamma, \Omega^r) = \begin{cases} \binom{n}{r}\binom{q}{p} & \textit{if } 1 \le p \le q \textit{ and } 0 \le r \le n \\ 0 & \textit{if } p > q \textit{ or } r > n. \end{cases}$$

(4) *Every* C^∞ $\overline{\partial}$*-closed* (r,p)*-form on* $\mathbb{C}^n/\Gamma$ *is* $\overline{\partial}$*-cohomologous to a constant form*

$$\frac{1}{r!p!}\sum_{J_r,I_p} c_{J_r,I_p}\, d\zeta_{J_r} \wedge d\overline{\zeta}_{I_p},$$

where c_{J_r,I_p}*'s are constants,* $r \ge 0$ *and* $p \ge 0$*.*

Let r, $p \ge 1$ and let φ be a d-exact C^∞ (r,p)-form on $\mathbb{C}^n/\Gamma$. Then there exists $(r+p-1)$-form $\psi = \psi_{(r-1,p)} + \psi_{(r,p-1)}$ such that

$$\varphi = d\psi = \partial\psi_{(r-1,p)} + \overline{\partial}\psi_{(r-1,p)} + \partial\psi_{(r,p-1)} + \overline{\partial}\psi_{(r,p-1)},$$

where $\psi_{(i,j)}$ denotes the component of type (i,j) of ψ. Since φ is (r,p)-form, then $\partial\psi_{(r,p-1)} = 0$ and $\overline{\partial}\psi_{(r-1,p)} = 0$. Then $\overline{\psi}_{(r,p-1)}$ and $\psi_{(r-1,p)}$ are a $\overline{\partial}$-closed form of type $(p-1,r)$ and a $\overline{\partial}$-closed form of type $(r-1,p)$ on $\mathbb{C}^n/\Gamma$, respectively. Now suppose that $\mathbb{C}^n/\Gamma$ is of cohomologically finite type. Then by Theorem 3.2, $\overline{\psi}_{(r,p-1)}$ and $\psi_{(r-1,p)}$ are $\overline{\partial}$-cohomologue to some constant forms, that is, there exist a $(r-1,p-1)$-form $\Psi^{(1)}$ and a $(p-1,r-1)$-form $\Psi^{(2)}$ such that

$$\begin{aligned} \psi_{(r-1,p)} &= \frac{1}{(r-1)!p!}\sum_{J_{r-1},I_p} c^{(1)}_{J_{r-1},I_p}\, d\zeta_{J_{r-1}} \wedge d\overline{\zeta}_{I_p} + \overline{\partial}\Psi^{(1)}, \\ \overline{\psi}_{(r,p-1)} &= \frac{1}{r!(p-1)!}\sum_{J_{p-1},I_r} c^{(2)}_{J_{p-1},I_r}\, d\zeta_{J_{p-1}} \wedge d\overline{\zeta}_{I_r} + \overline{\partial}\Psi^{(2)}. \end{aligned}$$

Since the constant forms are ∂-, $\overline{\partial}$-closed, we have

$$\begin{aligned} \varphi &= \partial\psi_{(r-1,p)} + \overline{\partial}\psi_{(r,p-1)} \\ &= \partial\overline{\partial}\Psi^{(1)} + \overline{\partial}\partial\overline{\Psi^{(2)}} \\ &= \partial\overline{\partial}(\Psi^{(1)} - \overline{\Psi^{(2)}}). \end{aligned}$$

This shows $\partial\overline{\partial}$-Lemma holds on toroidal groups of cohomologically finite type. We have the following theorem.

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THEOREM 3.3. *Let $\mathbb{C}^n/\Gamma$ be a toroidal group. Then*

(1) *If $\mathbb{C}^n/\Gamma$ is of cohomologically finite type and $r, p \geq 1$, then for any d-exact (r,p)-form φ there exists $(r-1,p-1)$-form Ψ such that $\varphi = \partial\overline{\partial}\Psi$ on $\mathbb{C}^n/\Gamma$. Further if $r = p$ and φ is a real form, one can choose the above Ψ so that $\sqrt{-1}\,\Psi$ is also real.*

(2) *If $\mathbb{C}^n/\Gamma$ is of non-Hausdorff type and $1 \leq r, p \leq q$, for some d-exact (r,p)-form φ there is no solution Ψ satisfying the $\partial\overline{\partial}$-equation $\varphi = \partial\overline{\partial}\Psi$ on $\mathbb{C}^n/\Gamma$.*

Proof. It remains only to prove the latter half of (1) and (2). Suppose $\varphi = \partial\overline{\partial}\Psi$ and φ is real. Then $\varphi = \overline{\varphi} = \overline{\partial}\partial\overline{\Psi} = \partial\overline{\partial}(-\overline{\Psi})$. We obtain

$$\varphi = \partial\overline{\partial}\left(\frac{\Psi - \overline{\Psi}}{2}\right).$$

Since $\sqrt{-1}\,(\Psi - \overline{\Psi})/2$ is real, we obtain the assertion of the latter half of (1).

Next to prove (2) we assume that $\mathbb{C}^n/\Gamma$ is of non-Hausdorff type. By Theorem 3.2 we have

$$\sup_{m\neq 0} \exp(-a\|m^*\|)/K_m = \infty \tag{3.6}$$

for any $a > 0$. For $m = (m_1, m_2, \ldots, m_{n+q}) \in \mathbb{Z}^{n+q}$ we put $\|m'\| := \max\{|m_i|, |m_{n+i}| \mid 1 \leq i \leq q\}$ and $\|m''\| := \max\{|m_j| \mid q+1 \leq j \leq n\}$. By (3.6) there exists $\varepsilon > 0$ such that we can choose a sequence $\{m_\mu \mid \mu \in \mathbb{N}\} \in \mathbb{Z}^{n+q} \setminus \{0\}$ satisfying $\exp(-\varepsilon\|m'_\mu\| - \mu\|m''_\mu\|)/K_{m_\mu} > \mu$ for any $\mu \in \mathbb{N}$ ([5, Lemma 4.2]). Put

$$\delta^m := \begin{cases} \exp(-\varepsilon\|m'_\mu\| - \mu\|m''_\mu\|)/K_{m_\mu} & m = m_\mu \text{ for some } \mu \geq 1, \\ 0 & \text{otherwise.} \end{cases}$$

We can find i_0 so that $1 \leq i_0 \leq q$ and $\sup\{\mu \mid K_m = |K_{m_\mu, i_0}|\} = \infty$. We may assume $i_0 = q$ without loss of generality. We take a $(r-1,p-1)$-form

$$\psi^m := \delta^m \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\right) \exp(2\pi\sqrt{-1}\,\langle m, t'\rangle) \times d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_{p-1}.$$

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By the choice of the sequence $\{m_\mu \mid \mu \in \mathbb{N}\}$ the formal series $\sum_m \psi^m$ cannot converge to any form. On the other hand

$$\overline{\partial}\psi^m = \sum_{\ell=1}^{q} K_{m,\ell}\delta^m \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\right) \exp(2\pi\sqrt{-1}\,\langle m, t'\rangle) \\ \times d\overline{\zeta}_\ell \wedge d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_{p-1}.$$

Since

$$K_{m_\mu,\ell}\,\delta^{m_\mu} = \frac{K_{m_\mu,\ell}}{K_{m_\mu}} \exp(-\varepsilon\|m'_\mu\| - \mu\|m''_\mu\|)$$

and $|K_{m_\mu,\ell}/K_{m_\mu}| \le 1$, $\sum_m \overline{\partial}\psi^m$ converges to a $\overline{\partial}$-closed $(r-1,p)$-form η ([5, Lemma 4.1]). We put $\varphi = d\eta = \partial\eta$. We suppose that there exists a $C^\infty(r-1,p-1)$-form λ satisfying

$$\partial\overline{\partial}\lambda = \varphi.$$

We can write

$$\lambda^m := \frac{1}{\pi(r-1)!(p-1)!} \sum_{J_{r-1},I_{p-1}} b^m_{J_{r-1},I_{p-1}}(t'') \\ \times \exp(2\pi\sqrt{-1}\,\langle m, t'\rangle)\, d\zeta_{J_{r-1}} \wedge d\overline{\zeta}_{I_{p-1}},$$

where $b^m_{J_{r-1},I_{p-1}}(t'')$'s are C^∞ functions in $t'' \in \mathbb{R}^{n-q}$ and $\lambda = \sum_m \lambda^m$. Then we have $\partial\overline{\partial}\lambda^m = \partial\overline{\partial}\psi^m$. Comparing the term of the left form to the right form of this equation involving only the differential $d\zeta_1 \wedge \cdots \wedge d\zeta_{r-1} \wedge d\zeta_q \wedge d\overline{\zeta}_1 \wedge \cdots \wedge d\overline{\zeta}_{p-1} \wedge d\overline{\zeta}_q$. We can obtain the same formula as (3.3) for C^∞ forms λ^m. Then we obtain

$$\begin{aligned}
&(-1)^{r+p}|K_{m_\mu,q}|^2\delta^{m_\mu} \exp\left(-2\pi \sum_{i=q+1}^{n} m_i t_{n+i}\right) \\
&\quad = (-1)^{r+p}|K_{m_\mu,q}|^2\, b^{m_\mu}_{1\cdots r-1,1\cdots p-1}(t'') \\
&\qquad + \sum_{k=1}^{p-1}\sum_{\ell=1}^{r-1}(-1)^{r+k+\ell} K_{m_\mu,\ell}\overline{K}_{m_\mu,k} b^{m_\mu}_{1\cdots\hat{\ell}\cdots r-1\,q,1\cdots\hat{k}\cdots p-1\,q}(t'') \\
&\qquad + \sum_{\ell=1}^{r-1}(-1)^{r+p+\ell} K_{m_\mu,\ell}\overline{K}_{m_\mu,q} b^{m_\mu}_{1\cdots\hat{\ell}\cdots r-1\,q,1\cdots p-1}(t'') \\
&\qquad + \sum_{k=1}^{p-1}(-1)^{k+1} K_{m_\mu,q}\overline{K}_{m_\mu,k} b^{m_\mu}_{1\cdots r-1,1\cdots\hat{k}\cdots p-1\,q}(t'').
\end{aligned}$$

Since we can choose a subsequence $\{m_{\mu_s}\}$ so that

$$|K_{m_{\mu_s},q}| = K_m,$$

we have, for $t'' = 0$,

$$\alpha|\delta^{m_{\mu_s}}| \leq |b^{m_{\mu_s}}_{1\cdots r-1,1\cdots p-1}(0)| + |b^{m_{\mu_s}}_{1\cdots\hat{\ell}\cdots r-1\,q,1\cdots\hat{k}\cdots p-1\,q}(0)| \\ +|b^{m_{\mu_s}}_{1\cdots\hat{\ell}\cdots r-1\,q,1\cdots p-1}(0)| + |b^{m_{\mu_s}}_{1\cdots r-1,1\cdots\hat{k}\cdots p-1\,q}(0)|,$$

for some positive constant α. Since the coefficients $b^m_{I_{r-1},I_{p-1}}(t'')$ of Fourier series converge to 0 ([2, Proposition 6]), this contradicts that $\lim_{\mu\to\infty}\delta^{m_\mu} = \infty$. □

Remark. In the statement (2) of Theorem 3.3 if $r = p$ and $1 \leq p \leq q$, one can choose φ as a real form. Under the assumption of Theorem 3.3, take any (p,p)-form φ satisfying (2) of Theorem 3.3 and put $\varphi_1 := (\varphi + \overline{\varphi})/2$, $\varphi_2 := (\varphi - \overline{\varphi})/(2\sqrt{-1})$. φ_1 and φ_2 are also d-exact and real (p,p)-forms. Suppose $\varphi_i = \partial\overline{\partial}\Psi_i$ for some $(p-1,p-1)$-form Ψ_i $(i = 1,2)$. Then $\varphi = \partial\overline{\partial}(\Psi_1 + \sqrt{-1}\,\Psi_2)$. This is a contradiction. Hence at least one of φ_1 and φ_2 satisfies the statement (2) of Theorem 3.3.

§4. Examples and related problems

We can show a very easy counter-example to the problem of the introduction of this paper.

Example 4.1. Let $\mathbb{T}_C := \mathbb{C}/\mathbb{Z}\{1,\sqrt{-1}\}$ be a complex torus of complex dimension 1. We put $X := \mathbb{T}_C \times \mathbb{C}$. Trivially X is weakly 1-complete and complete Kähler. Let z be a holomorphic local coordinate induced by the projection: $\mathbb{C} \to \mathbb{T}_C := \mathbb{C}/\mathbb{Z}\{1,\sqrt{-1}\}$ and w be a global coordinate of $\mathbb{C}$. We consider a $(0,1)$-form $\psi := w\,d\overline{z}$ and $\varphi := d\psi$. Suppose there exists a C^∞ function Ψ on X such that

$$\partial\overline{\partial}\Psi = \varphi. \tag{4.1}$$

Then $\partial(\overline{\partial}\Psi - \psi) = 0$. This means $\partial\overline{\Psi} - \overline{\psi} = (\partial\overline{\Psi}/\partial z - \overline{w})\,dz + (\partial\overline{\Psi}/\partial w)\,dw$ is $\overline{\partial}$-closed and then a holomorphic 1-form. Then $\partial\overline{\Psi}/\partial z - \overline{w}$ and $\partial\overline{\Psi}/\partial w$ are holomorphic on X. We have an entire holomorphic function $G(w) := \partial\overline{\Psi}/\partial z - \overline{w}$. We put $x := \operatorname{Re} z$, $y := \operatorname{Im} z$, $u := \operatorname{Re} w$ and $v := \operatorname{Im} w$. We can expand $\overline{\Psi}$ to Fourier series:

$$\overline{\Psi} := \sum_{m\in\mathbb{Z}^2} a^m(u,v)\exp(2\pi\sqrt{-1}\,(m_1x + m_2y)).$$

$\partial\overline{\Psi}/\partial z = \pi\sum_m(m_1\sqrt{-1} - m_2)a^m(u,v)\exp(2\pi\sqrt{-1}\,(m_1x + m_2y))$. Since $\partial\overline{\Psi}/\partial z = \overline{w} + G(w)$ is constant on variables x and y, $a^m = 0$ if $m \neq 0$. Then $\overline{\Psi} = a^0(u,v)$ and $\partial\overline{\Psi}/\partial z = 0$. This contradicts (4.1). By the same reason of Remark in §3 we can select a real $(1,1)$-form that has no solution of the $\partial\overline{\partial}$-equation on X.

Considering the fact that $H^1(X,\mathcal{O})$ is an infinite-dimensional Fréchet space and $\partial\overline{\partial}$-Lemma holds for toroidal groups of cohomologically finite type, we can give the following

Problem 4.2. Can one show $\partial\overline{\partial}$-Lemma on a weakly 1-complete Kähler manifold X with $\dim H^1(X,\mathcal{O}) < \infty$?

If X is strongly 1-convex in the sense of Andreotti and Grauert [2], then $\dim H^1(X,\mathcal{O}) < \infty$. Miyajima [7] considers another type of the $\partial\overline{\partial}$-equation on strongly pseudoconvex Kähler manifolds. In general case of strongly 1-convex Kähler manifolds the above problem still remains unsolved.

Further a weakly reformed problem of Problem 1.1 is posed in [10].

Problem 4.3. Let L be a holomorphic line bundle on a weakly 1-complete manifold X. We assume that the first Chern class $c_1(L)$ has a positive form. Then does L have a Hermitian metric with a positive curvature form?

We remark here that if $\partial\overline{\partial}$-Lemma holds on X, then Problem 4.3 can be solved affirmatively for X. The following example shows that $\partial\overline{\partial}$-Lemma does not hold even if $\mathbb{C}^n/\Gamma$ is a quasi-abelian variety.

Example 4.4. We consider a toroidal group of §2 in the case of $n = 2$ and $q = 1$.

Let Γ be the discrete subgroup generated by $\{e_1, e_2, v_1 := (\sqrt{-1}, \beta)\}$ over $\mathbb{Z}$, where β is an irrational real number. From (2.1) we have $K_m = \sqrt{(\beta m_2 - m_3)^2 + m_1^2}$ and $K_m > 0$ for $m \neq 0$. Then $\mathbb{C}^2/\Gamma$ is toroidal. We put $v_2 := (\beta, \sqrt{-1})$ and consider a complex torus $\mathbb{C}^2/\mathbb{Z}\{e_1, e_2, v_1, v_2\}$. Any such torus is an abelian variety ([3, §2.6 The Riemann Conditions]). We have the covering projection:

$$\mathbb{C}^2/\Gamma \longrightarrow \mathbb{C}^2/\mathbb{Z}\{e_1, e_2, v_1, v_2\}.$$

This means every $\mathbb{C}^2/\Gamma$ is a quasi-abelian variety for any β ([1, Theorem 4.6]). We obtain the following (1) and (2).

(1) If β is an algebraic number, then by Liouville's criterion there exists a positive number M and a positive integer ℓ such that $|\beta - m_3/m_2| > M/|m_2|^\ell$ for any integer m_3 and $m_2 \neq 0$. Since $K_m \geq |\beta m_2 - m_3| > M/|m_2|^{\ell-1}$ $(m_2 \neq 0)$,

$$\sup\left\{\frac{\exp(-\sqrt{m_1^2+m_2^2})}{K_m} \;\middle|\; m \in \mathbb{Z}^3 \setminus \{0\}\right\} < \infty.$$

By Theorem 3.2 $\mathbb{C}^2/\Gamma$ is of cohomologically finite type and then $\partial\overline{\partial}$-Lemma holds on it.

(2) If β is approximated by rational numbers very well, namely, satisfying for any $a > 0$

$$\sup\left\{\frac{\exp(-a|m|)}{|\beta - n/m|} \;\middle|\; m, n \in \mathbb{Z},\, m \neq 0\right\} = \infty,$$

(We find examples of such β in [6] and [12]), by Liouville's criterion such β must be a transcendental number and $\mathbb{C}^2/\Gamma$ is of non-Hausdorff type. Then $\partial\overline{\partial}$-Lemma does not hold on it.

Added in proof. After this paper was submitted, we obtained an answer to Problem 4.2 in the following form: There eixsts a 1-convex Kähler manifold on which the $\partial\overline{\partial}$-Lemma does not hold. This result will appear in our forthcoming paper.

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