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AN INTERNAL CHARACTERISATION OF STRONGLY REGULAR RINGS

LUOSHENG HUANG AND WEIMIN XUE

We show that a right duo ring R is strongly regular if and only if for each ideal I of R, the coset product of I in the factor ring R/I is the same as their set product.

In this note a ring means an associative ring with identity. If I is an ideal of a ring R, then the *coset product* of I in the factor ring R/I is defined as

$$(r_1 + I)(r_2 + I) = r_1 r_2 + I$$

where $r_1, r_2 \in R$. Since $r_1 + I$ and $r_2 + I$ are subsets of R, we can define their *set product* as

$$(r_1 + I) \circ (r_2 + I) = \{(r_1 + i_1)(r_2 + i_2) \mid i_1, i_2 \in I\}.$$

We always have that $(r_1 + I) \circ (r_2 + I) \subseteq (r_1 + I)(r_2 + I)$, since I is an ideal. We say that the ideal I is a *good ideal* in case

$$(r_1 + I) \circ (r_2 + I) = (r_1 + I)(r_2 + I)$$

for any $r_1, r_2 \in R$. A necessary condition for an ideal I to be good is that $I = I \circ I$, so the ideal $2\mathbb{Z}_4$ of $\mathbb{Z}_4 = \mathbb{Z}/4\mathbb{Z}$ is not a good ideal. We also say that a ring R is a *good ring* in case each ideal is good. Since the two trivial ideals are always good, each simple ring is a good ring. For this reason we consider *right duo rings* (after E.H. Feller), that is, rings whose right ideals are ideals. Our main result states that a right duo ring is good if and only if it is strongly regular.

LEMMA 1. *Let R be a right duo ring. If I is a finitely generated right ideal then the following are equivalent:*

(1) *$I = eR$ for some idempotent $e \in R$;*
(2) *I is a good ideal;*
(3) *$I = I \circ I$;*
(4) *$I = I^2$, where I^2 is defined as usual, that is, $I^2 = \{\sum_k i_k j_k \mid i_k, j_k \in I\}$.*

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PROOF: (1) $\Rightarrow$ (2). By [2, Theorem 1.3], each idempotent of R belongs to the centre of R. Let $L = (1-e)R$, then $R = I \oplus L$ where I and L are ideals. Take $r_1, r_2 \in R$ and $i \in I$, then $r_1 = j_1 + l_1$ and $r_2 = j_2 + l_2$ for some $j_1, j_2 \in I$ and $l_1, l_2 \in L$. Since $I = eR$ is generated by a central idempotent e, we have

$$I = I \circ I = (j_1 + I) \circ (j_2 + I).$$

Now $j_1 j_2 + i \in I$, so there are $j_1', j_2' \in I$ such that $j_1 j_2 + i = (j_1 + j_1')(j_2 + j_2')$. Then $i = j_1 j_2' + j_1' j_2 + j_1' j_2' = r_1 j_2' + j_1' r_2 + j_1' j_2'$ and hence $r_1 r_2 + i = (r_1 + j_1')(r_2 + j_2') \in (r_1 + I) \circ (r_2 + I)$. Therefore $(r_1 + I)(r_2 + I) = r_1 r_2 + I \subseteq (r_1 + I) \circ (r_2 + I)$, and then I is a good ideal.

(2) $\Rightarrow$ (3). Trivial.

(3) $\Rightarrow$ (4). Since $I = I \circ I \subseteq I^2 \subseteq I$, we have $I^2 = I$.

(4) $\Rightarrow$ (1). By [1, Proposition 4]. ∎

One notes that in the above lemma the implications (2) $\Rightarrow$ (3) $\Rightarrow$ (4) hold for an arbitrary ideal I of an arbitrary ring R, but the right duo assumption is essential for the implications (4) $\Rightarrow$ (3), (4) $\Rightarrow$ (1) and (2) $\Rightarrow$ (1), as shown by the following two examples.

EXAMPLE 2: The noetherian ring $R = \begin{bmatrix} \mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$ has an ideal $I = \begin{bmatrix} 2\mathbb{Z} & 2\mathbb{Z} \\ \mathbb{Z} & \mathbb{Z} \end{bmatrix}$. One sees that $I = I^2$. But we note that $I \neq I \circ I$, since $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \in I$ but $\begin{bmatrix} 2 & 4 \\ 0 & 1 \end{bmatrix} \notin I \circ I$. It follows that I is not a good ideal. One also checks that $I \neq eR$ and $I \neq Re$ for any idempotent $e \in I$.

EXAMPLE 3. Let F be a field and $R = \begin{bmatrix} F & F \\ 0 & F \end{bmatrix}$. It is easy to see that $I = \begin{bmatrix} 0 & F \\ 0 & F \end{bmatrix}$ is an ideal of R and $I = I \circ I$. To show that I is a good ideal, we may assume that $r_1 = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, $r_2 = \begin{bmatrix} b & 0 \\ 0 & 0 \end{bmatrix} \in R$, and $i = \begin{bmatrix} 0 & c \\ 0 & d \end{bmatrix} \in I$. We take $j = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \in I$ and have that $r_1 r_2 + i = (r_1 + i)(r_2 + j) \in (r_1 + I) \circ (r_2 + I)$. This proves that I is a good ideal. But $I \neq eR$ for any idempotent $e \in I$.

Similarly, the ideal $L = \begin{bmatrix} F & F \\ 0 & 0 \end{bmatrix}$ is also a good ideal. Now $I \cap L = IL = \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$, which is not a good ideal since $\begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix} \circ \begin{bmatrix} 0 & F \\ 0 & 0 \end{bmatrix}$. We conclude that the intersection or the product of two good ideals need not be a good ideal.

The next result asserts that to show a right duo ring is good we only need to check the principal right ideals.

LEMMA 4. *The following are equivalent for a right duo ring R:*

(1) *R is a good ring;*

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(2) *each finitely generated right ideal is a good ideal;*
(3) *each principal right ideal is a good ideal.*

PROOF: It remains to verify (3) $\Rightarrow$ (1). So let I be an ideal of R, $r_1, r_2 \in R$, and $i \in I$. Since the ideal iR is a principal right ideal, we have $r_1r_2 + i \in (r_1 + iR) \circ (r_2 + iR) \subseteq (r_1 + I) \circ (r_2 + I)$. Hence $r_1r_2 + I \subseteq (r_1 + I) \circ (r_2 + I)$, and then I is a good ideal. $\square$

Recall that a ring R is von Neuman regular if each principal right (or left) ideal of R is generated by an idempotent, and R is strongly regular if and only if it is regular and right duo (since idempotents in a right duo ring are central by [**2**, Theorem 1.3]). The following main result follows from Lemmas 1 and 4, which gives an internal characterisation of strongly regular rings.

THEOREM 5. *Let R be a right duo ring. Then R is (strongly) regular if and only if it is a good ring.*

Since any simple ring is a good ring, a good ring need not be regular. We do not know whether or not a regular ring must be a good ring.

Recall that a regular right (or left) noetherian ring is semisimple, and a semisimple right duo ring is a finite direct sum of division rings. Thus we have

COROLLARY 6. *Let R be a right duo ring. If R is either right or left noetherian, then R is a good ring if and only if it is a finite direct sum of division rings.*

In particular, we have

COROLLARY 7. *The ring $\mathbb{Z}_n$ is a good ring if and only if n has a square free factorisation.*

It is known that a right duo ring R is regular if and only if each simple right (left) R-module is injective (see [**4**, Theorem 1.3], or [**3**, Theorem 4.10]).

COROLLARY 8. *A right duo ring R is a good ring if and only if each simple right (left) R-module is injective.*

As we mention in the introduction, a necessary condition for an ideal I to be good is that $I = I \circ I$, but we do not know whether or not this condition is also sufficient. Our Lemma 1 and the following concluding proposition give some positive answers.

PROPOSITION 9. *Let J be the (Jacobson) radical of a local ring R. If $J = J \circ J$, then J is a good ideal.*

PROOF: Let $r_1, r_2 \in R$ and $j \in J$. Since $J = J \circ J$ we may assume that $r_2 \notin J$. Then r_2 is invertible, since R is local. So we have $r_1r_2 + j = (r_1 + jr_2^{-1})(r_2 + 0) \in (r_1 + J) \circ (r_2 + J)$. Therefore $r_1r_2 + J \subseteq (r_1 + J) \circ (r_2 + J)$, and J is a good ideal. $\square$

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Department of Mathematics
Fujian Normal University
Fuzhou
Fujian 350007
Peoples Republic of China

Current address:
Department of Mathematics
University of Iowa
Iowa City IA 52242
United States of America