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RESEARCH ARTICLE

Optimal surrender policy for reverse mortgage loans

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Abstract
This study conducts an optimal surrender analysis of reverse mortgage (RM) loans offered to elderly homeowners as a financing option. Recent market evidence on borrower early surrenders has raised concerns about the marketability of RM products and their impact on the program viability. In this article, we derive the borrower optimal surrender strategy as a function of the underlying value of the home used as collateral for RM contracts with tenure payment option. Using a probabilistic approach to American option pricing, we present a decomposition result for the value of the contract as the sum of its European counterpart without the surrendering provision and an early exercise premium. The methodology allows policymakers to assess the financial incentive of their policy design, from which we explain the existing market evidence about borrower rational lapse by means of the resulting surrender boundary and reference probabilities.

1. Introduction
A reverse mortgage (RM) is a financial scheme that allows elderly homeowners to cash out on their home equity without having to sell it via a variety of payment forms: a lump sum, a stream of annuity payment, or a line of credit (LOC). While RM-style products (a.k.a. equity release mortgages) are performing strongly in the United Kingdom and are poised to expand further across Europe, the merit of this product has not brought much success in the United States. In the United States, most RMs are federally insured home equity conversion mortgages (HECMs) administrated by the Federal Housing Administration (FHA). When the U.S. Department of Housing and Urban Development (HUD) reported 48,329 FHA-backed HECMs in Fiscal Year 2018, Urban Institute drew on statistics that other forms of tapping home equity, such as home equity loans, HELOCs, and cash-out refinances totaled 2.506 million for the same year. In practice, the relatively high cost of insurance, fees, and interest charges, and the inherent riskiness of the loan could be the major reasons for its limited marketability. Even for those having participated in the program, loan termination caused by some involuntary reasons, such as borrowers' health problem, relocation need, or their default on loan obligations, could also be a barrier of utilizing this product (see, e.g., Ji et al., 2012; Alai et al., 2014; Shao et al., 2015 and Moulton et al., 2015). According to FHA's annual report for Fiscal Year 2022, among those seniors who were qualified for HECMs, 92.77% of the pool preferred to take up the LOC payment option, while only 0.66% of them chose the tenure payment option. A serious concern over program viability has been raised by HECM authority due to the fact that HECM terminations have exceeded new originations every year since Fiscal Year 2016, and the number of terminated loans assigned to HECM's insurance fund has grown substantially since Fiscal Year 2014. As reported by United States Government Accountability Office (2019), among those terminations occurring in Fiscal Years 2014–2018, about 65% of them were

caused by non-death reasons, such as borrower defaults on property charges and maintenance (15%), loan repayment (9%), refinancing demands (8%), mobility issues (3%), and other reasons that cannot be determined (30%). Such a market experience could inspire a further investigation on the reasons for terminations with RM loans.

While most studies are focused on analyzing the risk from the uncertainty accompanying reverse RMs, such as the risk of house price drops, increase in interest rates and the longevity of the borrowers, a number of research emerge from the decision-making strategies about rational surrender adopted by RM borrowers. For instance, Nakajima and Telyukova (2017) and Blevins *et al.* (2020) develop several rational choice models to account for surrender decisions based on noneconomic factors, including borrower characteristics, refinancing options, and mobility issues. From market data of HECM loans, Jiang and Miller (2019) use regression analysis to find that borrowers who seek to refinance would be more likely to take advantage of rising housing prices, while younger borrowers tend to terminate their loans sooner than their older peers, through moving-out or refinancing. Although these models are helpful to explain borrowers' surrender incentives, it would be challenging to use them for RM pricing. Established on the option-pricing theory, some research come forth to analyze the financial impact of borrower surrender on RM guarantee. For example, Choi (2019) examines the RM loans from Korea market and concludes that a profitable surrender–and–refinance strategy requires a greater growth rate of housing prices in the presence of longevity risk. Using a binomial tree model, Lee and Shi (2021) investigate the borrower's prepayment decision based on utility maximization and the resulting impact on the entire risk profile of HECM loans. In their model, the borrower's utility is simplified as a function in which the only input is the underlying home value. Chiang and Tsai (2019) develop a micro-economic model to account for the rationality of borrowers' surrender decision caused by the desire to stay in their home, the uncertainty of their death time and the house price dynamic. In the presence of multiple risks for house price, interest rate, and borrower's prepayment, Shi and Lee (2021) employ an exogenous intensity-governed surrender model to capture both macro-economic and noneconomic drivers of surrender decisions.

In the notion that it can be of interest to policymakers to know the financial incentive of their policy design and hence improve borrower's welfare and discourage them from surrender, in this article, we provide an alternative way to tackle surrender analysis, which could be used as a tool for policymakers to assess the financial incentive of their policy design. In our approach, the decisions on borrower surrender are determined by financial optimization of the loan payout assessed from market conditions at loan origination. Due to the fact that the housing price is a non-tradable asset and hence arbitrage possibilities are quite limited, according to a stream of real estate literature on risk-neutral valuation, such as Van Bragt *et al.* (2015), the optimal surrender strategies may require the solution of optimal stopping problems akin to the valuation of American or Bermudan options. In actuarial literature, a number of contributions have taken this approach to evaluating various types of variable annuity (VA) contracts (see, for example, Grosen and Jørgensen 2000; Milevsky and Salisbury, 2006; Bauer *et al.*, 2008; Bacinello *et al.*, 2011 and references therein). In this article, we formulate with American option pricing principles an approach to valuation and optimal surrender problems in the context of reverse annuity mortgage (RAM). When compared to a VA rider, such as guaranteed minimum accumulation benefit (GMAB), the RAM has a more complex payout structure that requires a different approach of deriving core analytic formulas for surrender analyses (see, e.g., Venti and Wise, 1991; Lee *et al.,* 2012 and Bernard *et al.,* 2014b). We should note that the existing surrender analyses, such as Choi (2019) and Lee and Shi (2021), consider a surrender-and-refinance strategy in a discrete-time model, where the borrower has to refinance the loan to pay off the outstanding balance. This strategy requires additional assumptions, such as the knowledge of market parameters at the time of refinance and the level of principal limit factor used for identifying the accessibility of new loan. On the other hand, the surrender analyses considered by Chiang and Tsai (2019) and Lee and Shi (2021) involve borrower utility in the decision process. Different from prior methodologies, our approach builds on a continuous-time framework and produces analytic formulas for assessing borrower's financial incentives for surrender based on initial market conditions, without requiring borrower's refinance with a second loan when a surrender decision is made. In this article, we contribute to the literature by investigating a house-price-driven surrender strategy in line with the assumption of Chiang and Tsai (2019) and Lee and Shi (2021). In particular, we consider that rational RM borrowers tend to terminate the contract for the sake of receiving a surrender benefit when the house value rises above a certain level.

The major contribution of this article is that we formulate and perform an optimal surrender analysis in the context of RM loans. As an alternative approach to the existing surrender analysis from RM literature, it allows us to look more closely into the financial incentives of RM guarantees resulting from the prevalent valuation process, and explain the market evidence about borrowers' rational lapse from their financial optimization perspective. Second, assuming that the borrower acts rationally and follows an optimal strategy in order to receive a surrender benefit in the presence of home appreciation, we investigate the borrower surrender strategy and derive its boundary with RM guarantees in the spirit of Carr et al. (1992) and Bernard et al. (2014b). This technique can help us understand the impact of market parameters on the borrower surrender behavior, and thus draw policy implications. Using an optimal strategy to describe a borrower's behavior is a known approach in the context of market-linked insurance products or financial American options. However, our article is the first one that uses such an approach in the context of RM contracts. To relieve insurers' hedging difficulty at termination of the contract, we consider a surrender penalty charge, as widely used in VA markets. Finally, although our surrender analyses are focused on the government-issued HECM contracts in the United States, the methodologies and results presented in the article can be applied to any RM-style products around the world, including many proprietary RMs issued in the United Kingdom, North America, Europe, Australia, and Asian markets. The surrender analyses can help understand the low participation rate of the RM loans from the perspective of financial incentive embedded with this product design.

The remainder of this article is organized as follows. In Section 2, we introduce the model hypotheses and discuss the pricing condition for identifying the level of accessible loan amount for an RM. In Section 3, we first derive the value function of RM payouts based on borrower's financial incentive and price it with the surrender premiums. Then, we derive the optimal surrender boundary as a function of the value of the home over time. Meanwhile, a version of actuarial equivalence is also derived for identifying the fair level of loan amount inclusive of a rational surrender. We expect a mitigation of hedging difficulties for RM insurers and study borrowers' surrender incentive based on a proposed structure of penalty charges on the loan. Numerical results are presented in Section 4, where we perform a sensitivity analysis of borrower surrender behavior based on varying levels of market parameters. Section 5 concludes the article.

2. Problem setup

In this article, we formulate an approach to valuation and optimal surrender problems in the context of RAM with American option pricing principles. Assume that a borrower receives her/his T-year tenure loan from a lender who provides her/him with a nonrecourse loan in the form of an initial withdrawal ω and an ongoing annual rate of annuity payment c while entering a RM. At a terminating time $t \in [0, T]$, the borrower loses her/his home ownership and the lender sells the home used as collateral to pay off the loan. A nonnegative loan balance, $Bal(t) := (H_t - L_t)^+$, is then returned to the borrower or her/his heirs when the loan accumulation L_t is below the value of the home H_t. If the amount of the loan exceeds the value of the home, then the HECM program insurance covers the loss of the lender, that is $Loss(t) := (L_t - H_t)^+$, which is referred to as the crossover loss in RM literature. Insuring the loans allows the lender to reap additional profits with no risk by charging the borrower at an interest spread π_r, which are accrued with the loan over the years and will only be paid to the lender when the loan terminates. The borrower must pay MIPs to the HECM insurance fund, which can be financed as part of the HECM loan, including a lump-sum premium charged as a percentage of initial value of the home, that is $p_0 H_0$, as well as the ongoing mortgage insurance premiums (MIPs) at an annual percentage rate

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p_a on L_t. Moreover, the borrower is allowed to receive some equivalent rental income by depreciating the home appraisal, as well as she/he can choose to terminate the contract with no penalty. It is worth noting that the contract is structured using discrete-time periods, while the model uses continuous-time framework.

For simplicity, we assume a prevailing constant risk-free interest rate r. Then, for $t \in [0, T]$, the accumulated loan amount satisfies the ordinary differential equation

$$\frac{dL_t}{dt} = c + (r + \pi_r + p_a)L_t, \tag{2.1}$$

with the initial condition $L_0 = \omega + p_0 H_0$. By solving (2.1), we obtain

$$L_t = L_0 e^{mt} + \frac{c}{m}\left(e^{mt} - 1\right), \tag{2.2}$$

where $m = r + \pi_r + p_a$. The increase of L_t in (2.1) is contributed by the continuous-paying annuity payment c and accrued with the percentages of annual MIP and the interest charges at the total rate of m. Note that a discrete version of the loan dynamic (2.1) can be found in Lee et al. (2012). When $c \equiv 0$, the RAM loan L_t degenerates to a lump-sum loan with initial withdrawal ω. Our valuation process and surrender analyses derived in this article are applicable to this lump-sum payment option.

With a lifelong income backed by home equity used as collateral, in this study we identify the borrower's optimal surrender strategy for RAM loans with tenure payment option. This can be of interest to senior homeowners who want a steady stream of financing income since origination, and thus their surrendering decision can be influenced by the combined value of initial withdrawal, annuity payments, and the guarantee of the nonnegative balance, throughout the lifetime of the contract. Assume that the spot underlying value of the home used as collateral follows a geometric Brownian motion. Then, the price dynamics under the risk-neutral measure $\mathbb{Q}$ is given by

$$dH_t = (r - \delta)H_t dt + \sigma_H H_t dW_t^H, \tag{2.3}$$

where the home value is depreciated at a constant rate of rental discount/income δ over time. The constant σ_H is the volatility of housing prices and W_t^H is a Wiener process under the risk-neutral measure $\mathbb{Q}$. Note that HECM program does not restrict qualified borrowers from renting out rooms and space in the home for additional income as long as they are the primary resident, a rental discount is then applied to devalue the home appraisal for reflecting the equivalent rental income received by the borrower. Therefore, rental incomes can be viewed as some implicit benefits of equity release provided to borrowers. Similar to a number of studies that focus on theoretical development and implications of RM contracts, our research builds in the Black–Scholes framework, which is mathematically convenient and has been extensively used in the literature, such as Kau et al. (1992), Bardhan et al. (2006), Ji et al. (2012), Pu et al. (2014), Davidoff (2015), Han et al. (2017), Thomas (2021), and Lee and Shi (2021).

Under the Black–Scholes framework, the expectations of the discounted payoffs of the nonnegative balance $Bal(t)$ and the crossover event $Loss(t)$ can be characterized as the prices of a call and a put options, respectively, with the underlying H_t and the deterministic strike L_t, for any fixed time $t \in [0, T]$. We then evaluate these two quantities as follows:

$$\begin{aligned}\mathcal{C}(H_0, L_t, 0, t) := {}& e^{-rt}\mathbb{E}\left[Bal(t)\right] = e^{-rt}\mathbb{E}\left[(H_t - L_t)^+\right] \\ &= H_0 e^{-\delta t}\Phi\left(d_1\left(H_0, L_t, 0, t\right)\right) - L_t e^{-rt}\Phi\left(d_2\left(H_0, L_t, 0, t\right)\right),\end{aligned} \tag{2.4}$$

and

$$\begin{aligned}\mathcal{P}(H_0, L_t, 0, t) := {}& e^{-rt}\mathbb{E}\left[Loss(t)\right] = e^{-rt}\mathbb{E}\left[(L_t - H_t)^+\right] \\ &= L_t e^{-rt}\Phi(-d_2\left(H_0, L_t, 0, t\right)) - H_0 e^{-\delta t}\Phi(-d_1\left(H_0, L_t, 0, t\right)),\end{aligned} \tag{2.5}$$

where $\mathbb{E}[\cdot]$ denotes expectation operator under the risk-neutral measure $\mathbb{Q}$. $d_1(x, y, s, t) := \frac{\ln\left(\frac{x}{y}\right) + (r - \delta + \frac{\sigma_H^2}{2})(t - s)}{\sigma_H\sqrt{t - s}}$, $d_2(x, y, s, t) := \frac{\ln\left(\frac{x}{y}\right) + (r - \delta - \frac{\sigma_H^2}{2})(t - s)}{\sigma_H\sqrt{t - s}}$, and Φ denotes the standard normal cumulative distribution function.

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Mortality is another risk factor that plays an essential role in RM valuation and borrower's financial decision on the loan. Let τ_x denote the future lifetime of an individual aged (x) at time 0 with survival function under the risk-neutral probability measure $\mathbb{Q}$ given by

$$ {}_tp_x := \mathbb{Q}(\tau_x > t) = \mathbb{E}\left[e^{-\int_0^t \mu_{x+s}ds}\right], \tag{2.6} $$

where μ_{x+t} is the borrower's force of mortality at age $(x+t)$, $t \geqslant 0$. For illustrative purposes, we consider the same Makeham mortality model as in Bernard et al. (2014a), that is,

$$ \mu_x = A + B \cdot C^x, \text{ for } x \geqslant 0, \tag{2.7} $$

with $A = 0.0001$, $B = 0.00035$, and $C = 1.075$. In particular, for any $u \geqslant t$, the survival function for an individual aged $(x+t)$ can be written as

$$ {}_{u-t}p_{x+t} = e^{-\int_t^u A+B\cdot C^{x+s}ds} = \alpha^{u-t}\beta^{C^{x+t}\left(C^{u-t}-1\right)}, \tag{2.8} $$

where $\alpha = e^{-A}$ and $\beta = \exp(-B/\ln C)$.

3. Optimal surrender strategy

In this section, we formulate the optimal surrender strategy of a borrower who wants to maximize the value of her/his RM loan when the house price rises above a threshold barrier. We also derive a version of actuarial equivalence in conjunction with this rational surrender for the purpose of attaining the fair level of loan payment.

3.1. Optimal surrender region and reference probabilities

For a T-year contract, the issuer of an RM identifies the loan payments and MIPs based on the mortality table. Assume that during the underwriting, the mortality decrements about their ages are also recognized by RM borrowers with their life expectancy following what the mortality table suggests. Our goal is to determine the optimal surrender strategy, which a rational borrower should follow. In our model, the borrower's surrender is assumed to be driven by a home appreciation. To investigate the impact of all other market factors, such as interest rate and house price volatility, on the optimal strategy, sensitivity analyses of the surrender boundary can be done further with respect to these factors.

At a random death time $\tau_x \in [0, T]$, a borrower receives an accumulation of contract payouts over $[0, \tau_x]$, which includes initial withdrawal, annuity payments, and rental income by depreciating the value of the home used as collateral up to time τ_x. Then, the borrower's heirs receive a nonnegative balance $Bal(\tau_x)$. Therefore, the accumulated payout received by the borrower and her/his heirs from 0 to τ_x can be expressed as

$$ \Psi(H_{\tau_x}, 0, \tau_x; \tau_x) \ := \ \psi(H(0, \tau_x), 0, \tau_x; \tau_x) + Bal(\tau_x), \tag{3.1} $$

where

$$ \psi(H(0, \tau_x), 0, \tau_x; \tau_x) := \omega e^{r\tau_x} + \int_0^{\tau_x} e^{r(\tau_x - u)} \times \delta H_u du + c \times \int_0^{\tau_x} e^{r(\tau_x - u)} du, \tag{3.2} $$

which represents the accumulated value at time τ_x of the contract payouts the borrower received between time 0 and τ_x. Here, $H(0, \tau_x)$ is defined as a stochastic path of the home value over the time period $[0, \tau_x]$. As described in (3.2), the borrower receives by time τ_x an accumulated payout $\psi(H(0, \tau_x), 0, \tau_x; \tau_x)$, which includes the accumulation of initial withdrawals, rental income financed from the house appraisal, and annuity payments. Both the charges of MIP and interest reduce the amount of the borrower's nonnegative bequest $Bal(\tau_x)$ to their heirs.

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At each observation time t, $t \in [0, \tau_x)$, the accumulated payout between time 0 to τ_x can be broken into two components:

$$\psi(H(0, \tau_x), 0, \tau_x; \tau_x) = \psi(H(0, t), 0, t; \tau_x) + \psi(H(t, \tau_x), t, \tau_x; \tau_x), \tag{3.3}$$

where $H(0, t)$ and $H(t, \tau_x)$ are defined as the paths of the house value in time intervals $[0, t]$ and $[t, \tau_x]$, respectively. By $\psi(H(0, \tau_x), 0, t; \tau_x)$, we denote the accumulated contract payouts represented at time τ_x, which is the time-τ_x value of the borrower's loan over $[0, t]$, given by

$$\psi(H(0, t), 0, t; \tau_x) := \omega e^{r\tau_x} + \int_0^t e^{r(\tau_x - u)} \times \delta H_u du + c \times \int_0^t e^{r(\tau_x - u)} du.$$

Similarly, the value at time τ_x of the borrower's loan receipts over $[t, \tau_x]$ is defined as

$$\psi(H(t, \tau_x), t, \tau_x; \tau_x) := \int_t^{\tau_x} e^{r(\tau_x - u)} \times \delta H_u du + c \times \int_t^{\tau_x} e^{r(\tau_x - u)} du.$$

Note that we need the above decomposition of $\psi(H(0, \tau_x), 0, \tau_x; \tau_x)$ in order to represent the contract value of an RAM at arbitrary t, prior to the death time τ_x.

In what follows, we define the borrower surrender strategy based on the driving factor of home appreciation. For a situation when an immediate termination can lead to a substantial cash payout of the nonnegative balance, borrower's surrender of an RM could be worth more than her/his alternative strategy of staying in the program and continuing to pay ongoing charges of MIP and interest. In practice, a prudent RM borrower will consider surrender of the contract when the current home value is high. This observation motivates us to define an optimal surrender time as

$$\tau_B = \inf\{t \geqslant 0 : H_t \geqslant B_t\} = \inf\{t \geqslant 0 : H_t = B_t\}, \tag{3.4}$$

where the latter representation follows from the $\mathbb{Q}$–almost surely continuity of Brownian paths. In (3.4), the barrier B_t, $t \geqslant 0$, is a given function of time that separates the regions of surrender and no-surrender.

In the Appendix, we prove that the surrender option is a threshold strategy such that it is always worth for the borrower to lapse the contract when the value of the home reaches a prespecified deterministic barrier B_t. To be specific, we show that for any fixed time t less than or equal to τ_x, there is a value $H_t^* = B_t$ of the home above which the value of the contract for the borrower is less than the benefit available immediately by terminating the contract.

For a policymaker who wants to assess the surrender decision, it is tempting to find the probability $\mathbb{Q}(\tau_B \leqslant t)$, for borrower's surrender time τ_B occurring at or prior to the observation time t. Due to the fact that evaluating the surrender probability $\mathbb{Q}(\tau_B \leqslant t)$ is computationally difficult, for given times t_i, $i = 0, 1, \cdots, n$, the probability $\mathbb{Q}(\tau_B > t_i)$ can be approximated by the joint probability that $H_{t_j} < B_{t_j}$ for all $j \leqslant i$. Hence, the surrender probability for the contract in force can be approximated by

$$\mathbb{Q}(\tau_B \leqslant t_i) = 1 - \mathbb{Q}(\tau_B > t_i) \approx 1 - \mathbb{Q}\left(\bigcap_{j=0}^{i} \{H_{t_j} < B_{t_j}\}\right), \tag{3.5}$$

with $0 = t_0 < t_1 < \cdots < t_n = T$ for a sufficiently large n.

Note that the surrender probability (3.5) does not account for the effect of mortality decrement occurring by time t_i. For a market pool where the surrender experience is jointly considered with mortality decrement, RM policymakers may want to look at the joint distribution

$$\mathbb{Q}(\tau_B \leqslant t_i, \tau_x > t_i) = \mathbb{Q}(\tau_B \leqslant t_i | \tau_x > t_i) \times {}_{t_i}p_x. \tag{3.6}$$

An alternative way to show the financial incentive of an RM is based on the reference probability

$$\mathbb{Q}(H_t \geqslant B_t) = \Phi(d_2(H_0, B_t, 0, t)), \tag{3.7}$$

which, compared to the surrender probability (3.5), is computationally tractable and hence we use it for assessing the surrender decision in the numerical results of Section 4.3. For instance, given a prespecified level of MIP and loan payment, the borrower observes at loan origination and perceives the chance of the

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value of the home exceeding the corresponding barrier level of B_t over the future times. If the reference probability (3.7) is smaller and thus this chance is lower, the borrower will be more inclined to stay with the contract. We should note that the reference probability is a bound to the surrender probability by time t, in the sense that $\mathbb{Q}\left(H_t \geqslant B_t\right) \leqslant \mathbb{Q}\left(\tau_B \leqslant t\right)$. For larger values of t, this bound turns to be less accurate since the surrender probability for such t becomes larger. Since the borrower surrenders for the event $\{H_t \geqslant B_t\}$, the reference probability $\mathbb{Q}\left(H_t \geqslant B_t\right)$ can be also considered as a margin added to the surrender probability $\mathbb{Q}\left(\tau_B \leqslant t\right)$, whose steepness/rate of change of the curve then reflects the level of the reference probability at time t. For a real-world return of the underlying home equity $\mu_H > r$ (under $\mathbb{P}$), it is clear that the reference probability $\mathbb{P}\left(H_t \geqslant B_t\right)$ is greater than its risk-neutral counterpart (3.7).

A rational borrower only surrenders when the value of the home exceeds the loan amount; otherwise, she/he can stay in the home with small cost by paying home insurance and property tax, and even continue to receive annuity payments and rental incomes. In consequence, the value of the barrier B_t should be at least equal to the level of loan accumulation over time, that is $H_t \geqslant B_t \geqslant L_t$. That is to say, for a borrower who wants to terminate the contract at time $t \in [0, \tau_x)$, she/he will receive a surrender payout

$$\psi(H(0,t),0,t;t) + Bal(t) = \psi(H(0,t),0,t;t) + H_t - L_t, \quad \text{for } H_t \geqslant B_t. \tag{3.8}$$

Note that the nonnegative balance $Bal(t) = (H_t - L_t)^+$ in (3.8) will be immediately available to the borrower via foreclosure if the contract is surrendered at time $t \in [0, \tau_x)$. Meanwhile, the borrower stops receiving future annuity payment and rental income with the accumulated payout $\psi(H(0,t),0,t;t)$ by time t. Then, the borrower's obligation to continue the MIP and interest payments accrued to L_t ceases.

3.2. Derivation of the surrender boundary

In this section, we follow the approach of Carr et al. (1992) and derive the value function for determining the surrender boundary for RMs. We define the “book value" to reflect the fact that RM providers may want to know the values of the contract in their books of account, including the sum of any realized cash values and an unpaid “residual value" as their cost of ongoing liability.

As we state in the Theorem 3.1 below, the residual value $\widetilde{V}(H_t,t)$ of an RAM contract can be decomposed into a European loan guarantee in absence of a surrender activity until the event of death and a surrender premium charged against a potential rational termination by the borrower. To be specific, for $u > t \geqslant 0$, the time-t price of the European part is given by

$$\begin{aligned} v(H_t,t) = &\int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\left[\left(1-e^{-\delta(u-t)}\right)H_t + \frac{c}{r}\times\left(1-e^{-r(u-t)}\right)\right]du \\ &+ \int_t^T {}_{u-t}p_{x+t}\mu_{x+u}\left[H_t e^{-\delta(u-t)}\Phi\left(d_1(H_t,L_u,t,u)\right) - L_u e^{-r(u-t)}\Phi\left(d_2(H_t,L_u,t,u)\right)\right]du, \end{aligned} \tag{3.9}$$

where $f_{x+t}(u) := {}_{u-t}p_{x+t}\mu_{x+u}$ is the probability density function of a continuous lifetime of a borrower aged $(x+t)$.

To determine the level of the surrender premium charged against potential rational termination, we first derive the corresponding instantaneous rate of charges. By terminating the contract, the borrower can cease the increasing amount of MIPs on the loan, and the ongoing interest charges on the loan, MIPs and annuity payments that may no longer match the value of the contract guarantees. In exchange, an equivalent level of surrender premium is priced and added to the corresponding European part of the residual value of an RM loan, for offering borrower's surrender option. As shown in the Appendix, the instantaneous charge of surrender premium at time $u \in [t, \tau_x]$ is given by

$$\eta(u)du := (\pi_r + p_a)L_u du. \tag{3.10}$$

The instantaneous charge represented by Equation (3.10) implies that by means of an early termination, the borrower can save immediately on a surrender benefit, including the lender's interest spread and MIP charges at the rate of $\pi_r + p_a$ on accumulation of the loan. It is noteworthy here that at the time of termination, the borrower can save interest charges at the rate of $r + \pi_r$ on the loan amount, but

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also she/he has to early repay the lender so that the borrower will lose the time value of the loan at the risk-free rate r. Thus, we observe from (3.10) that the borrower earns her/his instantaneous surrender benefit from the loan interest at the rate of $\pi_r L_u$ in net.

Since a rational termination can only occur before the time of death, the surrender premium at $t < \tau_x$ is then based on the expectation of the instantaneous charges over time, that is

$$e\left(H_t, t\right) = \int_t^T {}_{u-t}p_{x+t} \times e^{-r(u-t)} \times \eta(u) \times \Phi\left(d_2(H_t, B_u, t, u)\right) du. \tag{3.11}$$

In (3.11), the expectation of the instantaneous charge of (3.10) is discounted with borrower's mortality decrement, while the payment can be triggered only when $H_u \geqslant B_u$ over time. Different from the MIPs financed from the loan and charged for crossover loss, the valuation of surrender premium (3.11) for a potential early exercise of nonnegative balance does not come with real cash flows. Note that rental incomes can be viewed as borrower's implicit benefit of releasing equity value by discounting an equivalent amount from home appraisal. For a borrower's rational surrender occurring when $H_u \geqslant B_u \geqslant L_u$, she/he stops to receive the rental income but can save an equal amount of rental discount from the nonnegative balance $Bal(u)$. As such, the borrower does not benefit or lose from ceasing rental income, and no rental benefit would be counted as part of the surrender premiums.

The formulas (3.9) and (3.11) lead us to the price of an RAM contract in Theorem 3.1, by which we identify the level of the boundary value B_t over time. A detailed derivation of Formulas (3.9)–(3.11) and the proof of Theorem 3.1 can be found in the Appendix.

Theorem 3.1 *For a filtered probability space* $\left(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geqslant 0}, P\right)$, *denote by* $\mathcal{T}_{[t,\tau_x]}$ *the set of all stopping times* τ *that are less or equal to* τ_x *and greater or equal to any observation time* $t \in [0, \tau_x]$. *Then, it is worth for a rational borrower to terminate the contract at the observation time t when the surrender payout in (3.8) is no less than the book value* $V(H_t, t)$ *of a surrenderable RAM given by*

$$\begin{aligned} V(h,t) &= \psi(H(0,t),0,t;t) + \sup_{\tau\in\mathcal{T}_{[t,\tau_x]}} \mathbb{E}\left[e^{-r(\tau-t)}\left(\psi(H(t,\tau),t,\tau;\tau) + Bal(\tau)\right) | H_t = h\right] \\ &= \psi(H(0,t),0,t;t) + \widetilde{V}(h,t), \end{aligned} \tag{3.12}$$

where the residual value of the contract by time-t admits the following decomposition

$$\widetilde{V}(H_t, t) = v(H_t, t) + e(H_t, t), \tag{3.13}$$

with $v(H_t, t)$ *and* $e(H_t, t)$ *denoting, respectively, a European part and an early surrender premium.*

Theorem 3.1 provides a way to calculate the value of an RAM with surrender option. However, since the surrender premium in (3.11) depends on the optimal surrender boundary $\{B_t, t \geqslant 0\}$, one needs to compute it first. In the following, we derive the optimal surrender boundary condition in analogy to Kim and Yu (1996) and Bernard et al. (2014b). For the determination of the surrender boundary, we observe that the time value of the contract guarantee fades away when it approaches T. Therefore, a rational borrower will terminate whenever the value of the home is greater than the loan accumulation at T. This implies that the barrier for the value of the home should be equal to the loan accumulation, that is, $B_T = L_T$. For any $t \in [0, T)$, along the surrender boundary, we have the following equation:

$$\Psi(H_t, 0, t; t) = \psi(H(0,t), 0, t; t) + H_t - L_t = \psi(H(0,t), 0, t; t) + B_t - L_t. \tag{3.14}$$

Note that in (3.14), $\Psi(H_t, 0, t; t)$ is a function of the value H_t, while $\psi(H(0,t), 0, t; t)$ is a function of the path $H(0, t)$. Thus, we have

$$B_t - L_t = \widetilde{V}(B_t, t) = v(B_t, t) + e(B_t, t). \tag{3.15}$$

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By substituting (3.9) and (3.11) into (3.15), we obtain

$$B_t - L_t = \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} \left[\left(1 - e^{-\delta(u-t)}\right) B_t + \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right) \right] du$$
$$+ \int_t^T {}_{u-t}p_{x+t}\mu_{x+u} \left[B_t e^{-\delta(u-t)} \Phi\left(d_1(B_t, L_u, t, u)\right) - L_u e^{-r(u-t)} \Phi\left(d_2(B_t, L_u, t, u)\right) \right] du,$$
$$+ \int_t^T {}_{u-t}p_{x+t} \times e^{-r(u-t)} \times \eta(u) \times \Phi\left(d_2(H_t, B_u, t, u)\right) du. \tag{3.16}$$

Based on Equation (3.16), in the following, we discuss methods of computing the barrier level B_t for practical use.

3.3. Pricing with actuarial equivalence for surrenderable RM loans

In Section 3.2, we describe how to calculate the barrier level B_t as a function of a given annuity payment c. Since the level of loan payment can also impact the solvency of HECM insurance fund, in this section, we develop an actuarial equivalence for identifying a fair level of loan payment in conjunction with the borrower's surrender option.

Due to the fact that an ongoing payment of annual MIP is ceased upon borrower's surrender at which the crossover loss becomes due for RM insurers, we consider a proxy loss function between the accumulated annual MIPs and the crossover loss occurring at the surrender time $\tau \in [0, \tau_x]$, which is defined by

$$\Psi_0^l\left(H_\tau, \tau\right) := Loss(\tau) - \int_0^\tau e^{r(\tau-u)} p_a L_u du = (L_\tau - H_\tau)^+ - \int_0^\tau e^{r(\tau-u)} p_a L_u du.$$

For an optimal stopping time $\tau^* \in \mathcal{T}_{[0,\tau_x]}$ that has maximized the value of the contract over the period $[0, \tau_x]$, we have recovered its associated surrender boundary $\{B_t, t \in [0, T]\}$ by Theorem 3.1. With an initial MIP, we want to identify a fair level of the loan payment by applying the assumption of zero expected loss-at-issue in the sense that

$$0 = \mathbb{E}\left[e^{-r\tau^*} \Psi_0^l\left(H_{\tau^*}, \tau^*\right)\right] - p_0 H_0,$$

or equivalently

$$p_0 H_0 + \mathbb{E}\left[\int_0^{\tau^*} e^{-ru} p_a L_u du\right] = \mathbb{E}\left[e^{-r\tau^*} Loss(\tau^*)\right], \tag{3.17}$$

which suggests a perfect match between the actuarial present value (APV) of MIPs and the cost of insurance accounting for the borrower's surrender option.

In Theorem 3.2, we derive the analytical formulas for actuarial equivalence (3.17) to identify the fair level of loan payment conforming to the rational surrender. The proof of Theorem 3.2 is similar to the one in Theorem 3.1 with its detail presented in Appendix.

Theorem 3.2 *For a surrenderable RM loan, a fair level of annuity payment c for tenure payment option or initial withdrawal ω (with assumed $c = 0$) for a lump-sum loan that satisfies the actuarial equivalence (3.17) can be solved from the following analytical representation*

$$p_0 H_0 = \widetilde{V}^l\left(H_0, 0\right) = \mathbb{E}\left[e^{-r\tau^*} \Psi_0^l\left(H_{\tau^*}, \tau^*\right)\right] = v^l(H_0, 0) + e^l\left(H_0, 0\right), \tag{3.18}$$

with the European price of the loss payout Ψ_0^l

$$v^l(H_0, 0) = \int_0^T {}_u p_x \mu_{x+u} \left[L_u e^{-ru} \Phi\left(-d_2\left(H_0, L_u, 0, u\right)\right) - H_0 e^{-\delta u} \Phi\left(-d_1\left(H_0, L_u, 0, u\right)\right) \right] du$$
$$- \int_0^T {}_u p_x \times p_a L_u e^{-ru} du, \tag{3.19}$$

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and an early surrender premium given by

$$e^l(H_0, 0) = \int_0^T {}_u p_x \times \left[p_a L_u e^{-ru} - \int_0^u r p_a L_s e^{-rs} ds \right] \times \Phi\left(d_2\left(H_0, B_u, 0, u\right)\right) du, \tag{3.20}$$

respectively. In (3.19) and (3.20), the surrender boundary $\{B_u, u \geqslant 0\}$ and the solution of loan payment (i.e., c or ω) conform to both Theorems 3.1 and 3.2.

In Theorem 3.2, the value of the discounted proxy loss Ψ_0^l can be written as a European part, which amounts to the cost of Ψ_0^l only paid out at the time of death, together with an early exercise premium for Ψ_0^l in the situation when the optimal surrender strategy in Theorem 3.1 is exercised. From the insurer's perspective, the early repayment of the loan allows the borrower to discontinue the ongoing annual MIP that should be paid until the death, while it also causes her/him to give up the time value of holding the past premium financing at the risk-free rate r. In consequence, as described in (3.20), additional cost of insurance needs to be added over the European part of the premium to compensate the insurer for the net surrender benefit received by the borrower. The purpose of Theorem 3.2 is to characterize a fair level of loan payment, together with the resulting boundary $\{B_t, t \geqslant 0\}$, that reconciles both Theorems 3.1 and 3.2. In the next section, we provide a recursive algorithm for the recovery of barrier level B_t as a function of annuity payment c. All other things being unchanged, we then use the Matlab "fzero" function to solve the constant c that complies with the equivalence (3.18).

3.4. Calculation of the surrender boundary

The integral equation in (3.16) can be used to compute the optimal surrender boundary $\{B_t, t \in [0, T]\}$. Observe, however, that in order to obtain the value of the barrier B_t at a specific time t, the optimal surrender barriers for future times must be known. Since $B_T = L_T$ at maturity T, we work backward through time to recursively recover the optimal surrender boundary. Because (3.16) does not have an analytic solution, numerical integration schemes must be used. Practically this is done by dividing the interval $[0, T]$ into n subintervals $0 = t_0 < t_1 < \cdots < t_n = T$ of equal lengths, that is, $\Delta_{t_i} = t_{i+1} - t_i = T/n$, where times t_i, $i = 0, \cdots, n-1$, represent the only possible times for the termination due to the event of borrower's surrender or death. For $k = 1, \cdots, n$, define

$$\begin{aligned} g_1(u, t_{n-k}) :=& \left(1 - e^{-\delta(u - t_{n-k})}\right) B_{t_{n-k}} + \frac{c}{r} \times \left(1 - e^{-r(u - t_{n-k})}\right) \\ &+ B_{t_{n-k}} e^{-\delta(u - t_{n-k})} \Phi\left(d_1(B_{t_{n-k}}, L_u, t_{n-k}, u)\right) - L_u e^{-r(u - t_{n-k})} \Phi\left(d_2(B_{t_{n-k}}, L_u, t_{n-k}, u)\right), \end{aligned} \tag{3.21}$$

and

$$g_2(u, t_{n-k}) := e^{-r(u - t_{n-k})} \times \eta(u) \times \Phi\left(d_2(B_{t_{n-k}}, B_u, t_{n-k}, u)\right). \tag{3.22}$$

Then, the surrender premium in (3.11) can be approximated by using the method of rectangular integration, which results in the following approximations $I_1(k)$ and $I_2(k)$, $k = 1, \cdots, n$, of the European value of the contract without surrender, and the surrender premiums over the time intervals (t_{n-k}, t_n), respectively:

$$I_1(k) := \sum_{i=0}^{k-1} {}_{t_{n-k+i+1} - t_{n-k}} p_{x + t_{n-k}} \times {}_{\Delta_{t_{n-k+i+1}}} q_{x + t_{n-k+i+1}} \times g_1(t_{n-k+i+1}, t_{n-k}), \tag{3.23}$$

and

$$I_2(k) := \frac{T}{n} \sum_{i=0}^{k-1} {}_{t_{n-k+i+1} - t_{n-k}} p_{x + t_{n-k}} \times g_2(t_{n-k+i+1}, t_{n-k}), \tag{3.24}$$

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for $i \leqslant k-1$ and $k = 1, 2, \cdots, n$, where n denotes the number of rectangles used for approximating the integral. In (3.23), we approximate the death probability ${}_{\Delta_{t_{n-k+i+1}}}q_{x+t_{n-k+i+1}} = 1 - {}_{\Delta_{t_{n-k+i+1}}}p_{x+t_{n-k+i+1}} \approx \mu_{x+t_{n-k+i+1}} \times \Delta_{t_{n-k+i+1}}$, for a sufficiently large n.

Note that for approximating both the European part (3.23) and the early surrender premium (3.24) in discrete-time periods, the time of borrower's death occurs after the observation time t_{n-k}. Furthermore, borrower's surrender can only happen prior to the death determination and at the beginning of each time interval, at which the ongoing annuity payment and MIP are paid. The numerical method of integration that we have employed in (3.23) and (3.24) has produced stable results, but it is possible that other quadrature methods may lead to more efficient algorithms. Next, we use the following iteration steps proposed by Bernard et al. (2014b) for the recovery of the borrower's surrender boundary, described in (3.15) and (3.16):

Step 1. $B_{t_n} = B_T = L_T$.

Step 2. Recursively, for $k = 1, \cdots, n$, compute $I_1(k)$ and $I_2(k)$ in (3.23) and (3.24), respectively, to approximate the right part of (3.16) and solve the following equation for the only unknown $B_{t_{n-k}}$:

$$B_{t_{n-k}} - L_{t_{n-k}} = I_1(k) + I_2(k). \tag{3.25}$$

3.5. Calculation of the surrender boundary with prepayment penalty

Although rational borrowers will only terminate the contract when $H_t \geqslant L_t$, RM insurers may still experience a loss due to a crossover event, when the contract is foreclosed due to the borrower's mobility issues such as relocation or the financial hardship of maintaining their contractual obligations. As suggested by some Canadian RM providers, such as Equitable Bank, insurers may impose additional management fees and financing costs caused by the lapse of the contract. Also, borrowers may surrender for refinance and/or home sale, which involve a transaction cost and/or a cost of moving and renting. To account for these expenses, we propose a prepayment penalty charge at the percentage rate of $\kappa_t = e^{\kappa t} - 1$, with constant $\kappa > 0$ at a surrender time t, representing an assessment of the additional equity release so that the borrower would get much less than $H_t - L_t$ through surrender. Due to the nonrecourse provision, the loan accumulation is alway increasing and hence the corresponding cost of a crossover event tends to increase over time. Thus, we assume that κ_t is an exponential function in t, which is levied on the loan accumulation L_t at the surrender time $t \in [0, T]$. In what follows, we describe a method of finding the surrender boundary with the proposed penalty charges. The details of the derivation can be found in the Appendix. The numerical approximation to determine the surrender boundary with prepayment charges follows a backward procedure, which can be summarized in two main steps:

Step 1. $B_{t_n} = B_T = L_T$.

Step 2. Recursively, for $k = 1, \cdots, n$, in analogy to the integral expressions in (3.23) and (3.24), compute $I_{1,\kappa}(k)$ and $I_{2,\kappa}(k)$ to approximate the right part of (3.26) and solve the following equation for the only unknown $B_{t_{n-k}}$:

$$B_{t_{n-k}} - e^{\kappa t_{n-k}} L_{t_{n-k}} = I_1(k) + I_{2,\kappa}(k), \tag{3.26}$$

where $I_1(k)$ is given in (3.23). $I_{2,\kappa}(k) := \frac{T}{n}\sum_{i=0}^{k-1} {}_{t_{n-k+i+1}-t_{n-k}}p_{x+t_{n-k}} \times g_{2,\kappa}(t_{n-k+i+1}, t_{n-k})$, for $i \leqslant k-1$ and $k = 1, 2, \cdots, n$. $g_{2,\kappa}(u, t_{n-k}) = e^{-r(u-t_{n-k})} \times \eta_\kappa(u) \times \Phi\left(d_2(B_{t_{n-k}}, B_u, t_{n-k}, u)\right)$. $\eta_\kappa(u) = (\pi_r + p_a + \kappa)\, e^{\kappa u} L_u + c \times (e^{\kappa u} - 1)$.

Note that the inclusion of prepayment charges in (3.26) increases the expected surrender costs, including the surrender premiums and penalty charges at the time of termination. However, it will not affect the value of the European part reflecting the value of the contract only with death termination.

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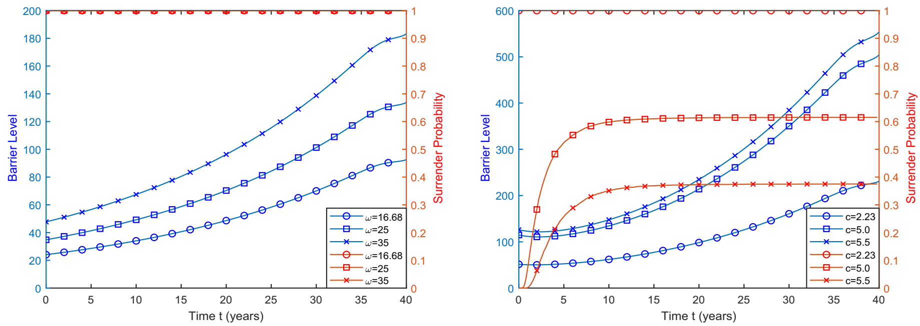

Figure 1. *Surrender boundary and probability with no mortality decrement for a lump-sum withdrawal (left panel) and tenure annuity payment (right panel) respectively.*

4. Numerical illustration

In this section, we present a sensitivity analysis of the evolution of the borrower's surrender motive based on the levels of some market parameters. We use the following parameters: $r = 0.02$, $\pi_r = 0.015$, $\delta = 0.01$, $\sigma_H = 0.083$, $H_0 = 100$, $T = 40$, $x = 70$, and $n = 200$. The mortality model with parameters is given in (2.7). Using the actuarial equivalence (3.17) with the HECM rates of $p_0 = 2\%$ and $p_a = 0.5\%$, we solve $\omega = 16.6780$ (with assumed $c = 0$) for the lump-sum option and $c = 2.2343$ (with $\omega = 0$) for the tenure annuity loan, respectively. We should mention that in practice, RM policymakers could determine the level of loan payment by adopting alternative pricing criteria other than the equivalence (3.17) with surrender option. For illustrative purposes, we next study the surrender boundary and the corresponding probabilities in response to different levels of loan payment.

4.1. Comparison of surrender boundary and surrender probability

In the following, we investigate the financial incentive of RMs by comparing the borrower surrender boundary and the probability of breaching it based on a variety of loan payment. Our findings are summarized as follows.

First, the level of the loan payment identified with the insurer's pricing criterion has a considerable impact on the value of the loan and hence the level of borrower's surrender boundary and probability. In Figure 1, we identify the respective surrender boundaries characterized in Theorem 3.1 (blue lines) and then use the Monte Carlo method with 10^6 repetitions for approximating the corresponding surrender probabilities excluding the effect of mortality decrement that have been defined in (3.5) (red lines). Since a standard RM loan generally bears a significant amount of interest and MIP expenses, it lowers the borrower accessibility of the loan. On top of the fair values of $\omega = 16.6780$ and $c = 2.2343$ identified by the pricing criterion (3.17), we consider different levels of loan payment for the sensitivity test. As displayed in the left panel of Figure 1, for different ω all the initial barrier values fall below the house price, that is, $B_0 < H_0$, and the corresponding surrender probability (3.5) is equal to one, at which it is financially unwise for a rational borrower to enter the loan. As explained by many authors, such as Szymanoski et al. (2007) and Haurin et al. (2016) HECM borrowers are a highly selected sample with high liquidity needs. When the liquidity needs fade away, the borrower may optimally terminate the contract for surrender benefit. In contrast, an increase in the loan payment could reduce the borrower's surrender incentive and lead to a higher initial barrier value than the house price. For instance, in the right panel of Figure 1, one can find that the surrender probability shifts downward for a sufficiently high level of the annuity payment that gives rise to a higher initial barrier value than the house price.

In addition, most market experience with rational surrender can happen in the early-through-mid term of the loan life. As displayed in the left panel of Figure 1, at all levels of ω the lower initial barrier level

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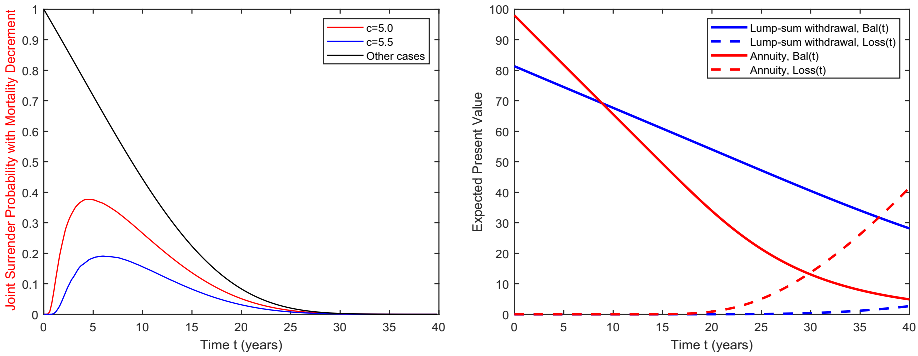

Figure 2. *Surrender probability with mortality decrement (left panel) and EPVs of Bal(t) and Loss(t) (right panel), respectively.*

than the value of the home should cause an immediate termination for rational borrowers. In the absence of mortality decrement, when the annuity level leads to a higher initial barrier level than the price of the home, we observe from the right panel of Figure 1 that the curves of the surrender probability (3.5) (red lines) strictly increase in time, while they are concave upward in the first several years and then turn to be concave downward in the later years of the contract. As explained in Section 3.1, this concavity of the curves can be further confirmed by the reference probability (3.7), indicating that the surrender event of $\{H_t \geqslant B_t\}$ is more likely to occur in the early years than that in the later years. In the case that the occurrence of mortality decrement is jointly considered after the surrender by time t, the left panel of Figure 2 depicts the joint probabilities (3.6) given different levels of lump-sum withdrawal and annuity payment. For the respective cases $c = 2.2343$, $\omega = 16.6780$, $\omega = 25$ and $\omega = 35$, compared to the probability curves (red lines) in the right panel of Figure 1, the joint distribution (3.6) illustrated in the left panel of Figure 2 is equivalent to the survival function (2.6), since the probability $\mathbb{Q}\,(\tau_B \leqslant t_i | \tau_x > t_i)$ is equal to one for all t_i. For the cases when $c = 5.0$ and $c = 5.5$, the probability curves slightly drop in the early years, while weighted by mortality decrements, they are significantly depressed in the later years, which leads to a right skewness of the entire distribution. In particular, it is impractical for a borrower to terminate for $t \geqslant 30$, as the joint probability (3.6) tends to be zero. This observation confirms that for a market pool of RM policies, one can anticipate that most surrender occurrences prior to the death termination will occur in the early-through-mid term of the contract.

Finally, the surrender probability generally decreases due to the accelerated accumulation in the later years of the loan. This reduces the borrower's surrender incentives but could also increase the inherent riskiness of the loan to the insurer. To assess the inherent riskiness of an RM loan over time, in the right panel of Figure 2, we depict the expected present values (EPVs) of nonnegative balance and crossover loss at a varying level of generic time t. Due to a greater possibility of a crossover event occurring in the later years of the contract, for example, at the maturity $T = 40$, we examine the level of crossover risk by checking the reserve of the nonnegative balance and the size of the crossover loss with the predetermined level of loan payment. As illustrated in the right panel of Figure 2, with the base cases $\omega = 16.6780$ and $c = 2.2343$, we obtain the corresponding EPVs of $Bal(T)$ and $Loss(T)$ in (2.4) and (2.5), respectively, as $\mathcal{C} = 28.1372$ and $\mathcal{P} = 2.6738$ for the lump-sum loan, and as $\mathcal{C} = 4.8813$ and $\mathcal{P} = 41.5151$ for the tenure annuity loan. These results show that the loan accumulation with annuity payment option exhausts the borrower's nonnegative balance more quickly than that with the lump-sum withdrawal and hence causes a significant level of crossover loss at T. In contrast, the lump-sum option reserves more cash value in the nonnegative balance than the annuity loan, with a substantial amount of $\mathcal{C}$ that leads to a negligible $\mathcal{P}$. One should note that the future mortality decrement of the borrower aged 70 has a significant influence on the actual risk level of the loan. As shown in the left panel of Figure 2 (black line), the survival probability is approaching zero for $t > 30$, indicating that the above significant cost of crossover loss $\mathcal{P}$ illustrated as examples can rarely happen from real market experience.

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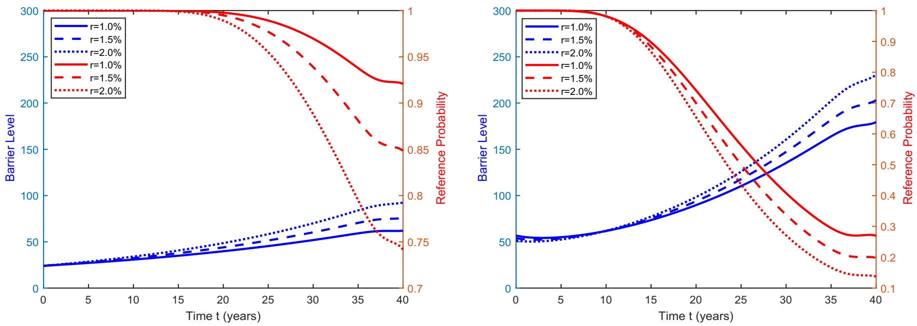

Figure 3. *Surrender boundary and reference probability versus the interest rate r for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

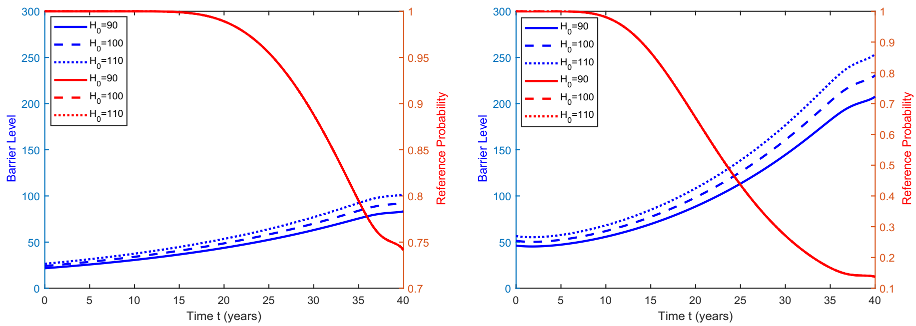

Figure 4. *Surrender boundary and reference probability versus the house price H_0 for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

4.2. General pattern of surrender boundary and reference probability

As stated in Theorem 3.1 and Equation (3.15), the value of the barrier B_t, for each $t \in [0, T]$, is a break-even point where borrower's surrender payout is exactly equal to the book value $V(H_t, t)$ with an ongoing liability of the contract. Instead of simulating the surrender probability (3.5) or (3.6), in Sections 4.2 and 4.3 we use the reference probability (3.7) based on the barrier level B_t to assess the likelihood of the borrower surrender, which, as indicated in Section 3.1, is more computationally tractable than the other two. For both types of RM such as the lump-sum withdrawal and the annuity loan, we examine the sensitivity of their surrender boundary with respect to the parameters r, H_0, δ, σ_H, π_r, p_0, p_a and κ. In this section, we delineate the general patterns of surrender boundary and reference probability based on the detailed sensitivity analyses followed in Section 4.3.

In our analyses, both curves of the surrender boundary and its corresponding reference probability are derived as a tool for identifying the timing and the likelihood of borrower surrender. In consequence, the general pattern for the curves of surrender boundary and its associated reference probability reflects the progression of the embedded financial value throughout the lifetime of the loan. With a fair level of loan payment, we observe from Figures 3–10 that the curve of the surrender boundary increases over time, while it turns to be much steeper in the later years than the early years of the contract. Correspondingly, the curve of the reference probability starts at a high level in the early years of the contract and then declines in the following years.

To understand the above pattern, we look more closely into both the intrinsic and time values of an option. It is well known that the intrinsic value of an option represents what it would be worth if the buyer exercised it immediately, while the time value represents the possibility that the option would

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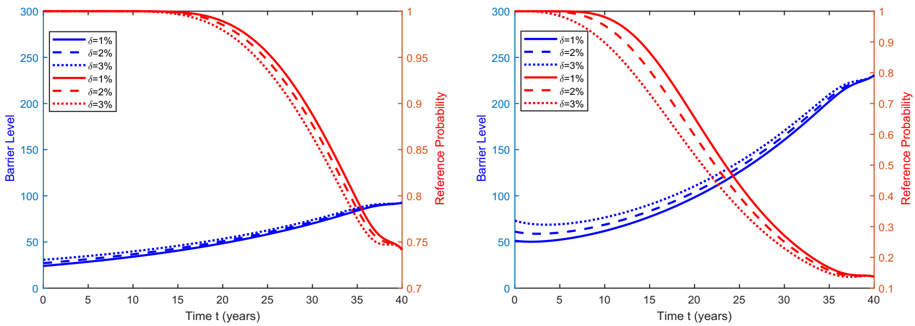

Figure 5. *Surrender boundary and reference probability versus the rental yield δ for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

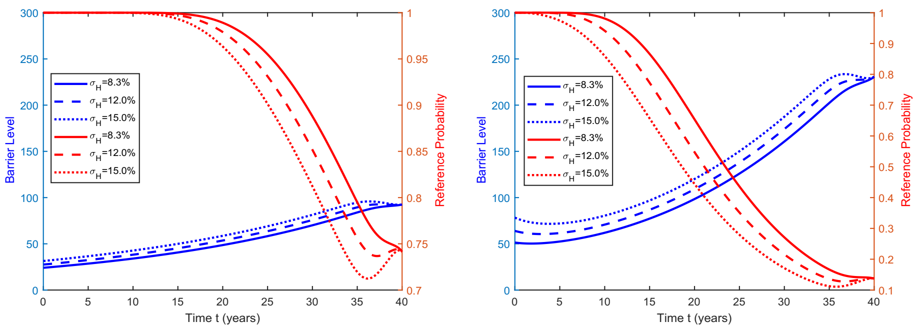

Figure 6. *Surrender boundary and reference probability versus the house price volatility σ_H for a lump-sum option (left panel) and tenure annuity payment (right panel) respectively.*

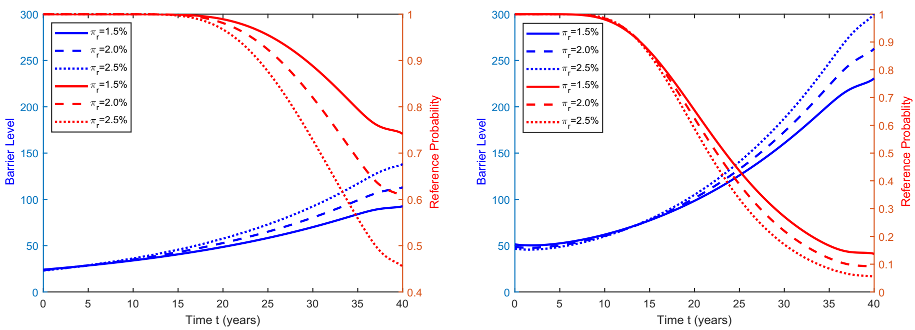

Figure 7. *Surrender boundary and reference probability versus the interest spread charge π_r for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

increase in value before its expiration date. In the early years of the contract, the nonnegative balance $Bal(t) = (H_t - L_t)^+$ is deeply in the money, with a relatively small loan accumulation. Since at this point the liability of a crossover event, $Loss(t) = (L_t - H_t)^+$, which can be viewed as a put option with its increasing strike L_t, for $t \in [0, T]$, is deeply out of the money, it is financially unwise for the borrower to stay in the program while adding financing charges of MIP and interest with no need for crossover loss protection, as it simply decreases the value of $Bal(t)$ that she/he can receive by surrender. During this

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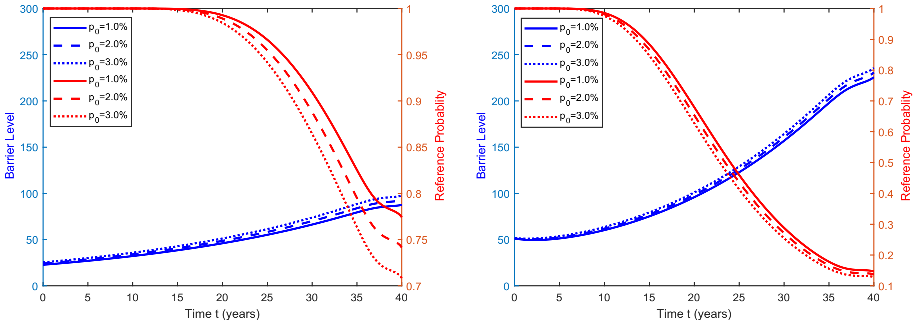

Figure 8. *Surrender boundary and reference probability versus the initial MIP rate p_0 for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

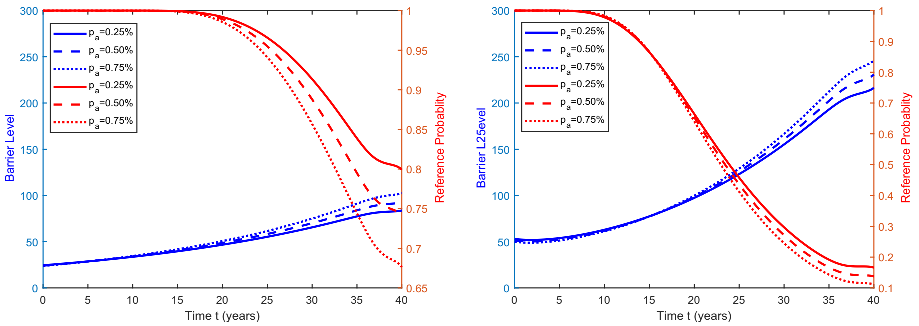

Figure 9. *Surrender boundary and reference probability versus the annual MIP rate p_a for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

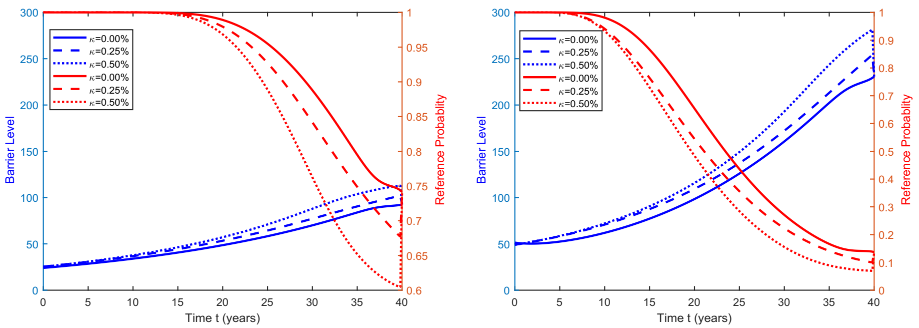

Figure 10. *Surrender boundary and reference probability versus the prepayment charge κ for a lump-sum option (left panel) and tenure annuity payment (right panel), respectively.*

period, when the liquidity needs fade away, the borrower is inclined to terminate the contract as long as the value of the home is sufficiently larger than the loan amount. We thus find that the curve of the boundary stays in a lower level and the corresponding reference probability stays at a high level in the first several years.

With the passage of time, the loan accumulation grows exponentially by payment contributions from annuity payments, premiums and interest. In the later years of the contract, although the increasing level of MIP and interest added to the loan decreases both intrinsic and time values of the nonnegative balance $Bal(t)$, the borrower tends to start receiving the nonrecourse benefit of her/his RM loan. Meanwhile, the put liability $Loss(t)$ to the insurer is still out of the money but its time value increases. Due to her/his

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increasing demand for crossover loss protection, these findings indicate that the borrower is more willing to pay the MIP and interest charges for a time value of the put liability $Loss(t)$ that potentially turns to be in-the-money soon. In the meantime, the borrower is more likely to stay in the program and take advantage of its out-of-the-money status of the nonnegative balance $Bal(t)$, since she/he can potentially benefit from the nonrecourse clause of an RM loan by receiving a loan amount greater than the value of the home, which is equivalent to the amount of $Loss(t)$ undertaken by the insurer. This trend lasts until the maturity of the contract, at which the time values of the nonnegative balance guarantee $Bal(t)$ and the put liability $Loss(t)$ both fade away. Consequently, the level of the reference probability decreases in time to a relative minimum at maturity, while the level of the corresponding surrender boundary increases more quickly and is capped at the level of L_T.

At this point, we have explained the pattern of the surrender boundary and its associated reference probability observed from the numerical examples based on the level of loan payment identified with the pricing condition (3.17). We should also mention that since both the level of loan payment and the distribution of death time can affect the initial barrier level and hence the surrender probability, some other patterns may be observed. As depicted in the right panel of Figure 1, for certain levels of loan payment that lead to a higher initial barrier value than the house price, that is, $c = 5.0$ or $c = 5.5$, the concavity of the surrender probability curve (3.5) changes from up to down. As explained in Section 3.1, this indicates that the reference probability (3.7) could first climb from zero to its relative maximum with a growing amount of the loan devaluing the borrower's nonnegative balance. Then, it decreases in time to a relative minimum at maturity as the nonrecourse feature of the loan takes into effect.

4.3. Sensitivity of surrender boundary and reference probability

In Figures 3–10, we perform the sensitivity test in regard to both types of RM loans by varying the levels of certain market parameters while keeping everything else unchanged. Unless otherwise specified, the numerical results are based on the base level of the loan payment, that is $\omega = 16.6780$ for the lump-sum withdrawal or $c = 2.2343$ for the tenure annuity loan. Generally, when compared to the lump-sum option, the annuitization of RM payout could save the interest expense and thus increase the loan accessibility to the borrower over time, which discourages borrower surrender with a relatively higher boundary level.

In Figure 3, we compute the surrender boundary for different levels of the interest rate r with the predetermined rates of MIP and loan payment. As indicated in the Introduction, Jiang and Miller (2019) account for the refinance termination due to appreciation of the value of the home and decreases in the interest rate. Their data also suggest that borrowers are more likely to take advantage of rising housing prices and declining interest rates through refinancing in the first few years after origination. While our model is assumed to capture borrower's surrender strategy driven by home appreciation, by performing a sensitivity test it also sheds lights on those surrender strategies driven by the interest rate. For example, we observe from Figure 3 that the surrender boundary shifts upward with the interest rate r. With the fair level of loan payment at the base rate $r = 2\%$, a decline in the rate of interest can slow down the accumulation of the loan, and then shifts the nonnegative balance value $Bal(t)$ to be more in-the-money at the time of the termination of the contract. Meanwhile, the put liability $Loss(t)$ becomes less valuable, which exacerbates the mismatch between the charges of MIPs and interest and the contract guarantees. Due to this value reduction, the curve of the surrender boundary (with an increasing reference probability) is pushed down, and thus, in a climate of low interest rates, a rational borrower is more inclined to terminate the contract. In general, borrower's surrender motive can be very sensitive to interest rates. When compared to the lump-sum option, the annuitization of the loan payout leads to a higher sensitivity of surrender boundary or the associated reference probability in response to the change of interest rate.

In practice, borrowers typically have different values of their equity at the point of entering the contract. On top of the base cases that $\omega = 16.6780$ and $c = 2.2343$ for $H_0 = 100$, we solve the fair level of loan payment for the lump-sum option as $\omega = 15.0102$ and $\omega = 18.3458$ for $H_0 = 90$ and $H_0 = 110$,

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respectively, while, for the annuity loan, $c = 2.0108$ and $c = 2.4577$, respectively. We observe that a borrower who has more initial equity values can receive a larger amount of loan payment, resulting that as observed from Figure 4 the surrender boundary is pushed up because of the increased value of the contract. Since the boundary improves with a larger level of loan release in proportion to the initial equity value, we observe that the reference probability does not change for both types of RM loan. With the base level of loan payment, Figure 5 displays the surrender boundaries for varying levels of the rental income δ as an equity discount. Since the rental discount reduces the appraised value of the home used as collateral over time, the higher discount rate exhausts the nonnegative balance more quickly. We thus observe that the surrender boundary shifts upward with the resulting lower curve of the reference probability over time. In Figure 6, we present the surrender boundary for varying levels of house price volatility σ_H. We are aware that increases in house price volatility push up the surrender boundary with a reduced reference probability, since the higher level of house price volatility makes the guaranteee of nonnegative balance $Bal(t)$ more valuable. Therefore, a rational borrower would prefer to stay in the program when the housing market is volatile.

In Figures 7–9, we present the surrender boundary for varying levels of interest spread charges π_r and rates of initial and annual MIP, p_0 and p_a, respectively. Generally, an increase from these parameters accelerates the loan accumulation over time. Hence, the surrender boundary is pushed upward, since the barrier level is not lower than that of the loan accumulation. All other things being unchanged, this encourages the borrower from staying in the program. Thus, we are aware that the curve of the reference probability is pushed down. As discussed in Section 3.5, for business operations RM lenders may not want their borrowers to lapse the loan and thus a penalty charge could be imposed by them. In Figure 10, we propose a penalty percentage charge in the form of $\kappa_t = e^{\kappa t} - 1$, where the force of the surrender penalty κ is constant. Then, we show the sensitivity of the optimal surrender boundary to the proposed penalty charges for $\kappa = 0\%$, $\kappa = 0.25\%$ and $\kappa = 0.50\%$. When a borrower terminates the contract, the lender sells the home used as collateral to reclaim the loan amount of $e^{\kappa t}L_t$ with penalty charges. Then, the borrower receives the balance in the amount of $H_t - e^{\kappa t}L_t$. We find that the surrender boundary shifts upward, while the corresponding reference probability decreases with κ. These penalty charges increase the borrower's costs and that reduces the incentives to terminate the contract. Since it is impossible for borrowers to lapse the contract in the limiting year, the surrender boundaries drop to the level of terminal loan amount at maturity. By charging this prepayment penalty, RM programs can control the surrender boundary effectively throughout the lifetime of the contract, and then use these charges to cover program losses whenever a crossover event occurs.

5. Concluding remarks

The world economy faces an unprecedented challenge with many countries facing rapid aging. In the United States, approximately 80% of households over 62 own their homes, and home equity makes up about one-half of their median net worth (see, e.g., Poterba et al. 2010). Elderly retirees may be classified as "house rich, and cash poor." RMs are useful instruments to alleviate the continuous and steady consumption needs of retirees. Although the prospect of RMs seems promising, only a small fraction of retirees have participated in RM markets so far. Moreover, many lenders are unwilling to offer RM contracts without the HECM insurance. In terms of both the supply and the demand, the stress on the profitability of the program and the corresponding value of the contract appear to be important reasons for its limited marketability. With an efficient policy design, new sources of retirement funding might be unlocked via RM contracts.

Different from prior research, we have proposed a framework accounting for the rationality of the decision by a borrower to terminate the contract in terms of optimizing the value of the RM payouts. On top of that, we have derived an actuarial equivalence in conjunction with the surrender option for a fair identification of loan payment. Our model has demonstrated that the RM borrower's surrender behavior is a threshold strategy based on a variety of market conditions, such as interest rates, housing prices, and volatility. Furthermore, we have investigated both the intrinsic value and the time value of the embedded option guarantees in RM contracts. The results have shown that rational borrowers are more inclined to terminate the contract during favorable market conditions, such as a home appreciation as the trigger for their house-price-driven surrender strategy, low interest rates, and a low level of volatility of housing prices. In addition, borrowers are sensitive to the status of the contract guarantees, costs and loan payment schedule, and some policy features, such as the permission for initial withdrawals and rental income. Moreover, the combined effect of the mortality decrement and the absence of liquidity needs could limit the use of contract guarantees and thus make RM contracts less appealing to the borrower. To improve the solvency of the program, we have proposed a prepayment charge, which is similar to the one used in today's annuity markets. Such charges not only improve the borrower's surrender boundary but also they can be used to account for the costs from borrower's refinance or/and home sale, or to mitigate the insurer's hedging difficulty when a crossover event occurs due to the borrower's terminating the contract. Our approach can help the RM policymakers understand the triggering factors of their borrower surrender behavior and encourage a policy design to improve the financial incentive of the program.

Following the model assumptions adopted by Chiang and Tsai (2019) and Lee and Shi (2021), we have quantitatively investigated and assessed the financial incentives of a house-price-driven surrender by RM borrowers. While home appreciation is arguably the primary driver for early surrender (e.g., Shan, 2011 and Davidoff and Welke, 2017), future research can address how interest rate movement affects surrender strategies and how to determine the fair level of MIPs and loan payment in the presence of more realistic models for the dynamics of house price and interest rate. Beyond the reverse annuity and lump-sum mortgages presented in the article, it would be fruitful to further investigate the surrender strategy for other types of RM loans. Although our formulation of the rational surrender problem that assesses the surrender decision and the reference probability has led to valuable economic insights on policy development of RMs, it is important to acknowledge that the surrender decision is the outcome of the interplay between supply and demand sides of the market. Therefore, for a comprehensive design of RM programs, policymakers should also consider other determinants, such as borrower characteristics, including change in financial or family situation and financial literacy, refinancing options, and changes in the housing market.

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Conflicts of interest. All authors certify that they have no affiliations with or involvement in any organization or entity with any financial interest or non-financial interest in the subject matter or materials discussed in this manuscript.

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Appendix

Optimal Surrender Region for Reverse Mortgages

Motivated by Bernard et al. (2014a,b), we prove here that the optimal surrender strategy for a reverse mortgage is a threshold strategy. That is, we show that for any time t before maturity, there exists a level H_t^* above which the value of the contract is less than the surrender benefit available immediately. As discussed in Section 2.3, a rational borrower will only terminate the contract when $H_t \geqslant L_t$. For $h = H_t$,

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we express the time-t price (book value) of the RAM contract $V(h,t)$ by

$$V(h,t)=\psi(H(0,t),0,t;t)+\sup_{\tau\in\mathcal{T}_{[t,\tau_x]}}\mathbb{E}\left[e^{-r(\tau-t)}\left(\psi(H(t,\tau),t,\tau;\tau)+Bal(\tau)\right)|H_t=h\right].$$

Consider an optimal stopping time $\tau^*\in\mathcal{T}_{[t,\tau_x]}$ that has maximized the value of the contract over the period $[t,\tau_x]$. Due to the fact that the contract can only be maximized when the borrower still survives, the expression for $V(h,t)$ then becomes

$$\begin{aligned}V(h,t,\tau^*)&:=V(h,t)\\&=\psi(H(0,t),0,t;t)+\mathbb{E}\left[{}_{\tau^*-t}p_{x+t}\times e^{-r(\tau^*-t)}\left(\psi(H(t,\tau^*),t,\tau^*;\tau^*)+Bal(\tau^*)\right)|H_t=h\right]\end{aligned}\tag{A1}$$

and for $h\geqslant L_t$ the surrender payout is

$$\psi(H(0,t),0,t;t)+h-L_t,\tag{A2}$$

which will be triggered in the surrender region $\mathcal{D}(t)$ defined as

$$\mathcal{D}(t)=\{H_t=h:V(h,t,\tau^*)\leqslant\psi(H(0,t),0,t;t)+h-L_t\}.\tag{A3}$$

To compare the amounts in (A1) and (A2), we define a difference function

$$\begin{aligned}\lambda(h,t,\tau^*)&:=\psi(H(0,t),0,t;t)+h-L_t-V(h,t)\\&=h-L_t-\mathbb{E}\left[{}_{\tau^*-t}p_{x+t}\times e^{-r(\tau^*-t)}\left(\psi(H(t,\tau^*),t,\tau^*;\tau^*)+Bal(\tau^*)\right)|H_t=h\right],\end{aligned}\tag{A4}$$

for $\tau^*\in[t,\tau_x]$, and then the optimal surrender region can be rewritten as $\mathcal{D}(t)=\{h:\lambda(h,t,\tau^*)\geqslant 0\}$.

We also define a threshold $h^*:=\inf\{h:\lambda(h,t,\tau^*)\geqslant 0\}=\inf\{h:\lambda(h,t,\tau^*)=0\}$, where the latter representation follows from the continuity of $\lambda(h,t,\tau^*)$. In order to identify the threshold strategy for a surrender, we break the proof into the following two steps:

Step 1. Check the behavior of the difference function $\lambda(h,t,\tau^*)$ and then show the form of surrender region, where a rational borrower will surrender at all values above the time-dependent threshold $h^*=H^*_t$.

Step 2. Demonstrate that the threshold h^* in Step 1 does exist for each time t from the period $[0,\tau_x]$.

In Step 1, at a fixed future time $u\in(t,T]$, which is prior to the event of death, we rewrite the difference function as

$$\begin{aligned}\lambda(h,t,u)&:=h-L_t-\mathbb{E}\left[{}_{u-t}p_{x+t}\times e^{-r(u-t)}\left(\psi(H(t,u),t,u;u)+Bal(u)\right)|H_t=h\right]\\&=h-L_t-{}_{u-t}p_{x+t}\times\left[h\left(1-e^{-\delta(u-t)}\right)+\frac{c}{r}\left(1-e^{-r(u-t)}\right)+e^{-r(u-t)}\mathbb{E}_t\left[\left(he^{X_{u-t}}-L_u\right)^+\right]\right],\end{aligned}$$

where $X_{u-t}=(r-\delta-\frac{\sigma_H^2}{2})(u-t)+\sigma_H(W^H_u-W^H_t)$.

Therefore, for any $\epsilon>0$, the difference

$$\begin{aligned}&\lambda(h+\epsilon,t,u)-\lambda(h,t,u)\\&=h+\epsilon-{}_{u-t}p_{x+t}\times(h+\epsilon)\left(1-e^{-\delta(u-t)}\right)-{}_{u-t}p_{x+t}\times e^{-r(u-t)}\mathbb{E}_t\left[\left((h+\epsilon)e^{X_{u-t}}-L_u\right)^+\right]\\&-\left(h-{}_{u-t}p_{x+t}\times h\left(1-e^{-\delta(u-t)}\right)-{}_{u-t}p_{x+t}\times e^{-r(u-t)}\mathbb{E}_t\left[\left(he^{X_{u-t}}-L_u\right)^+\right]\right)\\&=\epsilon-{}_{u-t}p_{x+t}\times\epsilon\left(1-e^{-\delta(u-t)}\right)-{}_{u-t}p_{x+t}\times e^{-r(u-t)}\mathbb{E}_t\left[\left((h+\epsilon)e^{X_{u-t}}-L_u\right)^+-\left(he^{X_{u-t}}-L_u\right)^+\right]\\&\geqslant(1-{}_{u-t}p_{x+t})\,\epsilon+{}_{u-t}p_{x+t}\times\epsilon e^{-\delta(u-t)}-{}_{u-t}p_{x+t}\times e^{-r(u-t)}\mathbb{E}_t\left[\epsilon e^{X_{u-t}}\right]\end{aligned}\tag{A5}$$

$$=(1-{}_{u-t}p_{x+t})\,\epsilon+{}_{u-t}p_{x+t}\times\epsilon e^{-\delta(u-t)}-{}_{u-t}p_{x+t}\times\epsilon e^{-\delta(u-t)}\geqslant 0,\tag{A6}$$

where the inequality in the last second line of (A5) results from the fact that for $a>b\geqslant 0$ and $c>0$, $(a-c)^+-(b-c)^+\leqslant a-b$. The equality in (A6) is only achieved at the limiting year $u=T$. Since any $\tau\in\mathcal{T}_{[t,\tau_x]}$ takes values in $[t,\tau_x]$ with probability 1, the inequality holds almost surely for any τ. Then, we

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have that

$$\lambda(h+\epsilon, t, \tau) > \lambda(h, t, \tau), \tag{A7}$$

which indicates that the function $\lambda(h, t, \tau)$ is monotonically increasing with respect to h for any $\tau < T$. We exclude the equality in (A7) for $\tau = T$, since no one can survive and hence surrender at the limiting year T.

Taking the supremum over all stopping times τ on $\lambda(h, t, \tau)$, we get τ^* at which $\lambda(h+\epsilon, t, \tau^*) > \lambda(h, t, \tau^*)$ for all $t \in [0, \tau]$. This indicates that if we can find $h^* = H_t^*$ such that $\lambda(h^*, t, \tau^*) = 0$, then for any $h = H_t \geqslant H_t^*$ we have $\lambda(h, t, \tau^*) \geqslant \lambda(h^*, t, \tau^*) = 0$, that is, $\psi(H(0, t), 0, t; t) + h - L_t \geqslant V(h, t, \tau^*)$, and for $h = H_t < H_t^*$, we have $\lambda(h, t, \tau^*) < \lambda(h^*, t, \tau^*) = 0$, that is, $\psi(H(0, t), 0, t; t) + h - L_t < V(h, t, \tau^*)$. Thus, the optimal surrender region $\mathcal{D}$ has the form $[H_t^*, \infty)$.

In Step 2, we want to show that it is possible to find h^* such that $\lambda(h^*, t, \tau^*) = 0$, that is, $V(h^*, t, \tau^*) = \psi(H(0, t), 0, t; t) + h^* - L_t$. For a fixed time $u \in [t, \tau_x]$, the time-t contract value $V(h, t, u)$ has the following form

$$\begin{aligned} V(h, t, u) &= \psi(H(0, t), 0, t; t) + \mathbb{E}\left[{}_{u-t}p_{x+t} \times e^{-r(u-t)} \left(\psi(H(t, u), t, u; u) + Bal(u)\right) | H_t = h\right] \\ &= \psi(H(0, t), 0, t; t) + {}_{u-t}p_{x+t} \times \left[c \times \int_t^u e^{-r(s-t)} ds + \mathbb{E}\left[\int_t^u e^{-r(s-t)} \times \delta H_s ds \middle| H_t = h\right]\right] \\ &+ {}_{u-t}p_{x+t} \times \mathbb{E}\left[e^{-r(u-t)} \left(H_u - L_u\right)^+ \middle| H_t = h\right] \\ &= \psi(H(0, t), 0, t; t) + {}_{u-t}p_{x+t} \times h - {}_{u-t}p_{x+t} \times y(h), \end{aligned} \tag{A8}$$

where $d_1(h, u) := d_1(h, L_u, t, u)$ and $d_2(h, u) := d_2(h, L_u, t, u)$. For the European payoff at u, Equation (A8) solves the time-t contract value in analogy to the formula (3.9). In (A8), we define the following function:

$$y(h) := L_u e^{-r(u-t)} \Phi(d_2(h, u)) + h e^{-\delta(u-t)} \Phi(-d_1(h, u)) - \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right). \tag{A9}$$

From (A9), we find the limiting cases: $\lim_{h\to 0} y(h) = -\frac{c}{r} \times \left(1 - e^{-r(u-t)}\right)$ and $\lim_{h\to\infty} y(h) = L_u e^{-r(u-t)} - \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right)$. Then, for $h \in (0, \infty)$, we obtain the (sub) range for the continuous function y:

$$y(h) \in \left(-\frac{c}{r} \times \left(1 - e^{-r(u-t)}\right), L_u e^{-r(u-t)} - \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right)\right). \tag{A10}$$

At time u, the difference function $\lambda(h, t, u)$ is given by

$$\lambda(h, t, u) = \psi(H(0, t), 0, t; t) + h - L_t - V(h, t, u) = (1 - {}_{u-t}p_{x+t}) \times h - L_t + {}_{u-t}p_{x+t} \times y(h). \tag{A11}$$

Since the survival probability ${}_{u-t}p_{x+t}$ takes values in [0,1) for any $u \in [t, \tau_x)$, by (A10), we obtain the limiting cases: $\lim_{h\to 0} \lambda(h, t, u) = -L_t - {}_{u-t}p_{x+t} \times \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right) < 0$ and $\lim_{h\to\infty} \lambda(h, t, u) = (1 - {}_{u-t}p_{x+t}) \times h - L_t + L_u e^{-r(u-t)} - \frac{c}{r} \times \left(1 - e^{-r(u-t)}\right) = \infty$. These results imply that for the continuous function $\lambda(h, t, u)$, we can always find positive values h_1 and h_2, such that $\lambda(h_1, t, u) < 0$ and $\lambda(h_2, t, u) > 0$. By Intermediate Value Theorem, there exists a point $h^* \in (h_1, h_2)$ so that $\lambda(h^*, t, u) = 0$. We thus have demonstrated the existence of a threshold level h^* such that

$$V(h^*, t, u) = \psi(H(0, t), 0, t; t) + h^* - L_t. \tag{A12}$$

Since the function $\lambda(h, t, u)$ is monotonically increasing with respect to h, for $h > h^*$, we have

$$V(h, t, u) < \psi(H(0, t), 0, t; t) + h - L_t. \tag{A13}$$

We find that the results in (A12) and (A13) hold almost surely for any $\tau \in \mathcal{T}_{[t,\tau_x]}$. Thus, we have that $V(h, t, \tau) \leqslant \psi(H(0, t), 0, t; t) + h - L_t$ for $h \geqslant h^*$. Taking the supremum over all stopping times on both sides, we get $V(h, t, \tau^*) \leqslant \psi(H(0, t), 0, t; t) + h - L_t$, which completes the proof.

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Proof of Theorem 3.1

In this section, we apply the method used by Carr et al. (1992) to calculate the surrender option for RM guarantees. With more complex payout functions than the American put option guarantee discussed in Carr et al. (1992), we show that by surrender a borrower could benefit from saving on the ongoing premium and interest charges to be added on the RM loan.

By time $t \in [0, T]$, the book value of the contract, $V(H_t, t)$, can be obtained by the sum of the payout $\psi\ (H(0, t), 0, t; t)$ and the residual value $\widetilde{V}(H_t, t)$. To derive the decomposition for $\widetilde{V}(H_t, t)$ in (3.13), at time $u \in [t, T]$, the accumulated contract payout received by the borrower and her/his heirs from t to u, representing at u, is given by

$$\Psi(H_u, u) \ := \ \psi(H(t, u), t, u; u) + Bal(u) = \int_t^u e^{r(u-s)} \times \delta H_s ds + c \times \int_t^u e^{r(u-s)} ds + Bal(u),$$

where $Bal(u) = (H_u - L_u)^+$. $\psi(H(t, u), t, u; u)$ is defined in (3.3).

In the region $\mathcal{D} = \{(H, u) : H \in [0, \infty), u \in [t, T]\}$, define the discount payout function $Z_u := Z(H_u, u) = e^{-r(u-t)}\Psi(H_u, u)$. Similarly to the analysis of McKean (1965), the payout function Ψ and the surrender boundary B_u jointly solve a free-boundary problem subject to the following boundary conditions:

(C1) $\Psi(H_T, T) = \int_t^T e^{r(T-s)} \times \delta H_s ds + c \times \int_t^T e^{r(T-s)} ds + (H_T - L_T)^+$.
(C2) $\lim_{H\uparrow\infty} \Psi(H_u, u) = \int_t^u e^{r(u-s)} \times \delta H_s ds + c \times \int_t^u e^{r(u-s)} ds + \lim_{H\uparrow\infty} (H_u - L_u)^+ = \infty$.
(C3) $\lim_{H\downarrow B_u} \Psi(H_u, u) = \int_t^u e^{r(u-s)} \times \delta H_s ds + c \times \int_t^u e^{r(u-s)} ds + B_u - L_u$.
(C4) $\lim_{H\downarrow B_u} \frac{\partial\Psi(H_u,u)}{\partial H} = 1$.

Condition (C1) states that the contract payout is European at time T. Condition (C2) shows that the contract payout tends to be infinitely large as the house price approaches infinity. Condition (C3) implies that the payout function Ψ is continuous across the surrender boundary. Condition (C4) further implies that the first derivative for Ψ is continuous in H.

We should note that the derivatives, $\frac{\partial\Psi}{\partial H}$ and $\frac{\partial^2\Psi}{\partial H^2}$, are discontinuous at the boundary B_u. Following the result of Carr et al. (1992) or Myneni (1992), for a random death time τ_x, we can extend the Ito's lemma to the process $Z(H_u, u)$ so that

$$Z_{\tau_x} = Z_t + \int_t^{\tau_x} \left[\frac{\partial Z(H_u, u)}{\partial u} + \frac{\sigma_H^2 H_u^2}{2} \frac{\partial^2 Z(H_u, u)}{\partial H^2} \right] du + \int_t^{\tau_x} \frac{\partial Z(H_u, u)}{\partial H} dH_u, \tag{A14}$$

where the discount payout function $Z(H, u)$ is convex in H for all $u \in [t, \tau_x]$, continuously differentiable in H, and almost everywhere twice differentiable in H for all u.

Substituting $Z_u = e^{-r(u-t)}\Psi(H_u, u)$ into (A14), we have

$$Z_{\tau_x} = Z_t + \int_t^{\tau_x} \left[e^{-r(u-t)} \frac{\partial\Psi(H_u, u)}{\partial u} - re^{-r(u-t)}\Psi(H_u, u) + e^{-r(u-t)} \frac{\sigma_H^2 H_u^2}{2} \frac{\partial^2\Psi(H_u, u)}{\partial H^2} \right] du + \int_t^{\tau_x} e^{-r(u-t)} \frac{\partial\Psi(H_u, u)}{\partial H} dH_u. \tag{A15}$$

By separating the payout Ψ over two regions, we get

$$\Psi(H_u, u) = \mathbb{1}_{\{H_u<B_u\}}\Psi(H_u, u) + \mathbb{1}_{\{H_u\geqslant B_u\}}\widetilde{\Psi}(H_u, u), \tag{A16}$$

where

$$\widetilde{\Psi}(H_u, u) = \int_t^u e^{r(u-s)} \times \delta H_s ds + c \times \int_t^u e^{r(u-s)} ds + H_u - L_u, \tag{A17}$$

which is a function of the value H_u.

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Then, we obtain the following derivatives: $\frac{\partial\widetilde{\Psi}(H_u,u)}{\partial H}=1$, $\frac{\partial^2\widetilde{\Psi}(H_u,u)}{\partial H^2}=0$, and

$$\frac{\partial\widetilde{\Psi}(H_u,u)}{\partial u}=\delta H_u+c-\frac{\partial L_u}{\partial u}+r\left[\int_t^u e^{r(u-s)}\times\delta H_s ds+c\times\int_t^u e^{r(u-s)}ds\right].$$

Replacing $\Psi(H_u,u)$ with the expression for two regions in (A16), Equation (A15) can be rewritten as

$$\begin{aligned}
Z_{\tau_x}&=Z_t+\int_t^{\tau_x}e^{-r(u-t)}\Bigg[\mathbb{1}_{\{H_u<B_u\}}\times\left(\frac{\partial\Psi(H_u,u)}{\partial u}-r\Psi(H_u,u)+\frac{\sigma_H^2H_u^2}{2}\frac{\partial^2\Psi(H_u,u)}{\partial H^2}\right)\\
&+\mathbb{1}_{\{H_u\geq B_u\}}\times\left(\frac{\partial\widetilde{\Psi}(H_u,u)}{\partial u}-r\widetilde{\Psi}(H_u,u)\right)\Bigg]du+\int_t^{\tau_x}e^{-r(u-t)}\Bigg[\mathbb{1}_{\{H_u<B_u\}}\\
&\times\frac{\partial\Psi(H_u,u)}{\partial H}+\mathbb{1}_{\{H_u\geq B_u\}}\times 1\Bigg]\times\left[(r-\delta)H_udu+\sigma_HH_udW_u^H\right]\\
&=Z_t+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u<B_u\}}\times\left(\frac{\partial\Psi(H_u,u)}{\partial u}+\frac{\sigma_H^2H_u^2}{2}\frac{\partial^2\Psi(H_u,u)}{\partial H^2}+(r-\delta)H_u\frac{\partial\Psi(H_u,u)}{\partial H}-r\Psi(H_u,u)\right)du\\
&+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u<B_u\}}\times\sigma_HH_u\times\frac{\partial\Psi(H_u,u)}{\partial H}dW_u^H\\
&+\int_t^{\tau_x}e^{-r(u-t)}\mathbb{1}_{\{H_u\geq B_u\}}\left[(r-\delta)H_u+\frac{\partial\widetilde{\Psi}(H_u,u)}{\partial u}-r\widetilde{\Psi}(H_u,u)\right]du+\int_t^{\tau_x}e^{-r(u-t)}\mathbb{1}_{\{H_u\geq B_u\}}\sigma_HH_udW_u^H.
\end{aligned}\tag{A18}$$

In the continuation region (i.e., $H_u<B_u$), the payout function $\Psi(H_u,u)$ satisfies the Black-Scholes partial differential equation

$$\frac{\partial\Psi(H_u,u)}{\partial u}+\frac{\sigma_H^2H_u^2}{2}\frac{\partial^2\Psi(H_u,u)}{\partial H^2}+(r-\delta)H_u\frac{\partial\Psi(H_u,u)}{\partial H}-r\Psi(H_u,u)=0.$$

Consequently, the terms multiplying $\mathbb{1}_{\{H_u<B_u\}}$ in (A18) sum to zero. Then, Equation (A18) becomes

$$\begin{aligned}
Z_{\tau_x}&=Z_t+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u\geqslant B_u\}}\times\left[(r-\delta)H_u+\frac{\partial\widetilde{\Psi}(H_u,u)}{\partial u}-r\widetilde{\Psi}(H_u,u)\right]du\\
&+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u<B_u\}}\times\sigma_HH_u\times\frac{\partial\Psi(H_u,u)}{\partial H}dW_u^H+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u\geqslant B_u\}}\times\sigma_HH_udW_u^H\\
&=Z_t+\int_t^{\tau_x}e^{-r(u-t)}\mathbb{1}_{\{H_u\geqslant B_u\}}\left[rL_u+c-\frac{\partial L_u}{\partial u}\right]du+\int_t^{\tau_x}e^{-r(u-t)}\mathbb{1}_{\{H_u<B_u\}}\sigma_HH_u\frac{\partial\Psi(H_u,u)}{\partial H}dW_u^H\\
&+\int_t^{\tau_x}e^{-r(u-t)}\times\mathbb{1}_{\{H_u\geqslant B_u\}}\times\sigma_HH_udW_u^H.
\end{aligned}\tag{A19}$$

Taking expectations with respect to the martingale measure $\mathbb{Q}$ establishes the result

$$\begin{aligned}
\widetilde{V}(H_t,t)&=v(t)+\mathbb{E}_t\left[\int_t^{\tau_x}e^{-r(u-t)}\left[\frac{\partial L_u}{\partial u}-rL_u-c\right]\times\Phi\left(d_2(H_t,B_u,t,u)\right)du\right]\\
&=v(t)+\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}\int_t^s e^{-r(u-t)}\times\eta(u)\times\Phi\left(d_2(H_t,B_u,t,u)\right)duds\\
&=v(H_t,t)+e(H_t,t),
\end{aligned}\tag{A20}$$

where $\frac{\partial L_u}{\partial u}=c+(r+\pi_r+p_a)L_u$ and $\eta(u)=(\pi_r+p_a)L_u$. Note that $\widetilde{V}(H_t,t)=\mathbb{E}_t[Z_t]$, $v(t):=v(H_t,t)=\mathbb{E}_t\left[Z_{\tau_x}\right]$ and $e(H_t,t)$ are defined in (3.13), (3.9) and (3.11), respectively. The function $\eta(u)$ describes the borrower's benefit/saving from without paying MIPs and interests immediately upon her/his surrender. For any $s\in[t,T]$, we consider a continuous and differentiable function $A(s)$ with $A(t)=0$. Knowing that ${}_{T-t}p_{x+t}=0$ for the limiting year T, the third equation in (A20) holds due to the fact that $\int_t^T {}_{s-t}p_{x+t}\mu_{x+s}A(s)ds=\int_t^T {}_{s-t}p_{x+t}A'(s)ds$.

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Derivation of Pricing Condition with Rational Surrender

In this section, we derive the pricing condition for finding the level of annuity payment c (or equivalently, the initial withdrawal ω for a lump-sum RM). On top of the initial payment of MIP, p_0H_0, the insurer charges an ongoing annual MIP until the time of borrower's surrender or her/his death. To measure the solvency of its premium collection over the cost of insurance, we thus define a proxy loss function between the accumulated annual MIPs and the crossover loss occurring at any future time $u \in [0, \tau_x]$, that is

$$\Psi_0^l\left(H_u, u\right) = Loss(u) - \int_0^u e^{r(u-s)} p_a L_s ds = (L_u - H_u)^+ - \int_0^u e^{r(u-s)} p_a L_s ds. \tag{A21}$$

At loan origination, an RM lender/insurer needs to identify a fair level of annuity payment c with a given initial withdrawal ω for a tenure payment option, or the level of ω with $c = 0$ for a lump-sum reverse mortgage. For the discounted premium loss $\mathcal{L}(\tau^*)$ at $\tau^* \in \mathcal{T}_{[0,\tau_x]}$, we apply the assumption of zero expected loss-at-issue to determine the fair level of loan payment with the following equivalence in conjunction with the borrower's surrender option

$$0 = \mathbb{E}\left[\mathcal{L}(\tau^*)\right] := \mathbb{E}\left[e^{-r\tau^*}\Psi_0^l\left(H_{\tau^*}, \tau^*\right)\right] - p_0H_0, \tag{A22}$$

where τ^* refers to the time of borrower's optimal surrender, which maximizes the contract value and has been identified from (3.15) when the house price reaches the barrier value B_t, for $t \geqslant 0$.

Similarly to the proof of Theorem 3.1, we next derive the price of the proxy loss Ψ_0^l in (A22) by separating its payout over two regions, that is

$$\Psi_0^l\left(H_u, u\right) = \mathbb{1}_{\{H_u < B_u\}}\Psi_0^l(H_u, u) + \mathbb{1}_{\{H_u \geqslant B_u\}}\widetilde{\Psi}_0^l(H_u, u), \tag{A23}$$

where, for the event of $\{H_u \geqslant B_u \geqslant L_u\}$ at which there is no crossover loss, the proxy loss at surrender can be simplified as

$$\widetilde{\Psi}_0^l(H_u, u) = -\int_0^u e^{r(u-s)} p_a L_s ds, \text{ with } \frac{\partial \widetilde{\Psi}_0^l(H_u,u)}{\partial H} = \frac{\partial^2 \widetilde{\Psi}_0^l(H_u,u)}{\partial H^2} = 0.$$

Similarly to the derivation in (A18), the discounted cash payouts in (A23) can be represented by

$$\begin{aligned}
Z_{\tau_x}^l &= Z_0^l + \int_0^{\tau_x} e^{-ru}\Bigg[\mathbb{1}_{\{H_u<B_u\}} \times \left(\frac{\partial \Psi_0^l(H_u,u)}{\partial u} - r\Psi_0^l(H_u,u) + \frac{\sigma_H^2 H_u^2}{2}\frac{\partial^2\Psi_0^l(H_u,u)}{\partial H^2}\right)\\
&+\mathbb{1}_{\{H_u\geq B_u\}} \times \left(\frac{\partial \widetilde{\Psi}_0^l(H_u,u)}{\partial u} - r\widetilde{\Psi}_0^l(H_u,u) + \frac{\sigma_H^2 H_u^2}{2}\frac{\partial^2\widetilde{\Psi}_0^l(H_u,u)}{\partial H^2}\right)\Bigg]du + \int_0^{\tau_x} e^{-ru}\Bigg[\mathbb{1}_{\{H_u<B_u\}}\\
&\times \frac{\partial \Psi_0^l(H_u,u)}{\partial H} + \mathbb{1}_{\{H_u\geq B_u\}} \times \frac{\partial \widetilde{\Psi}_0^l(H_u,u)}{\partial H}\Bigg] \times \left[(r-\delta)H_u du + \sigma_H H_u dW_u^H\right]\\
&= Z_0^l + \int_0^{\tau_x} e^{-ru}\mathbb{1}_{\{H_u\geq B_u\}}\left[\frac{\partial \widetilde{\Psi}_0^l(H_u,u)}{\partial u} - r\widetilde{\Psi}_0^l(H_u,u)\right]du\\
&= Z_0^l + \int_0^{\tau_x} e^{-ru}\mathbb{1}_{\{H_u\geq B_u\}}\left[-p_aL_u + r\int_0^u e^{r(u-t)}p_aL_t dt\right]du.
\end{aligned} \tag{A24}$$

Taking expectations on both sides of Equation (A24) under the pricing measure $\mathbb{Q}$, one obtains

$$\begin{aligned}
\widetilde{V}^l(H_0,0) &= v^l(H_0,0) + \mathbb{E}\left[\int_0^{\tau_x} e^{-ru}\mathbb{1}_{\{H_u\geqslant B_u\}}\left[p_aL_u - r\int_0^u e^{r(u-t)}p_aL_t dt\right]du\right]\\
&= v^l(H_0,0) + \int_0^T {}_sp_x\mu_{x+s}\left[\int_0^s\left[p_aL_ue^{-ru} - \int_0^u rp_aL_te^{-rt}dt\right] \times \Phi\left(d_2(H_0,B_u,0,u)\right)du\right]ds\\
&:= v^l(H_0,0) + e^l\left(H_0,0\right),
\end{aligned} \tag{A25}$$

where the initial value of Ψ_0^l can be computed as $\widetilde{V}^l(H_0,0) = \mathbb{E}\left[Z_0^l\right] = \mathbb{E}\left[e^{-r\tau^*}\Psi_0^l\left(H_{\tau^*},\tau^*\right)\right]$, and $v^l(H_0,0) = \mathbb{E}\left[Z_{\tau_x}^l\right]$ are defined in (3.18) and (3.19), respectively. The relation (A25) completes the proof of Theorem 3.2.

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Derivation of Surrender Boundary with Prepayment Percentage Charge

For $t \in [0, T]$, we propose a penalty percentage rate in the form of $\kappa_t = e^{\kappa t} - 1$, with a constant force of surrender penalty $\kappa > 0$. Note that the prepayment penalty κ_t on the loan accumulation is increasing in t, since the costs of crossover event can be exponentially larger by time. Thus, if a borrower terminates at time $t \in [0, T]$, the total loan amount with penalty charges is given by

$$L_t^\kappa := (1+\kappa_t)L_t = e^{\kappa t} L_t.$$

By the loan expression in (2.1), for $t \in [0, T]$, we have $\frac{dL_t^\kappa}{dt} = e^{\kappa t}\left[\frac{dL_t}{dt} + \kappa L_t\right]$. We should emphasize that the prepayment charges will only be applied against rational lapse of the contract, which is controlled by borrower's decision rather than her/his termination of death. In the surrender region, we thus rewrite the contract payout with inclusion of surrender penalty as follows

$$\widetilde{\Psi}(H_u, u) = \int_t^u e^{r(u-s)} \times \delta H_s ds + c \times \int_t^u e^{r(u-s)} ds + H_u - L_u^\kappa, \tag{A26}$$

while in the continuation region, there is no surrender charge and the corresponding contract payout $\Psi(H_u, u)$ is given in (A14).

Following the proof of Theorem 3.1 and the derivation in (A19), the time-t price of an RM with surrender penalty charges can be represented in the following way

$$\begin{aligned}
\widetilde{V}(H_t, t) &= v(t) + \int_t^T {}_{s-t}p_{x+t}\mu_{x+s} \int_t^s e^{-r(u-t)} \left[\frac{\partial L_u^\kappa}{\partial u} - rL_u^\kappa - c\right] \times \Phi\left(d_2(H_t, B_u, t, u)\right) duds \\
&= v(t) + \int_t^T {}_{s-t}p_{x+t}\mu_{x+s} \int_t^s e^{-r(u-t)} \times \eta_\kappa(u) \times \Phi\left(d_2(H_t, B_u, t, u)\right) duds \\
&= v(t) + \int_t^T {}_{u-t}p_{x+t} \times e^{-r(u-t)} \times \eta_\kappa(u) \times \Phi\left(d_2(H_t, B_u, t, u)\right) du,
\end{aligned}$$

where $\eta_\kappa(u) = (\pi_r + p_a + \kappa)\, e^{\kappa u} L_u + c \times (e^{\kappa u} - 1)$.

With the surrender penalty charge κ, we can modify $g_2(u, t_{n-k})$ in (3.22) as

$$g_{2,\kappa}(u, t_{n-k}) \;:=\; e^{-r(u-t_{n-k})} \times \eta_\kappa(u) \times \Phi\left(d_2(B_{t_{n-k}}, B_u, t_{n-k}, u)\right).$$

And hence, the integral function $I_2(k)$ in (3.24) can be rewritten as

$$I_{2,\kappa}(k) := \frac{T}{n}\sum_{i=0}^{k-1} {}_{t_{n-k+i+1}-t_{n-k}}p_{x+t_{n-k}} \times g_{2,\kappa}(t_{n-k+i+1}, t_{n-k}), \text{ for } i \leqslant k-1 \text{ and } k = 1, \cdots, n.$$

In (3.25), we replace $I_2(k)$ by $I_{2,\kappa}(k)$. We thus determine the surrender boundary with penalty charge κ. This completes the derivation.

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