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H_∞ FUNCTIONAL CALCULUS OF ELLIPTIC OPERATORS WITH C^∞ COEFFICIENTS ON L^p SPACES OF SMOOTH DOMAINS

XUAN THINH DUONG

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Abstract

The purpose of this paper is to show that higher order elliptic partial differential operators on smooth domains have an H_∞ functional calculus and satisfy quadratic estimates in L^p spaces on these domains.

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1. Introduction

Let A_B be the realization of an elliptic operator A which acts on an L^p space (with $1 < p < \infty$) and whose domain is defined by the boundary condition $Bu = 0$. In [4] and [5], Seeley obtained an asymptotic expansion of the resolvent $R_\lambda = (A_B - \lambda)^{-1}$ which he used to prove the boundedness of purely imaginary powers of A_B:

$$\|A_B^{iy}\| \leq Ce^{|\gamma|y}$$

where y is real and C and γ are positive constants.

An operator is of type ω if its spectrum is contained in the sector $S_\omega = \{\xi \in \mathbb{C} || \arg\xi| \leq \omega\}$ and its resolvent satisfies certain bounds (a precise definition will be given in Section 2). In [3], McIntosh defined a functional calculus for such operators for functions which are analytic (but not necessarily bounded) on S_μ^0 with $\mu > \omega$. He then established the equivalence of the following two properties for an operator T defined on a Hilbert space:

(i) $\|f(T)\| \le C\|f\|_\infty$, $f \in H_\infty(S_\mu^0)$ where $H_\infty(S_\mu^0) = \{g\colon S_\mu^0 \to \mathbb{C} | g$ is analytic and $\|g\|_\infty < \infty\}$ and $\|f\|_\infty = \sup\{|f(z)| | z \in S_\mu^0\}$,

(ii) $\{T^{iy} | y \in \mathbf{R}\}$ is a C^0 group and $\|T^{iy}\| \le C_\mu e^{\mu|y|}$ where μ and C_μ are positive constants.

Note that (i) is not equivalent to, and is in fact stronger than, (ii) for operators defined on Banach spaces. A natural question is whether the elliptic operators considered by Seeley satisfy property (i) in L^p spaces. The main result of this paper is to give an affirmative answer to this question.

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2. Operators of type ω and their functional calculus

We first give some notation and definitions:

$$S_\theta = \{z \in \mathbb{C} | z = 0 \text{ or } |\arg z| \le \theta\},$$
$$S_\theta^0 = \{z \in \mathbb{C} | z \ne 0 \text{ and } |\arg z| < \theta\}.$$

If $0 \le \omega < \pi$, then an operator T in a Banach space X is said to be of type ω if T is closed and densely defined, $\sigma(T) \subset S_\omega \cup \{\infty\}$ and for each $\theta \in (\omega, \pi]$ there exists $C_\theta < \infty$ such that $\|(T - zI)^{-1}\| \le C_\theta |z|^{-1}$ for all non-zero $z \notin S_\theta^0$.

If $0 < \omega \le \pi$, then

$$H_\infty(S_\mu^0) = \{f\colon S_\mu^0 \to \mathbb{C} | f \text{ is analytic and } \|f\|_\infty < \infty\}$$

where $\|f\|_\infty = \sup\{|f(z)| | z \in S_\mu^0\}$

$$\Psi(S_\mu^0) = \left\{ f \in H_\infty(S_\mu^0) | s > 0,\ c \ge 0 \text{ and } |f(z)| \le \frac{c|z|^s}{1 + |z|^{2s}} \text{ for all } z \in S_\mu^0 \right\}.$$

Let $\omega < \theta < \mu$ and Γ be the contour defined by the function

$$g(t) = \begin{cases} -te^{-i\theta}, & -\infty < t \le 0, \\ te^{i\theta}, & 0 \le t < \infty. \end{cases}$$

Let T be of type ω. For $\psi \in \Psi(S_\mu^0)$, we define $\psi(T) \in L(X)$ by

$$\psi(T) = \frac{1}{2\pi i} \int_\Gamma (T - \zeta I)^{-1} \psi(\zeta)\, d\zeta. \tag{2.1}$$

For $f \in H_\infty(S_\mu^0)$, $f(T)$ can be defined by

$$f(T) = (\psi(T))^{-1} (f\psi)(T) \quad \text{where } \psi(\zeta) = \frac{\zeta}{(1 + \zeta)^2}. \tag{2.2}$$

These definitions are well-defined and consistent. See [3] for details of this functional calculus.

We conclude this section with a convergence theorem which appeared in [3, Section 5]

THEOREM. *Let* $0 \leq \omega < \mu \leq \pi$. *Let* T *be an operator of type* ω *which is one-one with dense range. Let* (f_α) *be a uniformly bounded net in* $H_\infty(S_\mu^0)$. *Let* $f \in H_\infty(S_\mu^0)$, *and suppose, for some* $M < \infty$, *that*

(a) $\|f_\alpha(T)\| \leq M$ *and*

(b) *for each* $0 < \delta < \Delta < \infty$,

$$\sup\{|f_\alpha(\zeta) - f(\zeta)| | \zeta \in S_\mu^0 \text{ and } \delta \leq |\zeta| \leq \Delta\} \to 0.$$

Then $f(T) \in L(X)$ *and* $f_\alpha(T)u \to f(T)u$ *for all* $u \in X$. *So* $\|f(T)\| \leq M$.

3. Elliptic boundary value problems

In this paper, we investigate the same elliptic boundary value problems that Seeley studied in [5].

Let A be an elliptic operator of order m, with C^∞ coefficients, acting on q-tuples of functions (or sections of a vector bundle) on a compact manifold G ($\dim G = n$) with C^∞ boundary X. An example is A acting on the Banach space $L^p(G)$ with $1 < p < \infty$ where G is a bounded, open subset of $\mathbf{R}^n$ with smooth boundary. Near the boundary, A can be written in local coordinates as

$$A = \sum_{|\alpha| \leq m} a_\alpha(x,t)(D_{\alpha,t})^\alpha = \sum_{j=0}^{m} A_j(t) D_t^{m-j} \tag{3.1}$$

where $(x_1, \ldots, x_{n-1})$ are local coordinates on X, t is a normal variable with $t \geq 0$ in G, the a_α are C^∞ square matrices, and

$$(D_{x,t})^\alpha = (-i\partial/\partial x_1)^{\alpha_1} \cdots (-i\partial/\partial t)^{\alpha_n}.$$

The principal symbol of A is

$$\sigma_m(A)(x,t;\xi,\tau) = \sum_{|\alpha|=m} a_\alpha(x,t)(\xi,\tau)^\alpha. \tag{3.2}$$

A complex number λ is a symbolic eigenvalue of A if it is an eigenvalue of $\sigma_m(A)$ for some $(x,t;\xi,\tau)$. We assume that

(3.3) A is elliptic, that is, $\sigma_m(A)$ is nonsingular for $|\xi|^2 + \tau^2 > 0$,

(3.4) A has no symbolic eigenvalues λ with $|\arg\lambda| \geq \gamma$ where $\gamma < \pi$.

Consider the boundary operators

$$B_j = \sum_{k=1}^{m} B_{jk} D_t^{m-k}, \qquad j = 1, \dots, mq/2,$$

where B_{jk} is a system of C^∞ differential operators on X of order at most $m_j + k - m$; thus B has order at most m_j. We denote

$$\sigma_0'(B_j) = \sum_{k=1}^{m} \sigma_{m_j+k-m}(B_{jk}) D_t, \tag{3.5}$$

$$\sigma_m'(A) = \sum_{j=0}^{m} \sigma_j(A_j(0)) D_t^{m-j}. \tag{3.6}$$

The operator A_B is defined as the closure in L^2 of the operator A acting on the domain

$$C_B^\infty = \{u \in C^\infty(G) | B_j u = 0 \text{ on } X, \text{ for } j = 1, \dots, q/2\}.$$

A complex number $\lambda \neq 0$ is a symbolic eigenvalue of A_B unless, for every x and ζ and every choice of the constants g_j, there is a unique solution u of the system of ordinary differential equations

$$\begin{cases} \sigma_m'(A)u = \lambda u & \text{for } t > 0, \\ \sigma_0'(B_j)u = g_j & \text{for } t = 0, \ j = 1, \dots, mq/2, \\ u(+\infty) = 0. \end{cases} \tag{3.7}$$

In addition, the symbolic eigenvalues of A are symbolic eigenvalues of A_B. We assume that

(3.8) A_B is elliptic, that is (3.7) has a unique solution when $\lambda = 0$ and $\xi \neq 0$.

(3.9) A_B has no symbolic eigenvalues λ with $|\arg \lambda| \geq \gamma$ where γ is independent of the choice of coordinate system.

Under these conditions and by adding a constant C_0 if necessary, it can be verified that $(A_B + C_0)$ is an operator of type ω and the functional calculus of $(A_B + C_0)$ can be defined by (2.1) or (2.2). The estimate of the L^p norm $\|\psi(A_B + C_0)\|_P$ is based on the expansion of the resolvent $(A_B + C_0 - \lambda)^{-1}$ in the next section. From now on, for convenience we denote $(A_B + C_0)$ by A_B.

4. The expansion of $(A_B - \lambda)^{-1}$

Under the conditions of Section 3, Seeley proved in [4] that the resolvent $(A_B - \lambda)^{-1}$ exists for $|\arg \lambda| \geq \gamma$ and can be written, for any integer K, in the form

(4.1)
$$(A_B - \lambda)^{-1} = \sum_k \chi_k \left(\sum_{j=0}^{K} \mathrm{Op}_k(\theta_2 c_{-m-j}) + \sum \mathrm{Op}_k''(\theta_1 d_{-m-j}) \right) \Phi_k + R_K$$

where R_K is an integral operator

(4.2)
$$R_K f(x) = \int_G r_K(x,y) f(y)\, dy \quad \text{and}$$
$$|D_x^\alpha D_y^\beta r_K(x,y)| \leq C_{\alpha\beta}(1 + |\lambda|^{1/m})^{n+|\alpha|+|\beta|-K-m},$$

(4.3) φ_k is a partition of unity in X, $\phi_k \chi_k = \phi_k$ and χ_k has support in a neighborhood U_k of the support of ϕ_k.

The functions c_{-m-j} and d_{-m-j} are defined for $|\arg \lambda| \geq \gamma$ and for $|\lambda|/|\xi|$ sufficiently small, and satisfy the conditions

(4.4) $$c_{-m} = (\sigma_m(A) - \lambda)^{-1},$$

(4.5) $$|D_{x,t}^\alpha D_{\xi,\tau}^\beta c_{-m-j}(x,t,\xi,\tau,\lambda)| \leq C_{\alpha\beta}(|\xi| + |\tau| + |\lambda|^{1/m})^{-m-j-|\beta|},$$

(4.6) $$|D_\chi^\alpha D_{-m-j}^\beta(x,t,\xi,\tau,s,\lambda)|$$
$$\leq C_{\alpha\beta} e^{-c(t+s)(|\xi|+|\lambda|^{1/m})}(|\xi| + |\lambda|^{1/m})^{1-m-j-|\beta|}.$$

The operators $\mathrm{Op}(c)$ and $\mathrm{Op}''(d)$ are defined by

(4.7) $$\mathrm{Op}(c)f(x,t) = (2\pi)^{-n} \int \int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle + it\tau} c(x,t,\xi,\tau,\lambda) \hat{f}(\xi,\tau)\, d\xi\, d\tau,$$
$$\mathrm{Op}''(d)f(x,t) = (2\pi)^{1-n} \int_0^\infty \int_{\mathbf{R}^{n-1}} e^{i\langle x,\xi\rangle} d(x,t,\xi,s,\lambda) \tilde{f}(\xi,s)\, d\xi\, ds.$$

where $\hat{f}(\xi,\tau)$ and $\tilde{f}(\xi,t)$ are the full and tangential Fourier transforms of $f(x,t)$, respectively, understood to be taken after we transform G to R_+^n via the usual technique of local maps and partitions of unity.

(4.9) $$\begin{cases} \text{The functions } \theta_1(\xi,\lambda) \text{ is } C^\infty,\ \theta_1 \equiv 0 \text{ for } (|\xi|^2 + |\lambda|^{2/\omega}) \text{ small,} \\ \theta_1 \equiv 1 \text{ for } (|\xi|^2 + |\lambda|^{2/\omega}) \text{ large,} \\ \theta_2(\xi,\lambda) \text{ satisfies the same condition with } (|\xi|^2 + \tau^2) \text{ in} \\ \text{place of } |\xi|^2. \end{cases}$$

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5. H_∞ functional calculus for elliptic operators

We first need the following two lemmas 1 and 2 which are essentially due to Seeley [5].

LEMMA 1. *Let* $1 < p < \infty$, $0 < R < \infty$. *There is a constant* $C = C(p,n,R)$ *such that if* $k(x,\xi)$ *vanishes for* $|x| \geq R$ *and* $|D_x^\alpha |\xi|^{|\beta|} D_\xi^\beta k(x,\xi)| \leq 1$ *for* $|\alpha| \leq n+1$, $|\beta| \leq n$ *then* $\| \operatorname{Op}(k)\|_p \leq C$. *(Here,* x *and* ξ *are in* $\mathbf{R}^n$.*)*

PROOF. Let

$$k_\eta(x,\xi) = \int_{\mathbf{R}^n} e^{i\langle x,y\rangle} k(x,\xi)\,dx.$$

Then

$$k(x,\xi) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{-i\langle x,y\rangle} k_\eta(x,\xi)\,dy$$

and

$$\operatorname{Op}(k) = (2\pi)^{-n} \int_{\mathbf{R}^n} e^{i\langle x,y\rangle} \operatorname{Op}(k_\eta)\,dy.$$

Hence

$$\| \operatorname{Op}(k)\|_p \leq (2\pi)^{-n} \int_{\mathbf{R}^n} \| \operatorname{Op}(k_\eta)\|_p\,dy. \tag{5.2}$$

The Mikhlin multiplier theorem gives us the estimate

$$\| \operatorname{Op}(k_\eta)\|_p \leq C_p \sup_{\xi \in \mathbf{R}^n} \max_{|\beta| \leq n} |\xi^\beta D_\xi^\beta k_\eta(\xi)|, \qquad 1 < p < \infty.$$

Since

$$\eta^\alpha \xi^\beta D_\xi^\beta k_\eta(\xi) = \int_{\mathbf{R}^n} [D_x^\alpha \xi^\beta D_\xi^\beta k(x,\xi)] e^{i\langle x,\eta\rangle}\,dx$$

and $k(x,\xi)$ vanishes for $x \geq R$, we have

$$|\eta^\alpha| |\xi^\beta D_\xi^\beta k_\eta(\xi)| \leq C_{\alpha\beta} \quad \text{for } |\alpha| \leq n+1. \tag{5.3}$$

Thus $\| \operatorname{Op}(k_\eta)\|_p \leq C_p(1+|z|)^{-n-1}$ and it follows from (5.2) that

$$\| \operatorname{Op}(k)\|_p \leq (2\pi)^{-n} \int_{\mathbf{R}^n} \frac{C_p}{(1+|z|)^{n+1}}\,dz = C(p,n,R).$$

LEMMA 2. *Let* $1 < p < \infty$. *There is a constant* $C = C(p,n,R)$ *such that if* $k(x,t,\xi,s)$ *has support in* $|x| \leq R$ *and satisfies*

$$|\xi|^{|\beta|} |D_x^\alpha D_x^\beta k(x,t,\xi,s)| \leq \frac{1}{t+s} \tag{5.4}$$

for $s > 0$, $t > 0$, $|\beta| \leq n-1$, $|\alpha| \leq n$, *then* $\| \operatorname{Op}''(k)\|_p \leq C$. *(Here,* x *and* ξ *are in* $\mathbf{R}^{n-1}$.*)*

Proof. Let $k_{t,s}(x,\xi) = k(x,t,\xi,s)$ and $K_{t,s} = \mathrm{Op}(k_{t,s})$. Then

$$\mathrm{Op}''(k)f(x,t) = \int_0^\infty K_{t,s}f(x,s)\,ds.$$

Thus

(5.5)

$$\begin{aligned}
\int_0^\infty \int_{\mathbf{R}^{n-1}} &\left|\int_0^\infty K_{t,s}f(x,s)\,ds\right|^p dx\,dt \\
&\le \int_0^\infty \left(\int_0^\infty \left(\int_{\mathbf{R}^{n-1}} |K_{t,s}f(x,s)|^p\,dx\right)^{1/p} ds\right)^p dt \\
&\qquad\qquad\qquad\qquad \text{(Minkowski's inequality)} \\
&\le \int_0^\infty \left(\int_0^\infty \frac{C_1}{t+s}\left(\int_{\mathbf{R}^{n-1}} |f(x,s)|^p\,dx\right)^{1/p} ds\right)^p dt \quad \text{(Lemma 1)} \\
&= C_1^p \int_0^\infty \left(\int_0^\infty \frac{1}{t+s}\phi(s)\,ds\right)^p dt \\
&\qquad\qquad \text{where } \phi(s) = \left(\int_{\mathbf{R}^{n-1}} f(x,s)|^p\,dx\right)^{1/p}
\end{aligned}$$

For $1/p + 1/p' = 1$ and $0 < q < \min\{1/p', 1/p\}$, we also have

$$\begin{aligned}
\left\|\int_0^\infty \frac{\phi(s)}{t+s}\,ds\right\|_p &= \left\{\int_0^\infty \left|\int_0^\infty \left(\frac{1}{t+s}\right)^{1/p'}\left(\frac{t}{s}\right)^q \left(\frac{1}{t+s}\right)^{1/p}\left(\frac{s}{t}\right)^q \phi(s)\,ds\right|\right\}^{1/p} \\
&\le \left\{\int_0^\infty \left(\int_0^\infty \frac{1}{t+s}\left(\frac{t}{s}\right)^{qp'} ds\right)^{p/p'}\right. \\
&\qquad \times \left.\left(\int_0^\infty \frac{1}{t+s}\left(\frac{s}{t}\right)^{qp} |\phi(s)|^p\,ds\right) dt\right\}^{1/p} \\
&\le \left(\sup_t \int_0^\infty \frac{1}{t+s}\left(\frac{t}{s}\right)^{qp'} ds\right)^{1/p'} \\
&\qquad \times \left(\sup_s \int_0^\infty \frac{1}{t+s}\left(\frac{s}{t}\right)^{qp} dt\right)^{1/p} \|\phi(s)\|_p \\
&= \left(\int_0^\infty \frac{1}{1+\sigma_1}\left(\frac{1}{\sigma_1}\right)^{qp'} d\sigma_1\right)^{1/p'} \\
&\qquad \times \left(\int_0^\infty \frac{1}{1+\sigma_2}\left(\frac{1}{\sigma_2}\right)^{qp} d\sigma_2\right)^{1/p} \|\phi(s)\|_p \\
&= C_2\|\phi(s)\|_p. \qquad (5.5')
\end{aligned}$$

It follows from (5.5) and (5.5′) that $\|\mathrm{Op}''(k)\|_p \le C$.

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We now come to the L^p norm estimate for the H_∞ functional calculus corresponding to the operator Op$''$.

LEMMA 3. *Suppose that $k(x,t,\xi,s,\lambda)$ vanishes for $|x| \geq R$ and there is a positive constant c such that*

$$|D_x^\alpha D_x^\beta k| \leq e^{-c(t+s)(|\xi|,|\lambda|^{1/m})}(|\xi| + |\lambda|^{1/m})^{1-m-|\beta|} \tag{5.6}$$

for $m \geq 1$, $|\alpha| \leq n$ and $|\beta| \leq n$. Then for each p, $1 < p < \infty$, there is a constant C_p (depending on p, c, r and n) such that

$$\| \operatorname{Op}'' \left(\int_\Gamma \psi(\lambda) k(x,t,\xi,s,\lambda), d\lambda \right) \|_p \leq C_p \|\psi\|_\infty, \qquad \psi \in H_\infty(S_\mu^0).$$

PROOF. Let $K(x,t,\xi,s) = \int_\Gamma \psi(\lambda) k(x,t,\xi,s,\lambda)\, d\lambda$ with the curve Γ as in (2.1). Then

$$\begin{aligned}
|(t+s)D_x^\alpha |\xi|^{|\beta|} D_\xi^\beta K| &\leq (t+s)|\xi|^{|\beta|} \int_\Gamma e^{-c(t+s)(|\xi|+|\lambda|^{1/m})} \\
&\qquad\qquad \times (|\xi| + |\lambda|^{1/m})^{1-m-|\beta|} |\psi(\lambda)| d|\lambda| \\
&\leq m(t+s)|\xi|^{|\beta|} e^{-c(t+s)|\xi|} \int_0^\infty e^{-c(t+s)\lambda_1} \\
&\qquad\qquad \times (|\xi| + \lambda_1)^{1-m-|\beta|} \psi(\lambda)| \cdot (\lambda_1)^{m-1}\, d\lambda_1 \\
&\qquad\qquad\qquad \text{where } \lambda_1 = |\lambda|^{1/m} \\
&\leq m(t+s) e^{-c(t+s)|\xi|} \|\psi\|_\infty \int_0^\infty e^{-c(t+s)\lambda_1}\, d\lambda_1 \\
&= 2m(t+s) e^{-c(t+s)|\xi|} \|\psi\|_\infty \frac{1}{c(t+s)} \\
&\leq K \|\psi\|_\infty.
\end{aligned}$$

Hence Lemma 3 follows from Lemma 2.

The main result of this paper is the following theorem.

THEOREM. *There exists $M > 0$ such that*

$$\|\psi(A_B)\|_p \leq M \|\psi\|_\infty, \qquad \psi \in H_\infty(S_\mu^0),\ 1 < p < \infty. \tag{5.7}$$

PROOF. We first consider the case $\psi \in \Psi(S_\mu^0)$. For

$$R_K f(x) = \int_G r_K(x,y) f(y)\, dy,$$

we have

$$\left\{ \int_G |R_K f(x)|^p\, dx \right\}^{1/p} = \left\{ \int_G \left| \int_G r_K(x,y) f(y)\, dy \right|^p dx \right\}^{1/p}$$

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and it follows from (4.2) and Hölder's inequality that

$$\|R_k\|_p \le c(1+|\lambda|^{1/m})^{n-m-K}.$$

Thus

$$\begin{aligned}
&\left\{\int_G \left|\int_\Gamma |\psi(\lambda)[R_K f(x)]\, d\lambda\right|^p dx\right\}^{1/p} \\
&\qquad \le \int_\Gamma \left\{\int_G |\psi(\lambda)[R_K f(x)]|^p\, dx\right\}^{1/p} d\lambda \\
(5.8)\qquad &\qquad \le \left\{\int_\Gamma |\psi(\lambda)| \frac{c}{(1+|\lambda|^{1/m})^{K+m-n}}\, d\lambda\right\} \|f\|_p \\
&\qquad \le \left\{\int_\Gamma \frac{c}{(1+|\lambda|^{1/m})^{K+m-n}}\right\} \|\psi\|_\infty \|f\|_p \\
&\qquad = M_1 \|\psi\|_\infty \|f\|_p \quad \text{when we choose } K > n.
\end{aligned}$$

For the term $c_{-m} = (\sigma_m(A) - \lambda)^{-1}$, we have

$$\begin{aligned}
&\mathrm{Op}\left(\int_\Gamma \theta_2 c_{-m} \psi(\lambda)\, d\lambda\right) f(x,t) \\
&\quad = (2\pi)^{-n} \int\int_{\mathbf{R}^n} e^{i\langle x,\xi\rangle + it\tau} \left(\int_\Gamma \frac{\theta_2 \psi(\lambda)\, d\lambda}{\sigma_m(A) - \lambda}\right) \hat{f}(\xi, T)\, d\xi\, d\tau.
\end{aligned}$$

Since

$$\int_\Gamma \frac{\theta_2 \psi(\lambda)\, d\lambda}{\sigma_\omega(A) - \lambda} = \psi(\sigma_m(A)) \quad \text{for } |\xi|^2 + \tau^2 \text{ large}$$

and the analytic function ψ satisfies the conditions of the Mikhlin multiplier theorem, we obtain

$$\left\| \mathrm{Op}\left(\int_\Gamma \theta_2 c_{-m} \psi(\lambda)\, d\lambda\right)\right\|_p \le M_2 \|\psi\|_\infty.$$

For the term c_{-m-j} with $j \ge 1$, we apply Lemma 1 with $k = \theta_2 c_{-m-j}$, noting (4.5), and obtain

$$\begin{aligned}
&(|\xi|^2 + \tau^2)^{1/2|\beta|} |(D_{x,\tau})^\alpha (D_{\xi,\tau})^\beta k(x,t,\xi,\tau,\lambda)| \\
&\quad \le (|\xi|^2 + \tau^2)^{1/2|\beta|} C_{\alpha\beta} (1 + |\xi| + |\tau| + |\lambda|^{1/m})^{m-j-|\omega|} \\
&\quad \le C'_{\alpha\beta} (1 + |\lambda| +^{1/m})^{-m-j}
\end{aligned}$$

Therefore $\| \operatorname{Op}(\theta_2 c_{-m-j})\|_p \le C(1+|\lambda|^{1/m})^{-m-j}$. Thus

$$\begin{aligned}\left\|\int_\Gamma \psi(\lambda)\operatorname{Op}(\theta_2 c_{-m-j}) f\,d\lambda\right\|_p &= \left\{\int_G \left|\int_\Gamma \psi(\lambda)\operatorname{Op}(\theta_2 c_{-m-j}) f\,d\lambda\right|^p dx\right\}^{1/p}\\ &\le \int_\Gamma \left\{\int_G |\psi(\lambda)\operatorname{Op}(\theta_2 c_{-m-j}) f|^p\,dx\right\}^{1/p} d\lambda\\ &\le \left\{\int_\Gamma |\psi(\lambda)|\|\operatorname{Op}(\theta_2 c_{-m-j})\|_p\,d\lambda\right\}\|f\|_p\\ &\le \|\psi\|_\infty \left\{\int_\Gamma \frac{c}{(1+|\lambda|^{1/m})^{m+j}}\,d\lambda\right\}\|f\|_p.\end{aligned}$$

Therefore

$$\left\|\operatorname{Op}\left(\int_\Gamma \theta_2 c_{-m-j}\psi(\lambda)\,d\lambda\right)\right\|_p \le M_3\|\psi\|_\infty. \tag{5.10}$$

The estimate

$$\left\|\operatorname{Op}\left(\int_\Gamma \theta_1 d_{-m-j}\psi(\lambda)\,d\lambda\right)\right\|_p \le M_4\|\psi\|_\infty \tag{5.11}$$

follows from (4.6), Lemma 3 and the inequality $\theta_1(|\xi|+|\lambda|^{1/m})^{-j} \le$ constant.

It follows from (5.8), (5.9), (5.10), (5.11) and (4.1) that

$$\|\psi(A_B)\|_p \le M\|\psi\|_\infty, \qquad \psi \in \Psi(S^0_\mu),\ 1<p<\infty. \tag{5.12}$$

To extend the result for $\psi \in H_\infty(S^0_\mu)$, we define $\psi_{\varepsilon,R} \in \Psi(S^0_\mu)$ by

$$\psi_{\varepsilon,R}(\xi) = \int_\varepsilon^R (\psi\phi_t)(\xi)\frac{dt}{t}$$

where $\phi \in \Psi(S^0_\mu)$ such that $\int_0^\infty \phi(t)\,dt/t = 1$ and $\phi_t(\xi) = \phi(t\xi)$. The inequality (5.12) shows that $\|\psi_{\xi,R}(A_B)\|_p \le M\|\psi\|_\infty$ where M is independent of ε and R. It is not difficult to check that all the conditions of the theorem in Section 2 are satisfied. We conclude that

$$\|\psi(A_B)\|_p \le M\|\psi_\infty, \qquad \psi \in H_\infty(S^0_\mu),\ 1<p<\infty.$$

It is an interesting and still open question whether the H_∞ functional calculus property still holds for elliptic operators with milder assumptions on the smoothness of the coefficients or on the boundaries of the domains.

It is also worth noting that the H_∞ functional calculus property implies that the elliptic operator A_B satisfies quadratic estimates, that is, there exists a positive constant K_p such that

$$\left\|\left\{\int_0^\infty |\psi(tA_B)u(\cdot)|^2\frac{dt}{t}\right\}^{1/2}\right\|_p \le K_p\|u\|_p, \quad \text{for all } u \in L^p(G),\ 1<p<\infty$$

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for certain classes of functions ψ, including, for example

$$\psi(z) = z/(z - \bar{\alpha})(z - \alpha) \quad \text{where } \alpha \notin S_{\omega}.$$

Details can be found in the joint paper of M. Cowling, A. McIntosh and A. Yagi [2].

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School of Mathematics, Physics,
Computing and Electronics
Macquarie University
NSW, 2109
Australia