J. Aust. Math. Soc. 77 (2004), 1–16

ISOMETRIES BETWEEN MATRIX ALGEBRAS

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(Received 2 October 2001; revised 21 March 2003)

Communicated by G. Willis

Abstract

As an attempt to understand linear isometries between C^*-algebras without the surjectivity assumption, we study linear isometries between matrix algebras. Denote by M_m the algebra of $m \times m$ complex matrices. If $k \geq n$ and $\phi : M_n \to M_k$ has the form $X \mapsto U[X \oplus f(X)]V$ or $X \mapsto U[X^t \oplus f(X)]V$ for some unitary $U, V \in M_k$ and contractive linear map $f : M_n \to M_k$, then $\|\phi(X)\| = \|X\|$ for all $X \in M_n$. We prove that the converse is true if $k \leq 2n - 1$, and the converse may fail if $k \geq 2n$. Related results and questions involving positive linear maps and the numerical range are discussed.

2000 *Mathematics subject classification*: primary 15A04, 15A60.
Keywords and phrases: isometry, matrices, linear maps.

1. Introduction

In [6], Kadison characterized surjective linear isometries on C^*-algebras. The problem without surjectivity seems very difficult even in the finite dimensional case. In this paper, we study linear isometries from M_n to M_k, that is, linear maps $\phi : M_n \to M_k$ such that $\|\phi(A)\| = \|A\|$ for all $A \in M_n$, where M_m is the algebra of $m \times m$ complex matrices and $\|\cdot\|$ is the spectral norm. Clearly, if such a linear isometry ϕ exists, then $k \geq n$. If $k = n$, it follows from the result of Kadison [6] that ϕ has the form $X \mapsto UXV$ or $X \mapsto UX^tV$, for some unitary $U, V \in M_n$. One can modify the above maps to norm preserving linear maps $\phi : M_n \to M_k$ with $k > n$, namely, if $U, V \in M_k$ are unitary and $f : M_n \to M_{k-n}$ is a contractive linear map, then $\phi : M_n \to M_k$ defined by

$$X \mapsto U[X \oplus f(X)]V \quad \text{or} \quad X \mapsto U[X^t \oplus f(X)]V$$

https://doi.org/10.1017/S1446788700010119 Published online by Cambridge University Press

is a linear isometry. It is natural to ask whether the converse of this statement holds. We have the following result.

THEOREM 1.1. *Suppose* $k \le 2n-1$, *and* $\phi : M_n \to M_k$ *is linear such that* $\|\phi(X)\| = \|X\|$ *for all* $X \in M_n$. *Then* $k \ge n$, *and there exist* $U, V \in M_k$ *and a contractive linear map* $f : M_n \to M_{k-n}$ *such that* ϕ *has the form*

$$X \mapsto U[X \oplus f(X)]V \quad or \quad X \mapsto U[X^t \oplus f(X)]V. \tag{1.1}$$

Moreover, if $k \ge 2n \ge 4$, *then there exists a norm preserving linear map* $\psi : M_n \to M_k$ *that is not of the form* (1.1).

Recall that $B \in M_n$ is *essentially Hermitian* if $B = aA + bI$ for some Hermitian A and $a, b \in \mathbb{C}$, equivalently, B is normal and its eigenvalues lie on a straight line. It turns out that Theorem 1.1 can be deduced from the following result concerning unital linear maps $\phi : M_n \to M_k$ that preserve the norm of essentially Hermitian matrices.

THEOREM 1.2. *Suppose* $k \le 2n-2$, *and* $\phi : M_n \to M_k$ *is a linear map. Then* ϕ *satisfies* $\phi(I_n) = I_k$ *and* $\|\phi(X)\| = \|X\|$ *for all essentially Hermitian matrices* $X \in M_n$ *if and only if* $k \ge n$, *and there exist a unitary* $U \in M_k$ *and a unital positive linear map* $f : M_n \to M_{k-n}$ *such that* ϕ *has the form*

$$X \mapsto U[X \oplus f(X)]U^* \quad or \quad X \mapsto U[X^t \oplus f(X)]U^*. \tag{1.2}$$

Moreover, if $k \ge 2n-1 \ge 3$, *then there exists a linear map* $\psi : M_n \to M_k$ *which is not of the form* (1.2) *but satisfies* $\psi(I_n) = I_k$ *and* $\|\psi(X)\| = \|X\|$ *for all essentially Hermitian matrices* $X \in M_n$.

We prove some auxiliary results in the next section, and give the proofs of Theorems 1.1 and 1.2 in Section 3. Some related results and questions are discussed in the last section.

In our discussion, we let $\{e_1, \dots, e_n\}$ be the standard basis for $\mathbb{C}^n$, and $E_{ij} = e_i e_j^t$ be the standard matrix unit. Denote by $\mathcal{H}_n$ the real linear space of $n \times n$ Hermitian matrices, and $\lambda_1(A) \ge \cdots \ge \lambda_n(A)$ the eigenvalues of $A \in \mathcal{H}_n$; we write $A > 0$ if $\lambda_n(A) > 0$ and $A \ge 0$ if $\lambda_n(A) \ge 0$.

2. Auxiliary results

THEOREM 2.1. *Suppose* $\phi : M_n \to M_k$ *satisfies* $\|\phi(A)\| \le \|A\|$ *for all essentially Hermitian* $A \in M_n$ *and* $U^*\phi(I_n)V = I_p \oplus D$, *where* $U, V \in M_k$ *are unitary and* $D \in M_{k-p}$ *is a diagonal matrix with diagonal entries in the interval* $[0, 1)$. *Use the* *first p columns of U (respectively V) to form the matrix U_1 (respectively V_1). Then the mapping $\psi : M_n \to M_p$ defined by $\psi(X) = U_1^*\phi(X)V_1$ satisfies the following conditions*:

(1) $\psi(I_n) = I_p$.
(2) $\|\psi(A)\| \le \|A\|$ *for all essentially Hermitian* $A \in M_n$.
(3) $\psi(A) \ge 0$ *for all* $A \ge 0$.
(4) $\psi(A^*) = \psi(A)^*$ *for all* $A \in M_n$.

If, in addition, $\|\phi(A)\| = \|A\|$ *for all* $A \in \mathscr{H}_n$, *then*

(5) $\|\psi(B)\| = \|B\|$ *for all essentially Hermitian* $B \in M_n$.
(6) $\psi(A) \ge 0$ *if and only if* $A \ge 0$.
(7) *For every* $A \in \mathscr{H}_n$, $\lambda_1(A) = \lambda_1(\psi(A))$ *and* $\lambda_n(A) = \lambda_p(\psi(A))$.

PROOF. Conditions (1) and (2) follow from the definition.

For (3), suppose $A \ge 0$ and $\psi(A) = B + iC$, B, $C \in \mathscr{H}_p$. For any unit vector $x \in \mathbb{C}^p$, let $b = x^*Bx$ and $c = x^*Cx$. We are going to prove that $c = 0$. It will then follow that $C = 0$. To prove our claim, for each positive integer m, let $A_m = A - bI_n + i(mcI_n)$. Then

$$\begin{aligned}\|A - bI_n\|^2 + m^2c^2 &\ge \|(A - bI_n)^2 + m^2c^2I_n\| = \|A_mA_m^*\| = \|A_m\|^2 \\ &\ge \|\psi(A_m)\|^2 \ge |x^*\psi(A_m)x|^2 \\ &= |x^*(B - bI_n + i(mcI_n + C))x|^2 \\ &= |x^*Bx - b + i(mc + x^*Cx)|^2 = |(m+1)c|^2.\end{aligned}$$

Hence, $c = 0$ as asserted. So, $\psi(A) = B$. If $s > 0$ is small, then

$$\|I_p - sB\| = \|\psi(I_n - sA)\| \le \|I_n - sA\| \le 1.$$

Therefore, $B \ge 0$.

Condition (4) follows readily from (3).

Now, suppose that $\|\phi(A)\| = \|A\|$ for all $A \in \mathscr{H}_n$. Let $B \in M_n$ be essentially Hermitian, that is, $B = aA + bI$ for some $A \in \mathscr{H}_n$ and $a, b \in \mathbb{C}$. We are going to show that $\|\psi(B)\| = \|B\|$. The claim clearly holds if $a = 0$. So, without loss of generality, we assume that $a = 1$.

First, consider the case when $b = 0$. Hence, $B = A \in \mathscr{H}_n$. We may further assume that $\|A\| = \lambda_1(A)$; otherwise, replace A by $-A$. Then

$$\begin{aligned}1 + \lambda_1(A) = \|I_n + A\| &= \|\phi(I_n + A)\| = \|U^*\phi(I_n + A)V\| \\ &= \|(I_p \oplus D) + U^*\phi(A)V\|.\end{aligned}$$

So there exist unit vectors x and y in $\mathbb{C}^k$ such that

$$\begin{aligned}1+\lambda_1(A) &= y^*[(I_p \oplus D) + U^*\phi(A)V]x \\ &\leq |y^*(I_p \oplus D)x| + |y^*U^*\phi(A)Vx| \leq 1+\lambda_1(A).\end{aligned}$$

Therefore, $y = x = (I_p \oplus D)x$ and $U^*\phi(A)Vx = \lambda_1(A)x$. Hence, $x = \left[\begin{smallmatrix} x_1 \\ 0 \end{smallmatrix}\right]$, where $x_1 \in \mathbb{C}^p$ and $\psi(A)x_1 = \lambda_1(A)x_1$. As a result, $\|\psi(A)\| = \|A\|$.

For the general case, suppose $B = A + (a+ib)I_n$, where $A \in \mathscr{H}_n$ and $a, b \in \mathbb{R}$. Then

$$\begin{aligned}\|\psi(B)\|^2 &= \|\psi(A + aI_n) + ibI_p\|^2 = \|\psi(A+aI_n)\|^2 + |b|^2 \\ &= \|A + aI_n\|^2 + |b|^2 = \|(A+aI_n) + ibI_n\|^2 = \|B\|^2.\end{aligned}$$

This proves (5).

For (6), let $A \in M_n$ such that $\psi(A) \geq 0$. Let $A = B + iC$ where $B, C \in \mathscr{H}_n$. Then by (4), we have $\psi(B - iC) = \psi(A^*) = \psi(A)^* = \psi(A)$. Hence, $\psi(C) = 0$ implies $C = 0$, that is, $A \in \mathscr{H}_n$. For every $t > \|\psi(A)\|$, we have $t \geq \|tI_p - \psi(A)\| = \|tI_n - A\|$. Therefore, $A \geq 0$.

For (7), let $A \in \mathscr{H}_n$ and $t \in \mathbb{R}$. By (6), we have

$$t \geq \lambda_1(A) \iff tI_n - A \geq 0 \iff tI_p - \psi(A) \geq 0 \iff t \geq \lambda_1(\psi(A)).$$

Therefore, $\lambda_1(A) = \lambda_1(\psi(A))$. Similarly, $\lambda_n(A) = \lambda_p(\psi(A))$. □

REMARK 2.2. Note that one cannot weaken the hypothesis in Theorem 2.1 to $\|\phi(A)\| \leq \|A\|$ for all Hermitian $A \in M_n$. For example, suppose $\phi : M_2 \to M_3$ is given by

$$\phi(A) = A \oplus [(a+d+i(a-d))/2] \quad \text{if} \quad A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then $\phi(I_2) = I_3$ and $\|\phi(A)\| = \|A\|$ for all $A \in \mathscr{H}_2$. However, if $A = [1] \oplus 0 \in \mathscr{H}_2$, then $\phi(A) = A \oplus [(1+i)/2] \notin \mathscr{H}_3$ and

$$\|\phi(2A + 2iI_2)\| = \|(2A + 2iI_2) \oplus [1+3i]\| = \sqrt{10} > \sqrt{8} = \|2A + 2iI_2\|.$$

In fact, none of the conditions (2)–(7) holds.

Note also that the only place where we use the condition $\|\phi(A)\| \leq \|A\|$ for all essentially Hermitian $A \in M_n$ is in showing that $\psi(A) \in \mathscr{H}_k$ for all $A \in \mathscr{H}_n$. Hence, the proof of Theorem 2.1 also gives the equivalence of (a)–(c) in the following theorem.

https://doi.org/10.1017/S1446788700010119 Published online by Cambridge University Press

THEOREM 2.3. *Suppose $k \le 2n-2$, and $\phi : \mathscr{H}_n \to \mathscr{H}_k$ is a linear map satisfying $\phi(I_n) = I_k$. The following conditions are equivalent.*

(a) $\|\phi(X)\| = \|X\|$ *for all* $X \in \mathscr{H}_n$.

(b) $A \in \mathscr{H}_n$ *is positive semidefinite if and only if* $\phi(A)$ *is positive semidefinite.*

(c) *For every* $A \in \mathscr{H}_n$, $\lambda_1(A) = \lambda_1(\phi(A))$ *and* $\lambda_n(A) = \lambda_k(\phi(A))$.

(d) *We have $k \ge n$, and there exist a unitary $U \in M_k$ and a unital positive linear map $f : \mathscr{H}_n \to \mathscr{H}_{k-n}$ such that ϕ has the form*

$$X \mapsto U[X \oplus f(X)]U^* \quad \text{or} \quad X \mapsto U[X^t \oplus f(X)]U^*.$$

PROOF. By the discussion before the theorem, we see that (a), (b), (c) are equivalent. It is clear that (d) implies all the conditions (a)–(c). In the following, we assume that one, and hence all, of the conditions (a)–(c) holds, and prove condition (d) by induction on $n \ge 2$. By (a), we have $k \ge n$.

Suppose $n = k$. If $X_1 \in \mathscr{H}_n$ is a rank one orthogonal projection, then there exist rank one orthogonal projections $X_2, \dots, X_n$ such that $\sum_{j=1}^n X_i = I_n$. By condition (c), $\phi(X_j)$ is positive semi-definite with largest eigenvalue equal to one for $j = 1, \dots, n$. Moreover,

$$\sum_{j=1} \operatorname{tr} \phi(X_j) = \operatorname{tr} \phi\left(\sum_{j=1}^n X_j\right) = \operatorname{tr} I_n = n.$$

Thus, $\phi(X_j)$ has eigenvalues $1, 0, \dots, 0$, that is, $\phi(X_j)$ is a rank one orthogonal projection, for $j = 1, \dots, n$. Hence, ϕ maps rank one orthogonal projections to rank one orthogonal projections. By [3, Theorem 3], we conclude that there exists a unitary $S \in M_n$ such that ϕ has the form

$$X \mapsto SXS^* \quad \text{or} \quad X \mapsto SX^tS^*. \tag{2.1}$$

Thus, condition (d) holds if $n = k$. Note that if $n = 2$, then $n \le k \le 2n-2$ implies that $n = k = 2$. So, condition (d) holds. Now, suppose $n \ge 3$ and $n < k \le 2n-2$, and the result is true for linear maps from $\mathscr{H}_r$ to $\mathscr{H}_s$ for any $r < n$ and $s \le 2r-2$. We shall establish the following.

CLAIM. *There exist unitary matrices $V \in M_n$ and $U \in M_k$ such that the mapping*

$$A \mapsto U\phi(VAV^*)U^* \tag{2.2}$$

has the form

$$X \mapsto g(X) \oplus \tilde{f}(X), \tag{2.3}$$

where $\tilde{f} : \mathscr{H}_n \to \mathscr{H}_{k-n}$ is a unital positive linear map, and $g : \mathscr{H}_n \to \mathscr{H}_n$ is unital, linear, and maps rank one orthogonal projections to rank one orthogonal projections.

Once the claim is proved, we can apply [3, Theorem 3] to g and conclude that g has the form (2.1) for some unitary $S \in M_n$. Consequently, the original map ϕ will satisfy condition (d). □

Note that we only need to show that there exist unitary matrices U and V such that the mapping in (2.2) is a direct sum of two linear maps in the form (2.3). It will then follow (say, from (b)) that $\tilde{f}$ is a unital positive linear map as asserted.

We establish several assertions to prove our claim.

ASSERTION 1. *For each $j \in \{1, \dots, n\}$, $\phi(E_{jj})$ has largest and smallest eigenvalues equal to* 1 *and* 0, *respectively. Moreover, if $v \in \mathbb{C}^k$ is a unit vector such that $v^*\phi(E_{jj})v = 1$, then $v^*\phi(X)v = 0$ for any $X \in \mathcal{H}_n$ with (j, j) entry equal to* 0.

PROOF. The first statement follows from (c). To prove the second statement, we may assume that $j = 1$. Suppose $v \in \mathbb{C}^n$ is a unit vector such that $v^*\phi(E_{11})v = 1$. If $Y = [0] \oplus Y_1$ with $Y_1 \in \mathcal{H}_{n-1}$, then for any $t \in [-1, 1]$,

$$v^*[\phi(E_{11}) + t\phi(Y)]v \le \|E_{11} + tY\| = 1.$$

Thus, $v^*\phi(Y)v = 0$. If $Z = e_1z^* + ze_1^*$ for some unit vector $z \in \operatorname{span}\{e_2, \dots, e_n\}$, then there exists a unitary matrix $U = [1] \oplus U_1$ with $U_1 \in M_{k-1}$ such that $UZU^* = E_{12} + E_{21}$. Therefore, for every $t \in [-1, 1]$,

$$\begin{aligned} v^*[\phi(E_{11}) + t\phi(Z)]v &\le \|E_{11} + tZ\| = \|U(E_{11} + tZ)U^*\| \\ &= \|E_{11} + t(E_{12} + E_{21})\| \le \sqrt{1 + 2t^2}. \end{aligned}$$

Again, we have $v^*\phi(Z)v = 0$. Consequently, if X is any (real) linear combination of two matrices Y and Z of the above form, we have $v^*\phi(X)v = 0$. □

ASSERTION 2. *There exists a rank one orthogonal projection X such that $\phi(X)$ is unitarily similar to $[1] \oplus O_q \oplus D_1$, where $q + 1 < k$ and D_1 is a diagonal matrix with diagonal entries in the interval* (0, 1).

PROOF. By Assertion 1, each $\phi(E_{jj})$ has largest and smallest eigenvalues equal to 1 and 0, respectively. Since $n < k \le 2n - 2$ and

$$k = \operatorname{tr} I_k = \operatorname{tr} \phi(I_n) = \sum_{j=1}^{n} \operatorname{tr} \phi(E_{jj}), \tag{2.4}$$

we see that there exist at least two matrices $\phi(E_{jj})$ with exactly one eigenvalue equal to 1. If one of these matrices, say, $\phi(E_{jj})$, is not an orthogonal projection in $\mathcal{H}_k$, then E_{jj} is a desired matrix X.

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Suppose each matrix $\phi(E_{jj})$ with one eigenvalue equal to 1 is an orthogonal projection, and $\phi(E_{11})$ is one of them. Since $n < k$, by (2.4) again there exists $\phi(E_{pp})$ with at least two eigenvalues equal to 1. Without loss of generality, we may assume that $p = 2$. By Assertion 1, there exists a unitary $U \in M_k$ such that

$$\phi(E_{11}) = U(E_{11} \oplus O_{k-n})U^* \quad \text{and} \quad \phi(E_{22}) = U([0] \oplus I_r \oplus C_2)U^*$$

so that $r > 1$ and $\|C_2\| < 1$. For simplicity, assume that $U = I_k$; otherwise, replace ϕ by the mapping $X \mapsto U^*\phi(X)U$. So,

$$\phi(E_{11}) = E_{11} \oplus O_{k-n} \quad \text{and} \quad \phi(E_{22}) = [0] \oplus I_r \oplus C_2. \tag{2.5}$$

Let $Y_1 = \phi(E_{11} + E_{22})$, $Y_2 = \phi(E_{12} + E_{21})$, and

$$Y = (Y_1 + Y_2)/2 = \phi(E_{11} + E_{22} + E_{12} + E_{21})/2.$$

Since $\|\phi(Z)\| = \|Z\|$, we have $1 = \|Y\| = \|Y_1\| = \|Y_2\|$. Applying Assertion 1 to the matrices $\phi(E_{11})$ and Y_2, and also to $\phi(E_{22})$ and Y_2, we see that

$$Y_2 = \begin{pmatrix} 0 & u_1^* & u_2^* \\ u_1 & O_r & * \\ u_2 & * & * \end{pmatrix} \tag{2.6}$$

for some $u_1 \in \mathbb{C}^r$ and $u_2 \in \mathbb{C}^{k-1-r}$. If $v \in \mathbb{C}^k$ is a unit vector so that $v^*Yv = 1$, then

$$2 = 2v^*Yv = v^*Y_1v + v^*Y_2v \le \|Y_1\| + \|Y_2\| = 2,$$

and hence $v^*Y_1v = 1 = v^*Y_2v$. Since $Y_1 = I_{1+r} \oplus C_2$ with $\lambda_1(C_2) < 1$ by (2.5), we see that $v \in \operatorname{span}\{e_1, \dots, e_{1+r}\} \subseteq \mathbb{C}^k$. Thus, if P is obtained from I_k by taking its first $1 + r$ columns, then $1 = \|Y_2\| = v^*Y_2v \le \|P^*Y_2P\| \le \|Y_2\|$. It follows that

$$1 = \|P^*Y_2P\| = \left\| \begin{pmatrix} 0 & u_1^* \\ u_1 & O_r \end{pmatrix} \right\|;$$

thus, u_1 is a unit vector. Since Y_2 in the form (2.6) has norm 1, we see that $u_2 = 0$ and there exists a unitary matrix $W = [1] \oplus W_1 \oplus I_{k-1-r}$ such that $WY_2W^* = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \oplus Z_2$. Hence, $W(Y_1 + Y_2)W^* = \left(\begin{smallmatrix} 1 & 1 \\ 1 & 1 \end{smallmatrix}\right) \oplus Z_0$, and Z_0 is nonzero positive semidefinite such that

$$\|Z_0\| \le \left\| \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \oplus Z_0 \right\| = \|\phi(E_{22} + E_{12} + E_{21})\| = \|E_{22} + E_{12} + E_{21}\| < 2.$$

Thus, Y is unitarily similar to the direct sum of a rank one orthogonal projection and a non-trivial D with $0 < \lambda_1(D) < 1$. So, $X = (E_{11} + E_{22} + E_{12} + E_{21})/2$ is a desired rank one orthogonal projection. □

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ASSERTION 3. *There exist unitary $U \in M_k$ and $V \in M_n$ such that the mapping $\tilde{\phi}$ defined by*

$$X \mapsto U\phi(VXV^*)U^* \tag{2.7}$$

satisfies

$$\tilde{\phi}(Y) = Y \oplus \tilde{f}(Y) \quad \textit{for all } Y = [a] \oplus Y_1, \quad \textit{or} \tag{2.8}$$

$$\tilde{\phi}(Y) = Y^t \oplus \tilde{f}(Y) \quad \textit{for all } Y = [a] \oplus Y_1, \tag{2.9}$$

where $\tilde{f} : \mathscr{H}_n \to \mathscr{H}_{k-n}$ is a unital positive linear map satisfying $0 < \|\tilde{f}(E_{11})\| < 1$.

PROOF. By Assertion 2, we may replace ϕ by a mapping of the form (2.7) and assume that $\phi(E_{11}) = [1] \oplus O_q \oplus D_1$, where $q + 1 < k$ and D_1 is a diagonal matrix with diagonal entries in the interval (0, 1). Let $Y = [0] \oplus Y_1 \in \mathscr{H}_n$, where $Y_1 \in \mathscr{H}_{n-1}$, $\|Y_1\| = 1$. By Assertion 1, the (1, 1) entry of $\phi(Y)$ is 0. Since $\|\phi(E_{11} + Y)\| = \|E_{11} + Y\| = 1$, the first row and column of $\phi(Y)$ are all zero. Therefore,

$$\phi(Y) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \psi(Y_1) & * \\ 0 & * & * \end{pmatrix} \quad \text{with } \psi(Y_1) \in \mathscr{H}_q.$$

Since $1 = \|\phi(Y)\|$, there exists a unit vector v such that $1 = |v^*\phi(Y)v| = \|\phi(Y)\|$. Clearly, the first entry of v must be zero. Suppose $v = \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$ with $v_1 \in \mathbb{C}^q$ and $v_2 \in \mathbb{C}^{k-1-q}$. Since $\phi(E_{11}) = [1] \oplus O_q \oplus D_1$ and

$$|v_2^* D_1 v_2 \pm v^*\phi(Y)v| = |v^*\phi(E_{11} \pm Y)v| \le \|E_{11} \pm Y\| = 1,$$

we see that $v_2 = 0$ and $|v_1^*\psi(Y_1)v_1| = 1 = \|Y_1\|$. Hence, the mapping from M_{n-1} to M_q defined by $Y_1 \mapsto \psi(Y_1)$ is unital and satisfies $\|\psi(Y_1)\| = \|Y_1\|$ for all $Y_1 \in \mathscr{H}_{n-1}$. Since $q \le k - 2 \le 2n - 4$, we can apply induction assumption to ψ and conclude that ψ on $\mathscr{H}_{n-1}$ has the standard form:

$$Y_1 \mapsto U^*[Y_1 \oplus f(Y_1)]U \quad \text{or} \quad Y_1 \mapsto U^*[Y_1^t \oplus f(Y_1)]U$$

for some unitary $U \in M_q$. Now, the mapping $\tilde{\phi}$ defined by

$$X \mapsto ([1] \oplus U \oplus I_{k-q-1})\phi(X)([1] \oplus U^* \oplus I_{k-q-1})$$

satisfies (2.8) or (2.9). $\square$

PROOF OF THE CLAIM. By Assertion 3, we can modify ϕ to $\tilde{\phi}$ that satisfies (2.8) or (2.9), where $\tilde{f}$ is a unital linear map satisfying $0 < \|\tilde{f}(E_{11})\| < 1$. We may further assume that $\tilde{\phi}$ satisfies (2.8); otherwise, replace ϕ by the mapping $A \mapsto \phi(A^t)$. For simplicity, we assume that $\phi = \tilde{\phi}$.

To prove the claim, we note that every matrix in $\mathscr{H}_n$ is a linear combination of rank one orthogonal projections. Therefore, we only need to show that if $X \in \mathscr{H}_n$ is a rank one orthogonal projection, then

$$\phi(X) = g(X) \oplus \tilde{f}(X), \tag{2.10}$$

where $g(X)$ is a rank one $n \times n$ orthogonal projection.

If $X = E_{11}$ or X has the form $[0] \oplus X_1$, then we are done because $\phi = \tilde{\phi}$ satisfies (2.8). Now, suppose X is not of these forms. Then $X = uu^*$, where $u = ae_1 + bv \in \mathbb{C}^n$ is a unit vector such that $v \in e_1^\perp$ and a, b are nonzero complex numbers satisfying $|a|^2 + |b|^2 = 1$. Replacing u by $\xi_1 u$ for a suitable complex unit ξ_1, we may assume that $a > 0$; then replacing v by $\xi_2 v$ for a suitable complex unit ξ_2, we may assume that $b > 0$ as well. So, $(a, b) = (\cos\theta, \sin\theta)$ for some $\theta \in (0, \pi/2)$. Suppose $V \in M_n$ is a unitary matrix with e_1 and v as the first two columns. Then V has the form $[1] \oplus V_1$ and satisfies

$$V^* X V = \cos^2\theta E_{11} + \cos\theta\sin\theta(E_{12} + E_{21}) + \sin^2\theta E_{22}.$$

Consider the mapping ϕ_V defined by $A \mapsto (V^* \oplus I_{k-n})\phi(VAV^*)(V \oplus I_{k-n})$. Note that the mapping ϕ_V inherits all the properties we have established in Assertions 1–3, (2.8) for ϕ. Moreover, if we can show that ϕ_V sends the matrix

$$\cos^2\theta E_{11} + \cos\theta\sin\theta(E_{12} + E_{21}) + \sin^2\theta E_{22}$$

to a matrix of the form $Z_1 \oplus Z_2$ so that $Z_1 \in M_n$ is a rank one orthogonal projection, then $\phi(X) = (V \oplus I_{k-n})\phi_V(V^* X V)(V^* \oplus I_{k-n}) = VZ_1V^* \oplus Z_2$, where VZ_1V^* is a rank one orthogonal projection as desired. So, we focus on ϕ_V. For simplicity, we write ϕ_V as ϕ in the rest of our proof. For $j \in \{1, \ldots, n\}$, let $\phi(E_{jj}) = E_{jj} \oplus C_j$. Then $C_1 = \tilde{f}(E_{11})$ satisfies $0 < \|C_1\| < 1$ and

$$C_1 + \cdots + C_n = I_{k-n}. \tag{2.11}$$

We consider two cases.

Case 1. Suppose $\lambda_1(C_1 + C_2) < 1$, that is, $\phi(E_{11} + E_{22})$ only has two eigenvalues equal to 1. If $v \in \mathbb{C}^k$ satisfies $v^*(\phi(E_{11} + E_{22}))v = 1$, then only the first two entries of v can be nonzero. Now,

$$\begin{aligned} 2 = \|(e_1 + e_2)(e_1 + e_2)^*\| &= \|\phi((e_1 + e_2)(e_1 + e_2)^*)\| \\ &\le \|\phi(e_1e_1^* + e_2e_2^*)\| + \|\phi(e_1e_2^* + e_2e_1^*)\| = 2. \end{aligned}$$

So, there is a unit vector $v \in \mathbb{C}^n$ such that

$$v^*(\phi(e_1e_1^* + e_2e_2^*))v = 1 = v^*(\phi(e_1e_2^* + e_2e_1^*))v.$$

Thus, the leading 2×2 principal submatrix of $\phi(e_1e_2^* + e_2e_1^*)$ has norm one. By Assertion 1, the (1, 1) and (2, 2) entries of $\phi(e_1e_2^* + e_2e_1^*)$ are zero. Hence, there is a complex unit μ such that $\phi(e_1e_2^* + e_2e_1^*) = (\mu e_1e_2^* + \bar{\mu}e_2e_1^*) \oplus D$. Therefore,

$$\phi\big((\cos\theta e_1 + \sin\theta e_2)(\cos\theta e_1 + \sin\theta e_2)^*\big) = \begin{pmatrix} \cos^2\theta & \mu\cos\theta\sin\theta \\ \bar{\mu}\cos\theta\sin\theta & \sin^2\theta \end{pmatrix} \oplus \tilde{D}.$$

Since

$$\left\| \phi\left((\cos\theta e_1 + \sin\theta e_2)(\cos\theta e_1 + \sin\theta e_2)^* \pm \sum_{j=3}^{n} e_j e_j^* \right) \right\| = 1,$$

we see that $\tilde{D}$ has the form $O_{n-2} \oplus \hat{D}$. Hence, $\phi((\cos\theta e_1 + \sin\theta e_2)(\cos\theta e_1 + \sin\theta e_2)^*)$ has the desired form (2.10).

Case 2. Suppose $\lambda_1(C_1 + C_2) = 1$. We shall prove that there exists a sequence of unit vectors $\{v_r\}$ in the linear span of $\{e_2, \dots, e_n\} \subseteq \mathbb{C}^n$ such that $v_r \to e_2$, and for each r, $\phi(E_{11} + v_r v_r^*)$ has only two eigenvalues equal to 1. By the result in Case 1,

$$\phi\big((\cos\theta e_1 + \sin\theta v_r)(\cos\theta e_1 + \sin\theta v_r)^*\big)$$

has the desired form (2.10). By continuity, we see that

$$\phi\big((\cos\theta e_1 + \sin\theta e_2)(\cos\theta e_1 + \sin\theta e_2)^*\big)$$

has the desired form (2.10) as well.

To construct our sequence $\{v_r\}$, note that by (2.11) and the fact that $0 < \|C_1\| < 1$, we have

$$(I_{k-n} - C_1)^{-1/2}(C_2 + \cdots + C_n)(I_{k-n} - C_1)^{-1/2} = I_{k-n}.$$

Since $k - n \leq n - 2$, comparing traces, we see that there exists $j \geq 3$ such that

$$(I_{k-n} - C_1)^{-1/2}C_j(I_{k-n} - C_1)^{-1/2}$$

is a strict contraction, equivalently, $\lambda_1(C_1 + C_j) < 1$. Without loss of generality, we may assume that $j = 3$. Let $\phi(E_{23} + E_{32}) = (E_{23} + E_{32}) \oplus C_{23}$. For $t \in [0, \pi/2]$, let

$$F(t) = v(t)v(t)^* \quad \text{with} \quad v(t) = \cos t e_2 + \sin t e_3 \in \mathbb{C}^n.$$

Then

$$\phi(E_{11} + F(t)) = [E_{11} + F(t)] \oplus [C_1 + \cos^2 t C_2 + \sin^2 t C_3 + \cos t \sin t C_{23}].$$

If $\phi(E_{11} + F(t))$ has more than two eigenvalues equal 1, then

$$\begin{aligned}(2.12)\quad 0 &= \det(I_{k-n} - (C_1 + \cos^2 t C_2 + \sin^2 t C_3 + \cos t \sin t C_{23}))\\ &= \det(\cos^2 t[(1 + \tan^2 t)(I_{k-n} - C_1) - C_2 - \tan^2 t C_3 - \tan t C_{23}])\\ &= \cos^{2(k-n)} t \det((I_{k-n} - C_1 - C_2) - \tan t C_{23} + \tan^2 t(I_{k-n} - C_1 - C_3)).\end{aligned}$$

Since $\phi(E_{11} + E_{33})$ has only two eigenvalues equal to 1, $\det(I_{k-n} - C_1 - C_3) \neq 0$. It follows that (2.12) only has finitely many roots in the interval $[0, \pi/2]$. Thus, we can find a sequence $\{t_r\} \to 0$ such that $\{v_r\} = \{v(t_r)\} \to e_2$, and for each r, $E_{11} + v_r v_r^*$ has only two eigenvalues equal to 1 as desired. $\square$

3. Proof of the main theorems

PROOF OF THEOREM 1.2. The 'if' part of the theorem is clear. Suppose $k \leq 2n-2$, $\phi(I_n) = I_k$ and $\|\phi(X)\| = \|X\|$ for all essentially Hermitian X. By Theorem 2.1, $\phi(X)$ is Hermitian whenever X is Hermitian. Now, the result follows from Theorem 2.3.

For the last statement, suppose $n \geq 2$ and $k \geq 2n - 1$. Let

$$W = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & I_{n-1} & 0 & 0 \\ 0 & 0 & 0 & I_{n-1} \end{pmatrix}.$$

Define $\phi : M_n \to M_k$ by

$$\begin{aligned}\phi(A) &= W[A \oplus A^t]W^* \oplus (\operatorname{tr} A/n) I_{k-2n+1}\\ &= \begin{pmatrix} A_{11} & A_{12}/\sqrt{2} & A_{21}^t/\sqrt{2} \\ A_{21}/\sqrt{2} & A_{22} & 0 \\ A_{12}^t/\sqrt{2} & 0 & A_{22}^t \end{pmatrix} \oplus (\operatorname{tr} A/n) I_{k-2n+1}\end{aligned}$$

for any

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{with } A_{22} \in M_{n-1}.$$

Since $WW^* = I_{2n-1}$, by the interlacing inequalities for eigenvalues of Hermitian matrices [4, Theorem 4.3.6], if $A \in \mathscr{H}_n$ and $B = W[A \oplus A^t]W^*$, then $\lambda_1(B) = \lambda_1(A)$ and $\lambda_{2n-1}(B) = \lambda_n(A)$. Consequently, $\|\phi(X)\| = \|X\|$ for all essentially Hermitian $X \in M_n$.

If ϕ has the standard form (1.2), then there exist a contractive linear map $f : M_n \to M_{k-n}$ and a unitary matrix $U \in M_k$ such that $U\phi(A) = (A^\dagger \oplus f(A))U$, where $A^\dagger = A$

or A^t. Partition U into $U = (U_{ij})_{i,j=1}^2$, where $U_{22} \in M_{k-1}$, then we have

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix} \begin{pmatrix} A_{11} & A_{12}/\sqrt{2} & A_{21}^t/\sqrt{2} & 0 \\ A_{21}/\sqrt{2} & A_{22} & 0 & 0 \\ A_{12}^t/\sqrt{2} & 0 & A_{22}^t & 0 \\ 0 & 0 & 0 & (\operatorname{tr} A/n) I_{k-2n+1} \end{pmatrix}$$
$$= \begin{pmatrix} A^\dagger & 0 \\ 0 & f(A) \end{pmatrix} \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}.$$

Let $A = E_{11}$ and consider the first row on both sides. We have $\begin{pmatrix} U_{11} & 0 \end{pmatrix} = \begin{pmatrix} U_{11} & U_{12} \end{pmatrix}$. Hence, $U_{11} = e^{i\theta}$ for some real number θ and U_{12}, U_{21} are both zero. Consider the first row on both sides in the general case, we have

$$e^{i\theta} \begin{pmatrix} A_{11} & A_{12}/\sqrt{2} & A_{21}^t/\sqrt{2} & 0 \end{pmatrix} = \begin{pmatrix} A_{12}^\dagger & 0 \end{pmatrix} U_{22},$$

for all $A \in M_n$, which is impossible. Hence, ϕ is not of the standard form (1.2). □

Proof of Theorem 1.1. Suppose $k \le 2n-1$, and $\|\phi(X)\| = \|X\|$ for all $X \in M_n$. Clearly, we have $k \ge n$. If $k = n$ then (1.1) follows from Kadison's result [6]. So we may assume that $n < k \le 2n-1$. By the result in [1], it is impossible that $\phi(U)$ is unitary for every unitary $U \in M_n$. Thus, there exists a unitary $X \in M_n$ such that $\phi(X)$ is not unitary. By replacing ϕ with the map $A \mapsto \phi(XA)$, if necessary, we may assume that $X = I$. Therefore, ϕ satisfies all conditions in Theorem 2.1 with $1 \le p \le 2n-2$. Let U, $V \in M_k$, and ψ be as given by Theorem 2.1. Then ψ satisfies the conditions for Theorem 1.2 (with ϕ, k replaced by ψ, p). So, there exists a unitary $W_1 \in M_p$ and a unital positive linear map $\tilde{f} : M_n \to M_{p-n}$ such that ψ has the form

$$A \mapsto W_1[A^\dagger \oplus \tilde{f}(A)]W_1^*,$$

where $A^\dagger = A$ or A^t. Let $W = W_1 \oplus I_{k-p}$. Then the mapping $\phi_0 : M_n \to M_k$ defined by $A \mapsto W^* U^* \phi(A) V^* W$ has the form

$$A \mapsto \begin{pmatrix} A^\dagger & 0 & * \\ 0 & \tilde{f}(A) & * \\ * & * & g(A) \end{pmatrix}.$$

If $A \in M_n$ is unitary, then $\|\phi_0(A)\| = \|A\|$ implies that

$$\phi_0(A) = A^\dagger \oplus \begin{pmatrix} \tilde{f}(A) & * \\ * & g(A) \end{pmatrix}. \tag{3.1}$$

Since this is true for n^2 linearly independent unitary matrices A, it follows that (3.1) holds for any $A \in M_n$. Consequently, the original map ϕ has the form (1.1) as asserted.

For the last statement, suppose $n \geq 2$ and $k \geq 2n$. Let

$$W = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & I_{n-1} & 0 & 0 \\ 0 & 0 & 0 & I_{n-1} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \end{pmatrix} \quad \text{and} \quad P = I_{2n-1} \oplus O_{k-2n+1}.$$

Define $\phi : M_n \to M_k$ by

$$\begin{aligned} \phi(A) &= P(W \oplus I_{k-2n})[A \oplus A^t \oplus O_{k-2n}](W \oplus I_{k-2n})^* \\ &= \begin{pmatrix} A_{11} & A_{12}/\sqrt{2} & A_{21}^t/\sqrt{2} & 0 & 0 \\ A_{21}/\sqrt{2} & A_{22} & 0 & A_{21}/\sqrt{2} & 0 \\ A_{12}^t/\sqrt{2} & 0 & A_{22}^t & -A_{12}^t/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned}$$

for any

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \quad \text{with } A_{22} \in M_{n-1}.$$

Note that $\phi(A)\phi(A)^* = B \oplus O_{k-2n+1}$, where $B \in \mathscr{H}_{2n-1}$ is a leading principal submatrix of $W(AA^* \oplus (AA^*)^t)W^*$. By the interlacing inequalities for eigenvalues of Hermitian matrices [4, Theorem 4.3.6], we have $\|\phi(A)\| = \|A\|$. By an argument similar to the one in the proof of Theorem 1.2, we can show that ϕ is not of the form (1.1). □

4. Related results and questions

Motivated by Theorem 1.1 and the example constructed in its proof, we have the following.

PROPOSITION 4.1. *Suppose P and Q are $n(p+q) \times m$ matrices such that*

$$I - PP^* \geq 0, \quad I - QQ^* \geq 0,$$

and $\operatorname{rank}(I - PP^*) + \operatorname{rank}(I - QQ^*) < p + q$. *Let $W_1, W_2 \in M_k$ be unitary, and let $f : M_n \to M_{k-m}$ be a contractive linear map. If $\phi : M_n \to M_k$ is defined by*

$$\phi(X) = W_1\{P^*[(X \otimes I_p) \oplus (X^t \otimes I_q)]Q \oplus f(X)\}W_2,$$

then $\|\phi(X)\| = \|X\|$ for all $X \in M_n$.

https://doi.org/10.1017/S1446788700010119 Published online by Cambridge University Press

PROOF. It is clear that $\|\phi(X)\| \le \|X\|$. To prove the reverse inequality, suppose P has singular value decomposition UDV, where $U \in M_{n(p+q)}$ and $V \in M_m$ are unitary, and the singular values of P lie in the $(1, 1), (2, 2), \dots$ positions of D in descending order. Let $\tilde{D}$ be obtained from D by setting all the entries in $(0, 1)$ to 0, and let $\tilde{P} = U\tilde{D}V$. Apply a similar construction to Q to get $\tilde{Q}$. Then

$$\text{(4.1)} \qquad \begin{aligned} &\operatorname{rank}(I - \tilde{P}\tilde{P}^*) + \operatorname{rank}(I - \tilde{Q}\tilde{Q}^*) \\ &\quad = \operatorname{rank}(I - PP^*) + \operatorname{rank}(I - QQ^*) < p + q. \end{aligned}$$

If the largest singular value of X is $s_1 = \|X\|$, then s_1 is a singular value of $(X \otimes I_p) \oplus (X^t \otimes I_q)$ with multiplicity at least $p + q$. By (4.1) and a result of Thompson [7], the matrix $\tilde{P}^*[(X \otimes I_p) \oplus (X^t \otimes I_q)]\tilde{Q}$ has largest singular value equal to s_1 also. Thus, we have $\|X\| = s_1 = \|\tilde{P}^*[(X \otimes I_p) \oplus (X^t \otimes I_q)]\tilde{Q}\| \le \|\phi(X)\|$. $\square$

Using a similar argument as in the proof of Proposition 4.1 and the interlacing inequalities on Hermitian matrices (see [4, Theorem 4.3.6]), we have the following.

PROPOSITION 4.2. *Suppose P is an $n(p + q) \times m$ matrix such that $P^*P = I_m$, where $0 \le n(p + q) - m < p + q$. Let $U \in M_k$ be unitary, and $f : M_n \to M_{k-m}$ be a unital positive linear map. If $\phi : M_n \to M_k$ is defined by*

$$\phi(X) = U^*\{P^*[(X \otimes I_p) \oplus (X^t \otimes I_q)]P \oplus f(X)\}U,$$

then $\|\phi(X)\| = \|X\|$ for all essentially Hermitian $X \in M_n$.

Recall that the *numerical range* of a matrix $A \in M_n$ is the set

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\},$$

which is a useful concept in matrix and operator theory, and has been studied extensively; see [5, Chapter 1]. We have the following proposition.

PROPOSITION 4.3. *Let $V_m = M_m$ or $\mathcal{H}_m$. Suppose $\phi : V_n \to V_k$ is linear. If ϕ has the form given in Proposition 4.2, then*

$$\text{(4.2)} \qquad W(\phi(X)) = W(X) \quad \text{for all } X \in V_n.$$

When $k \le 2n - 2$, (4.2) holds if and only if ϕ has the form (1.2) in Theorem 1.2.

PROOF. Suppose ϕ has the form in Proposition 4.2. If $X \in \mathcal{H}_n$, then X and $\phi(X)$ have the same largest and smallest eigenvalues; since $W(X)$ is the convex hull of the largest and smallest eigenvalues of X, it follows that $W(\phi(X)) = W(X)$.

https://doi.org/10.1017/S1446788700010119 Published online by Cambridge University Press

Suppose $V_n = M_n$ and $X \in M_n$ is not Hermitian. Then $X = H + iG$ for some Hermitian H and G. Now, $\phi(\cos tH + \sin tG)$ and $\cos tH + \sin tG$ have the same largest and smallest eigenvalues for all $t \in [0, 2\pi)$, we see that the two convex sets $W(X)$ and $W(\phi(X))$ have the same support lines; see [5, Theorem 1.5.11]. Thus, the two sets are equal.

Suppose $k \leq 2n - 2$. If $V_n = \mathcal{H}_n$, the result follows readily from Theorem 2.3. If $V_n = M_n$, one can use the fact that $W(X) \subseteq \mathbb{R}$ if and only if X is Hermitian to conclude that $\phi(\mathcal{H}_n) \subseteq \mathcal{H}_k$. Then the result follows from the Hermitian case. $\square$

There are several related problems that deserve further investigation.

(1) If $\phi : M_n \to M_k$ has the form in Proposition 4.1, then $\|\phi(A)\| = \|A\|$ for all $A \in M_n$. It would be nice to know whether the converse is true.

(2) If $\phi : M_n \to M_k$ has the form in Proposition 4.2, then $\|\phi(A)\| = \|A\|$ for all essentially Hermitian $A \in M_n$. It would be nice to know whether the converse is true. Note that by Theorem 2.1 and Theorem 2.3 (b), the problem is equivalent to studying positive linear maps ϕ such that A is positive definite whenever $\phi(A)$ is positive definite.

(3) One can ask whether the converse of the first statement in Proposition 4.3 is true.

(4) One can study the above problems under the additional assumption that ϕ is a decomposable or completely positive linear map.

Acknowledgment

We thank the referee for some helpful suggestions. Also, we thank Professor Ngai-Ching Wong for helpful comments and for drawing our attention to the problem, which was motivated by the work in [2].

The second author would like to thank Professors Leiba Rodman and Peter Šemrl for helpful discussion on linear maps between matrix algebras preserving numerical ranges.

Research of the first author was supported by a PIMS Postdoctoral Fellowship, and the research of the second author was partially supported by NSF.

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