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FREELY QUASICONFORMAL MAPS AND DISTANCE RATIO METRIC

YAXIANG LI, SAMINATHAN PONNUSAMY✉ and MATTI VUORINEN

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Abstract

Suppose that E and E' denote real Banach spaces with dimension at least 2 and that $D \subset E$ and $D' \subset E'$ are domains. Let $\varphi : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\varphi(t) \geq t$. We say that a homeomorphism $f : D \to D'$ is φ-FQC if for every subdomain $D_1 \subset D$, we have $\varphi^{-1}(k_D(x, y)) \leq k_{D'}(f(x), f(y)) \leq \varphi(k_D(x, y))$ holds for all $x, y \in D_1$. In this paper, we establish, in terms of the j_D metric, a necessary and sufficient condition for a homeomorphism $f : E \to E'$ to be FQC. Moreover, we give, in terms of the j_D metric, a sufficient condition for a homeomorphism $f : D \to D'$ to be FQC. On the other hand, we show that this condition is not necessary.

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1. Introduction and main results

During the past few decades, modern mapping theory and the geometric theory of quasiconformal maps have been studied from several points of view. These studies include Heinonen's work on metric measure spaces [5], Koskela's study of maps with finite distortion [7] and Väisälä's work about quasiconformality in infinite-dimensional Banach spaces [15–19]. Our study is motivated by Väisälä's theory of freely quasiconformal maps in the setup of Banach spaces [15–17]. The basic tools in Väisälä's theory are metrics and the notion of uniform continuity between metric spaces; in particular, the norm metric, the quasihyperbolic metric and the distance ratio metric are used. We begin with some basic definitions and the statements of our results.

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Throughout the paper, we always assume that E and E' denote real Banach spaces with dimension at least 2, and that $D \subset E$ and $D' \subset E'$ are domains. The norm of a vector z in E is written as $|z|$ and, for each pair of points z_1, z_2 in E, the distance between them is denoted by $|z_1 - z_2|$. The distance from $z \in D$ to the boundary ∂D of D is denoted by $d_D(z)$.

For each pair of points z_1, z_2 in D, the *distance ratio metric* $j_D(z_1, z_2)$ between z_1 and z_2 is defined by

$$j_D(z_1, z_2) = \log\left(1 + \frac{|z_1 - z_2|}{\min\{d_D(z_1), d_D(z_2)\}}\right).$$

The *quasihyperbolic length* of a rectifiable arc or a path α in the norm metric in D is the number (compare with [2, 3, 15])

$$\ell_k(\alpha) = \int_\alpha \frac{|dz|}{d_D(z)}.$$

For each pair of points z_1, z_2 in D, the *quasihyperbolic distance* $k_D(z_1, z_2)$ between z_1 and z_2 is defined in the usual way:

$$k_D(z_1, z_2) = \inf \ell_k(\alpha),$$

where the infimum is taken over all rectifiable arcs α joining z_1 to z_2 in D. Gehring and Palka [3] introduced the quasihyperbolic metric of a domain in $\mathbb{R}^n$. Many of the basic properties of this metric may be found in [15]. We remark that the quasihyperbolic metric has been recently studied by many people (compare with [4, 6, 8–12]).

Definition 1.1. Let $D \subsetneq E$ and $D' \subsetneq E'$ be domains, and let $\varphi : [0, \infty) \to [0, \infty)$ be a growth function, that is, a homeomorphism with $\varphi(t) \geq t$. We say that a homeomorphism $f : D \to D'$ is *φ-semisolid* if

$$k_{D'}(f(x), f(y)) \leq \varphi(k_D(x, y))$$

for all $x, y \in D$, and *φ-solid* if both f and f^{-1} satisfy this condition.

The special case $\varphi(t) = Mt$ ($M > 1$) gives the M-quasihyperbolic maps, or briefly M-QH. More precisely, f is called M-QH if

$$\frac{k_D(x, y)}{M} \leq k_{D'}(f(x), f(y)) \leq M k_D(x, y)$$

for all x and y in D.

We say that f is *fully φ-semisolid* (respectively *fully φ-solid*) if f is φ-semisolid (respectively φ-solid) on every subdomain of D. In particular, when $D = E$, the subdomains are taken to be proper ones in D. Fully φ-solid mappings are also called *freely φ-quasiconformal mappings*, or briefly φ-FQC mappings.

If $E = E' = \mathbb{R}^n$, then f is FQC if and only if f is quasiconformal (compare with [15]). See [1, 14, 21] for definitions and properties of K-quasiconformal mappings, or briefly K-QC mappings.

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It is well known that for all z_1, z_2 in D, we have (compare with [15])

$$j_D(z_1, z_2) \le \inf_{\alpha\in\Gamma} \log\Big(1 + \frac{\ell(\alpha)}{\min\{d_D(z_1), d_D(z_2)\}}\Big) \le \inf_{\alpha\in\Gamma} \ell_k(\alpha) = k_D(z_1, z_2), \tag{1.1}$$

where Γ is the class of all rectifiable arcs joining z_1 and z_2 in D. Hence, in the study of FQC maps, it is natural to ask whether we could use the j_D metric to describe FQC or not. In fact, we get the following conditions for a homeomorphism to be FQC.

Theorem 1.2. *A homeomorphism $f : E \to E'$ is φ_1-FQC if and only if for every proper subdomain $D \subset E$*

$$\varphi_2^{-1}(j_D(x, y)) \le j_{D'}(f(x), f(y)) \le \varphi_2(j_D(x, y)) \tag{1.2}$$

for all $x, y \in D$, where φ_1 and φ_2 are self-homeomorphisms of $[0,\infty)$ with $\varphi_i(t) \ge t$ $(i = 1, 2)$ for all t, and φ_1, φ_2 depend only on each other.

Theorem 1.3. *Let $\varphi : [0,\infty) \to [0,\infty)$ be a homeomorphism with $\varphi(t) \ge t$ for all t, and let $f : D \subsetneq E \to D' \subsetneq E'$ be a homeomorphism. If, for every subdomain $D_1 \subset D$,*

$$\varphi^{-1}(j_{D_1}(x, y)) \le j_{D_1'}(f(x), f(y)) \le \varphi(j_{D_1}(x, y)) \tag{1.3}$$

for all $x, y \in D_1$, then f is φ_1-FQC with $\varphi_1 = \varphi_1(\varphi)$. Moreover, if $D_1 = D$ and $\varphi(t) = Mt$ $(M \ge 1)$, then f is M-QH.

Theorem 1.4. *The converse of Theorem 1.3 is not true.*

The proofs of Theorems 1.2–1.4 will be given in the next section.

2. The proofs of Theorems 1.2–1.4

For convenience, in what follows, we always assume that $x, y, z, \ldots$ denote points in D and $x', y', z', \ldots$ denote the images in D' of $x, y, z, \ldots$ under f, respectively.

Before the proofs of our main results, we list a series of results which are critical to our proofs.

Definition 2.1 [15, Definition 3.6]. Suppose that $f : D \to D'$ is a homeomorphism with $D \subsetneq E$, $D' \subsetneq E'$. Let $0 < t_0 < 1$ and let $\theta : [0, t_0) \to [0,\infty)$ be an embedding with $\theta(0) = 0$. We say that f is (θ, t_0)-relative if

$$\frac{|x' - y'|}{d_{D'}(x')} \le \theta\Big(\frac{|x - y|}{d_D(x)}\Big)$$

whenever $x, y \in D$ and $|x - y| < t_0 d_D(x)$. If $t_0 = 1$, we simply say that f is θ-relative.

Theorem A [15, Corollary 3.8]. For a homeomorphism $f : D \to D'$, the following conditions are quantitatively equivalent:

(1) f and f^{-1} are θ-relative;
(2) f and f^{-1} are (θ, t_0)-relative;
(3) f is φ-solid.

Definition 2.2 [13, 15]. Let X and Y be two metric spaces, and let $\eta : [0,\infty) \to [0,\infty)$ be a homeomorphism. An embedding $f : X \to Y$ is said to be *η-quasisymmetric*, or briefly *η-QS*, if $|x' - y'|/|x' - z'| \le \eta(|x - y|/|x - z|)$ for all x, y, z in X.

Theorem B [15, Theorem 5.13]. Suppose that $f : E \to D' \subset E'$ is fully φ-semisolid. Then:

(1) $D' = E'$;
(2) f is ψ-FQC with $\psi = \psi_\varphi$;
(3) f is η-quasisymmetric with $\eta = \eta_\varphi$.

Theorem C [15, Theorem 4.6]. Let $f : D \to D'$ be a homeomorphism with $D \subsetneq E$, $D' \subsetneq E'$. Then f is M-QH if and only if

$$\frac{L(x, f)d_D(x)}{d_{D'}(x')} \le M \quad and \quad \frac{L(x', f^{-1})d_{D'}(x')}{d_D(x)} = \frac{d_{D'}(x')}{l(x, f)d_D(x)} \le M$$

for all $x \in D$, where

$$L(x, f) = \limsup_{y\to x} \frac{|x' - y'|}{|x - y|}, \quad l(x, f) = \liminf_{y\to x} \frac{|x' - y'|}{|x - y|}.$$

Lemma 2.3. Let $\varphi : [0, \infty) \to [0, \infty)$ be a homeomorphism with $\varphi(t) \ge t$ for all t, and let $f : D \subsetneq E \to D' \subsetneq E'$ be a homeomorphism. If, for every $x, y \in D$,

$$j_{D'}(f(x), f(y)) \le \varphi(j_D(x, y)),$$

then f is (θ, s)-relative, where $\theta(t) = e^{\varphi_2(\log(1+t/(1-s)))} - 1$ and $s \in (0, 1)$ is a constant.

Proof. Let $0 < s < 1$ and $x, y \in D$ with $|x - y|/d_D(x) = t < s$. Then

$$d_D(y) \ge d_D(x) - |x - y| > (1 - s)d_D(x),$$

whence

$$\frac{|x - y|}{\min\{d_D(x), d_D(y)\}} \le \frac{|x - y|}{(1 - s)d_D(x)},$$

and therefore

$$\frac{|x' - y'|}{d_{D'}(x')} \le e^{j_{D'}(x',y')} - 1 \le e^{\varphi(j_D(x,y))} - 1 \le e^{\varphi(\log(1+t/(1-s)))} - 1 = \theta(t),$$

which shows that Lemma 2.3 holds. □

2.1. The proof of Theorem 1.2. We first prove the sufficient part and we assume that $f : E \to E'$ is φ_1-FQC. By Theorem B, we know that f is η-quasisymmetric, where $\eta = \eta_{\varphi_1}$. Let D be a proper subdomain in E. For $x, y \in D$, by symmetry, to prove (1.2) we only need to prove the right-hand-side inequality. Without loss of generality, we may assume that $d_{D'}(x') \le d_{D'}(y')$. Choose a point $z \in \partial D$ such that $|x' - z'| \le 2d_{D'}(x')$. Now

$$\frac{|x' - y'|}{d_{D'}(x')} \le 2\frac{|x' - y'|}{|x' - z'|} \le 2\eta\Big(\frac{|x - y|}{|x - z|}\Big) \le 2\eta\Big(\frac{|x - y|}{d_D(x)}\Big),$$

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whence

$$j_{D'}(x', y') = \log\Big(1 + \frac{|x' - y'|}{d_{D'}(x')}\Big) \leq \log\Big(1 + 2\eta\Big(\frac{|x - y|}{d_D(x)}\Big)\Big) \leq \varphi(j_D(x, y)),$$

where $\varphi(t) = \log(1 + 2\eta(e^t - 1))$.

In the following, we prove the necessary part. Let D be a proper subdomain of E. By the assumption, we know that there is a self-homeomorphism φ_2 such that (1.2) holds for every $x, y \in D$. Then Lemma 2.3 shows that $f : D \to D'$ is (θ, s)-relative with $\theta(t) = e^{\varphi_2(\log(1+t/(1-s)))} - 1$. Hence, by Theorem A, we know that $f : D \to D'$ is φ-solid with $\varphi = \varphi_{\varphi_2}$. By the arbitrariness of D, we get that f is φ-FQC. □

2.2. The proof of Theorem 1.3. We first prove the first part. Let D_1 be a subdomain of D. By the assumption, we know that there is a self-homeomorphism φ such that (1.2) holds for every $x, y \in D_1$. Then we get from Lemma 2.3 that $f : D_1 \to D'_1$ is (θ, s)-relative with $\theta(t) = e^{\varphi(\log(1+t/(1-s)))} - 1$ and $s \in (0, 1)$ is a constant. Hence, Theorem A shows that $f : D_1 \to D'_1$ is φ_1-solid with φ_1 depending on φ. By the arbitrariness of D_1, we get that f is φ_1-FQC.

Now we are going to prove the second part of Theorem 1.3. By Theorem C, it suffices to show that

$$\frac{L(x, f)d_D(x)}{d_{D'}(x')} \leq M \tag{2.1}$$

for each $x \in D$. Let $0 < s < 1$ and let $\theta(t)$ be as in Lemma 2.3 for $0 < t < s$. Now

$$\theta(t) = \Big(1 + \frac{t}{(1 - s)}\Big)^M - 1,$$

whence $\theta'(0) = M/(1 - s)$. Let $x, y \in D$ be points with $|x - y|/d_D(x) = t$. By Lemma 2.3,

$$\frac{|x' - y'|d_D(x)}{|y - x|d_{D'}(x')} \leq \frac{\theta(t)}{t} \to \frac{M}{1 - s}$$

as $t \to 0$. As $s > 0$ is arbitrary, this implies (2.1). □

2.3. The proof of Theorem 1.4. We prove this theorem by presenting two examples.

Example 2.4. Let $E = \mathbb{R}^2 \cong \mathbb{C}$ and f be a conformal mapping of the unit disc $\mathbb{B}(0, 1) = \{z : |z| < 1\}$ $(= D)$ onto $D' = \mathbb{B}(0, 1) \setminus [0, 1)$. There is no self-homeomorphism φ of $[0, \infty)$ such that (1.3) holds.

Proof. By [2, Theorem 3], we know that conformal mapping is an M-QH mapping for some constant $M \geq 1$. Hence, f is φ_0-FQC with $\varphi_0(t) = Mt$. Therefore, for each pair of points $x, y \in D$,

$$\frac{k_D(x, y)}{M} \leq k_{D'}(x', y') \leq Mk_D(x, y). \tag{2.2}$$

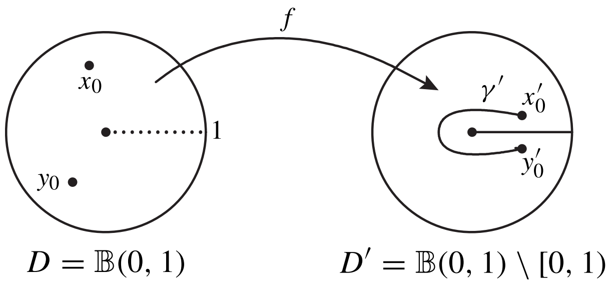

FIGURE 1. γ' is a rectifiable arc joining x_0' and y_0' in D'.

Take $x_0', y_0' \in D'$ with $x_0' = (\frac{1}{2}, t)$ and $y_0' = (\frac{1}{2}, -t)$ (see Figure 1). Let Γ be the class of all rectifiable arcs joining x_0' and y_0' in D'. Then, by (1.1),

$$k_{D'}(x_0', y_0') \geq \inf_{\gamma' \in \Gamma} \log\Big(1 + \frac{\ell(\gamma')}{d_{D'}(x_0')}\Big) \geq \log\Big(1 + \frac{1}{t}\Big)$$

and

$$j_{D'}(x_0', y_0') = \log\Big(1 + \frac{|x_0' - y_0'|}{\min\{d_{D'}(x_0'), d_{D'}(y_0')\}}\Big) = \log 3.$$

But, by [21, page 35] and (2.2),

$$\begin{aligned} j_D(x_0, y_0) &\geq \frac{1}{2} k_D(x_0, y_0) \geq \frac{1}{2M} k_{D'}(x_0', y_0') \\ &\geq \frac{1}{2M} \log\Big(1 + \frac{1}{t}\Big) \to \infty \quad \text{as } t \to 0. \end{aligned}$$

Hence, there is no self-homeomorphism φ of $[0, \infty)$ such that (1.3) holds. The proof of Example 2.4 is complete. □

EXAMPLE 2.5. Let E be an infinite-dimensional separable real Hilbert space with an orthonormal basis $(e_j)_{j \in \mathbb{Z}}$. Setting $\gamma_j' = [e_{j-1}, e_j]$, we obtain the infinite broken line $\gamma' = \cup\{\gamma_j' : j \in \mathbb{Z}\}$. Let γ denote the line spanned by e_1, let $D = \gamma + \mathbb{B}(r)$ with $r \leq \frac{1}{10}$ and let f be a locally M-bilipschitz homeomorphism from D onto a neighbourhood D' of γ' (for a detailed explanation, we refer the reader to [15, Section 4.12] or [20]). There is no self-homeomorphism φ of $[0, \infty)$ such that (1.3) holds.

PROOF. By [15, Theorem 4.8], we obtain that f is M^2-QH. Let $m \geq 2$ be an integer and let $x, y \in D$ with $x = \sqrt{2}e_1$, $y = m\sqrt{2}e_1$. Then $d_D(x) = d_D(y) = r$. Because f is locally M-bilipschitz,

$$d_{D'}(x') \geq \frac{r}{M} \quad \text{and} \quad d_{D'}(y') \geq \frac{r}{M}.$$

Since $D' \subset \mathbb{B}(0, 2)$,

$$j_{D'}(x', y') = \log\Big(1 + \frac{|x' - y'|}{\min\{d_{D'}(x'), d_{D'}(y')\}}\Big) \leq \log\Big(1 + \frac{4M}{r}\Big),$$

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but

$$j_D(x, y) = \log\left(1 + \frac{|x - y|}{\min\{d_D(x), d_D(y)\}}\right) = \log\left(1 + \frac{\sqrt{2}(m - 1)}{r}\right) \to \infty$$

as $m \to \infty$. Hence, (1.3) does not hold. □

REMARK 2.6. In support of Theorem 1.4, we include here another simple planar example which is indeed similar to Example 2.4. Let D be the semi unit disc $D = \{z \in \mathbb{B}(0, 1) : |\arg z| < \pi/2\}$ and let $z = re^{i\varphi} \in D$ $(0 < r < 1)$ in polar coordinates. Then $f(r, \varphi) = (r, 2\varphi)$ defines a locally 2-bilipschitz homeomorphism of D onto $D' = \mathbb{B}(0, 1) \setminus (-1, 0]$. For $0 < t < \pi/4$, let $x = (1/2, \pi/2 - t)$, $y = (1/2, -\pi/2 + t)$ in polar coordinates. Then $j_D(x, y) = \log(1 + 2 \cot t) \to \infty$ as $t \to 0$, while $j_{D'}(x', y') = \log 3$ for all t, whence there is no self-homeomorphism φ of $[0, \infty)$ such that (1.3) holds.

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YAXIANG LI, College of Science,
Central South University of Forestry and Technology,
Changsha, Hunan 410004, PR China
e-mail: yaxiangli@163.com

SAMINATHAN PONNUSAMY, Indian Statistical Institute (ISI) Chennai Centre,
SETS (Society for Electronic Transactions and Security),
MGR Knowledge City, CIT Campus,
Taramani, Chennai 600 113, India
e-mail: samy@isichennai.res.in, samy@iitm.ac.in

MATTI VUORINEN, Department of Mathematics and Statistics,
University of Turku, FIN-20014 Turku, Finland
e-mail: vuorinen@utu.fi