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EXISTENCE RESULT FOR A CLASS OF NONLINEAR THIRD-ORDER TWO-POINT BOUNDARY-VALUE PROBLEMS

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Abstract The upper and lower solutions method and Leray–Schauder degree theory are employed to establish the existence result for a class of nonlinear third-order two-point boundary-value problems with a sign-type Nagumo condition.

Keywords: sign-type Nagumo condition; upper and lower solutions method; Leray–Schauder degree theory

2000 *Mathematics subject classification:* Primary 34B15

1. Introduction

Third-order boundary value problems have been discussed in many papers in recent years (see, for example, [**1**–**4**, **6**]). But most of them considered linear boundary conditions. Recently, Grossinho [**5**] established an existence and location result for the nonlinear differential equation

$$x''' = f(t, x, x', x''),$$

with two types of boundary conditions:

$$x(a) = A, \quad \phi(x'(b), x''(b)) = 0, \quad x''(a) = B,$$

or

$$x(a) = A, \quad \psi(x'(a), x''(a)) = 0, \quad x''(b) = C.$$

In this work, we extend the study to a more general case, since we consider the third-order nonlinear differential equation

$$x''' = f(t, x, x', x''), \quad a < t < b, \tag{1.1}$$

with nonlinear boundary conditions

$$x(a) = A, \quad g(x'(a)) - [x''(a)]^p = B, \quad \phi(x(b), x'(b), x''(b)) = C, \tag{1.2}$$

or

$$x(a) = A, \quad \psi(x(a), x'(a), x''(a)) = B, \quad h(x'(b)) + [x''(b)]^q = C. \tag{1.3}$$

The function $f(t, x, y, z) : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ is continuous, $g, h : \mathbb{R} \to \mathbb{R}$ is continuous, $\phi, \psi : \mathbb{R}^3 \to \mathbb{R}$ are continuous and monotone on the first and third variables, and p and q are odd numbers.

By the use of the upper and lower solutions method and Leray–Schauder degree theory, we show existence results with a sign-type Nagumo condition, which is weaker than the one in [**5**].

This work is organized as follows. In § 2, some notation and preliminaries are introduced. The existence results are discussed in § 3. As applications of our results, an example is given in the last section.

2. Preliminaries

Definition 2.1. Function $\alpha(t) \in C^3[a, b]$ is said to be a lower solution of the boundary-value problem (BVP) (1.1), (1.2) if

$$\alpha'''(t) \geqslant f(t, \alpha(t), \alpha'(t), \alpha''(t)), \quad t \in [a, b], \tag{2.1}$$

and

$$\alpha(a) \leqslant A, \; g(\alpha'(a)) - [\alpha''(a)]^p \leqslant B, \qquad \phi(\alpha(b), \alpha'(b), \alpha''(b)) \leqslant C. \tag{2.2}$$

Function $\beta(t) \in C^3[a, b]$ is said to be an upper solution of the BVP (1.1), (1.2) if it satisfies the reversed inequalities.

Definition 2.2. Given a subset $D \subset [a, b] \times \mathbb{R}^3$, a function $f : D \to \mathbb{R}$ is said to satisfy the sign-type Nagumo condition (N^*_+) in D if there exists $\Phi \in C(\mathbb{R}^+_0, (0, +\infty))$ such that

$$f(t, x, y, z) \operatorname{sgn}(z) \leqslant \Phi(|z|) \quad \text{for all } (t, x, y, z) \in D \tag{2.3}$$

and

$$\int_0^{+\infty} \frac{s}{\Phi(s)} \, \mathrm{d}s = +\infty. \tag{2.4}$$

If (2.3) is replaced by

$$f(t, x, y, z) \operatorname{sgn}(z) \geqslant -\Phi(|z|) \quad \text{for all } (t, x, y, z) \in D, \tag{2.5}$$

we say that f satisfies the sign-type Nagumo condition (N^*_-).

Lemma 2.3. *Let $\alpha_i, \beta_i \in C[a, b]$ satisfy*

$$\alpha_i(t) \leqslant \beta_i(t), \quad i = 0, 1, \; t \in [a, b],$$

and consider the set

$$E = \{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : \alpha_0(t) \leqslant x \leqslant \beta_0(t), \; \alpha_1(t) \leqslant y \leqslant \beta_1(t)\}.$$

Let $f : [a,b] \times \mathbb{R}^3 \to \mathbb{R}$ be a continuous function that satisfies the sign-type Nagumo condition (N_+^*) in E. Then for every $\rho > 0$ there exists $K > 0$ (depending on $\alpha_1(t)$, $\beta_1(t)$, Φ and ρ) such that, for every solution $x(t)$ of (1.1) verifying

$$|x''(a)| \leqslant \rho \tag{2.6}$$

and

$$\alpha_0(t) \leqslant x(t) \leqslant \beta_0(t), \quad \alpha_1(t) \leqslant x'(t) \leqslant \beta_1(t) \quad \text{for all } t \in [a,b], \tag{2.7}$$

we have

$$\|x''\|_\infty < K.$$

Proof. This result can be easily proved by using the analogous technique of Lemma 2 from [**5**]. □

Remark 2.4. The above result still holds if we replace condition (N_+^*) by (N_-^*) and assumption (2.6) by $|x''(b)| \leqslant \rho$.

Lemma 2.5. *The boundary-value problem*

$$x''' = x'\Phi(|x''|), \tag{2.8}$$

$$x(a) = 0, \quad x'(a) = [x''(a)]^p, \quad x'(b) = 0 \tag{2.9}$$

has only the trivial solution, where $\Phi \in C(\mathbb{R}_0^+, (0, +\infty))$.

Proof. Assume, by contradiction, that $x_0(t)$ be a non-trivial solution of BVP (2.8), (2.9). Then there exists $t \in [a,b]$ such that $x_0'(t) > 0$ or $x_0'(t) < 0$. Suppose the first case holds. Define

$$\max_{t\in[a,b]} x_0'(t) = x_0'(t_1) > 0.$$

If $t_1 \in (a,b)$, then $x_0''(t_1) = 0$ and $x_0'''(t_1) \leqslant 0$. From (2.8) we have the following contradiction:

$$0 \geqslant x_0'''(t_1) = x_0'(t_1)\Phi(|x_0''(t_1)|) > 0.$$

If $t_1 = a$, then $x_0'(a) > 0$ and $x_0''(a) \leqslant 0$, which contradicts (2.9).

If $t_1 = b$, from (2.9) we can get the contradiction.

Thus, BVP (2.8), (2.9) has only the trivial solution. □

3. Main results

Theorem 3.1. *Assume that*

(i) *there exist lower and upper solutions of BVP (1.1), (1.2), $\alpha(t)$, $\beta(t)$, such that*

$$\alpha'(t) \leqslant \beta'(t), \quad t \in [a,b],$$

(ii) *$f(t,x,y,z)$ is continuous on $[a,b] \times \mathbb{R}^3$ and decreasing on x,*

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(iii) *$f(t,x,y,z)$ satisfies the sign-type Nagumo condition (N_+^*) in*

$$D_* = \{(t,x,y,z) \in [a,b] \times \mathbb{R}^3 : \alpha(t) \leqslant x \leqslant \beta(t),\ \alpha'(t) \leqslant y \leqslant \beta'(t)\},$$

(iv) *$g(y)$ is continuous on $\mathbb{R}$, $\phi(x,y,z)$ is continuous on $\mathbb{R}^3$, decreasing on x and increasing on z.*

Then BVP (1.1), (1.2) has at least one solution $x(t) \in C^3[a,b]$ such that

$$\alpha(t) \leqslant x(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant x'(t) \leqslant \beta'(t), \quad t \in [a,b].$$

Proof. For $i = 0, 1$, define

$$w_i(t,x_i) = \begin{cases} \beta^{(i)}(t), & x_i > \beta^{(i)}(t), \\ x_i, & \alpha^{(i)}(t) \leqslant x_i \leqslant \beta^{(i)}(t), \\ \alpha^{(i)}(t), & x_i < \alpha^{(i)}(t). \end{cases}$$

For $\lambda \in [0,1]$, we consider the auxiliary equation

$$x'''(t) = \lambda f(t, w_0(t,x(t)), w_1(t,x'(t)), x''(t)) + [x'(t) - \lambda w_1(t,x'(t))]\Phi(|x''(t)|), \tag{3.1}$$

where Φ is decided by the sign-type Nagumo condition (N_+^*), with the boundary condition

$$x(a) = \lambda A, \tag{3.2}$$

$$x'(a) = \lambda[B - g(w_1(a,x'(a))) + w_1(a,x'(a))] + [x''(a)]^p, \tag{3.3}$$

$$x'(b) = \lambda[C - \phi(w_0(b,x(b)), w_1(b,x'(b)), x''(b)) + w_1(b,x'(b))]. \tag{3.4}$$

Then we can select $M_1 > 0$ such that for every $t \in [a,b]$,

$$-M_1 < \alpha'(t) \leqslant \beta'(t) < M_1, \tag{3.5}$$

$$f(t,\alpha(t),\alpha'(t),0) - [M_1 + \alpha'(t)]\Phi(0) < 0, \tag{3.6}$$

$$f(t,\beta(t),\beta'(t),0) + [M_1 - \beta'(t)]\Phi(0) > 0, \tag{3.7}$$

$$B - g(\alpha'(a)) + \alpha'(a) > -M_1, \qquad |C - \phi(\alpha(b),\alpha'(b),0) + \alpha'(b)| < M_1, \tag{3.8}$$

$$B - g(\beta'(a)) + \beta'(a) < M_1, \qquad |C - \phi(\beta(b),\beta'(b),0) + \beta'(b)| < M_1. \tag{3.9}$$

In the following, we shall complete the proof in four steps.

Step 1. Every solution $x(t)$ of BVP (3.1)–(3.4) satisfies

$$|x'(t)| < M_1, \quad t \in [a,b], \tag{3.10}$$

independently of λ.

We suppose that the estimate is not true. Then there exists some $t \in [a,b]$ such that $x'(t) \geqslant M_1$ or $x'(t) \leqslant -M_1$. Suppose the first case holds. Define

$$\max_{t\in[a,b]} x'(t) := x'(t_0) \ (\geqslant M_1 > 0).$$

If $t_0 \in (a, b)$, then $x''(t_0) = 0$ and $x'''(t_0) \leqslant 0$. For $\lambda \in (0, 1]$, by condition (ii) and (3.7), we have the following contradiction

$$\begin{aligned} 0 &\geqslant x'''(t_0) \\ &= \lambda f(t_0, w_0(t_0, x(t_0)), w_1(t_0, x'(t_0)), x''t_0) + [x'(t_0) - \lambda w_1(t_0, x'(t_0))]\Phi(|x''(t_0)|) \\ &= \lambda f(t_0, w_0(t_0, x(t_0)), w_1(t_0, x'(t_0)), 0) + [x'(t_0) - \lambda \beta'(t_0)]\Phi(0) \\ &\geqslant \lambda\{f(t_0, \beta(t_0), \beta'(t_0), 0) + [M_1 - \beta'(t_0)]\Phi(0)\} \\ &> 0 \end{aligned}$$

and, for $\lambda = 0$, we have

$$0 \geqslant x'''(t_0) = x'(t_0)\Phi(0) \geqslant M_1\Phi(0) > 0.$$

If $t_0 = a$, then

$$\max_{t \in [a,b]} x'(t) = x'(a) \ (\geqslant M_1 > 0),$$

and $x''(a) \leqslant 0$. For $\lambda = 0$, by (3.3) we have the following contradiction:

$$0 < M_1 \leqslant x'(a) = [x''(a)]^p \leqslant 0.$$

For $\lambda \in (0, 1]$, by (3.3) and (3.9) we can obtain the following contradiction:

$$\begin{aligned} M_1 &\leqslant x'(a) \\ &= \lambda[B - g(w_1(a, x'(a))) + w_1(a, x'(a))] + [x''(a)]^p \\ &\leqslant \lambda[B - g(\beta'(a)) + \beta'(a)] < M_1. \end{aligned}$$

If $t_0 = b$, then

$$\max_{t \in [a,b]} x'(t) = x'(b) \ (\geqslant M_1 > 0),$$

and $x''(b) \geqslant 0$. For $\lambda = 0$, by (3.4) we have the following contradiction:

$$0 < M_1 \leqslant x'(b) = 0.$$

For $\lambda \in (0, 1]$, by (3.4), (3.9) and condition (iv) we can obtain the following contradiction:

$$\begin{aligned} M_1 &\leqslant x'(b) \\ &= \lambda[C - \phi(w_0(b, x(b)), w_1(b, x'(b)), x''(b)) + w_1(b, x'(b))] \\ &\leqslant \lambda[C - \phi(\beta(b), \beta'(b), 0) + \beta'(b)] < M_1. \end{aligned}$$

Thus, $x'(t) < M_1$ for $t \in [a, b]$. In a similar way, we prove that $x'(t) > -M_1$ for $t \in [a, b]$. From (3.2) we have

$$|x(t)| < M_0 = (b - a)M_1 + |A|, \quad t \in [a, b].$$

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Step 2. There exists $M_2 > 0$ such that every solution $x(t)$ of BVP (3.1)–(3.4) satisfies

$$|x''(t)| < M_2, \quad t \in [a, b],$$

independently of $\lambda \in [0, 1]$.

Consider the set

$$D_{**} = \{(t, x, y, z) \in [a, b] \times \mathbb{R}^3 : |x| \leqslant M_0,\ |y| \leqslant M_1\}$$

and the function $F_\lambda : [a, b] \times \mathbb{R}^3 \to \mathbb{R}$ defined by

$$F_\lambda(t, x, y, z) = \lambda f(t, w_0(t, x), w_1(t, y), z) + [y - \lambda w_1(t, y)]\Phi(|z|).$$

In the following, we show that F_λ satisfies the sign-type Nagumo condition in D_{**}, independently of $\lambda \in [0, 1]$. In fact, since f satisfies the sign-type Nagumo condition in D_{**}, we have

$$\begin{aligned} F_\lambda(t, x, y, z) \operatorname{sgn}(z) &= \lambda f(t, w_0(t, x), w_1(t, y), z) \operatorname{sgn}(z) + [y - \lambda w_1(t, y)]\Phi(|z|) \operatorname{sgn}(z) \\ &\leqslant [2M_1 + 1]\Phi(|z|) \\ &:= \Phi^*(|z|). \end{aligned}$$

Furthermore, we obtain

$$\int_0^{+\infty} \frac{s}{\Phi^*(s)} \, \mathrm{d}s = \int_0^{+\infty} \frac{s}{(2M_1 + 1)\Phi(s)} \, \mathrm{d}s = +\infty.$$

Thus, F_λ satisfies the sign-type Nagumo condition (N_+^*) in D_{**}, independently of $\lambda \in [0, 1]$.

Let

$$\rho := [|B| + G + 2M_1]^{1/p},$$

where

$$G = \max_{y \in [-M_1, M_1]} |g(y)|.$$

From (3.3), every solution $x(t)$ of BVP (3.1)–(3.4) satisfies

$$\begin{aligned} |x''(a)| &= |x'(a) - \lambda[B - g(w_1(a, x'(a))) + w_1(a, x'(a))]|^{1/p} \\ &\leqslant [|B| + G + 2M_1]^{1/p} \\ &= \rho. \end{aligned}$$

Define

$$\alpha_0(t) = -M_0, \quad \beta_0(t) = M_0, \quad \alpha_1(t) = -M_1, \quad \beta_1(t) = M_1, \quad t \in [a, b].$$

In view of Step 1 and applying Lemma 2.3, there then exists $M_2 > 0$ (independent of λ) such that $|x''(t)| < M_2$, $t \in [a, b]$.

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Step 3. For $\lambda = 1$, BVP (3.1)–(3.4) has at least one solution $x_1(t)$.
Define the operators

$$L : C^3[a,b] \subset C^2[a,b] \to C[a,b] \times \mathbb{R}^3$$

by

$$Lx = (x''', x(a), x'(a), x'(b))$$

and

$$N_\lambda : C^2[a,b] \to C[a,b] \times \mathbb{R}^3$$

by

$$N_\lambda x = (\lambda f(t, w_0(t,x(t)), w_1(t,x'(t)), x''(t)) + [x'(t) - \lambda w_1(t,x'(t))]\Phi(|x''(t)|), A_\lambda, B_\lambda, C_\lambda)$$

with

$$\begin{aligned}
A_\lambda &= \lambda A,\\
B_\lambda &= \lambda[B - g(w_0(a,x(a)), w_1(a,x'(a))) + w_1(a,x'(a))] + [x''(a)]^p,\\
C_\lambda &= \lambda[C - \phi(w_0(b,x(b)), w_1(b,x'(b)), x''(b)) + w_1(b,x'(b))].
\end{aligned}$$

As L^{-1} is compact, we can define the completely continuous operator

$$T_\lambda : C^2[a,b] \to C^2[a,b]$$

by

$$T_\lambda(x) = L^{-1}N_\lambda(x).$$

Consider the set

$$\Omega = \{x \in C^2[a,b] : \|x\|_\infty < M_0,\ \|x'\|_\infty < M_1,\ \|x''\|_\infty < M_2\}.$$

By Steps 1 and 2, the degree $\deg(I - T_\lambda, \Omega, \theta)$ is well defined for every $\lambda \in [0,1]$ and, by homotopy invariance, we get

$$\deg(I - T_0, \Omega, 0) = \deg(I - T_1, \Omega, 0).$$

As the equation $x = T_0(x)$ has only the trivial solution from Lemma 2.5, by degree theory,

$$\deg(I - T_1, \Omega, 0) = \deg(I - T_0, \Omega, 0) = \pm 1.$$

Hence, the equation $x = T_1(x)$ has at least one solution. That is, the problem

$$x'''(t) = f(t, w_0(t,x(t)), w_1(t,x'(t)), x''(t)) + [x'(t) - w_1(t,x'(t))]\Phi(|x''(t)|) \tag{3.11}$$

with the boundary condition

$$x(a) = A, \tag{3.12}$$

$$x'(a) = [B - g(w_1(a,x'(a))) + w_1(a,x'(a))] + [x''(a)]^p, \tag{3.13}$$

$$x'(b) = [C - \phi(w_0(b,x(b)), w_1(b,x'(b)), x''(b)) + w_1(b,x'(b))] \tag{3.14}$$

has at least one solution $x_1(t)$ in Ω.

Step 4. In fact, the solution $x_1(t)$ of the above problem will also be a solution of BVP (1.1), (1.2) since it satisfies

$$\alpha(t) \leqslant x_1(t) \leqslant \beta(t), \quad \alpha'(t) \leqslant x_1'(t) \leqslant \beta'(t), \quad t \in [a,b]. \tag{3.15}$$

Suppose, by contradiction, that there exists $t \in [a,b]$ such that $x_1'(t) > \beta'(t)$ and define

$$\max_{t\in[a,b]}[x_1'(t) - \beta'(t)] := x_1'(t_1) - \beta'(t_1) > 0.$$

If $t_1 \in (a,b)$, then $x_1''(t_1) = \beta''(t_1)$ and $x_1'''(t_1) \leqslant \beta'''(t_1)$. By condition (ii), we have the contradiction

$$\begin{aligned}
0 &\geqslant x_1'''(t_1) - \beta'''(t_1) \\
&\geqslant f(t_1, w_0(t_1, x_1(t_1)), w_1(t_1, x_1'(t_1)), x_1''(t_1)) \\
&\qquad + [x_1'(t_1) - w_1(t_1, x_1'(t_1))]\Phi(|x_1''(t_1)|) - f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) \\
&\geqslant f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) + [x_1'(t_1) - \beta'(t_1)]\Phi(|x_1''(t_1)|) - f(t_1, \beta(t_1), \beta'(t_1), \beta''(t_1)) \\
&= [x_1'(t_1) - \beta'(t_1)]\Phi(|x_1''(t_1)|) \\
&> 0.
\end{aligned}$$

If $t_1 = a$, we have

$$\max_{t\in[a,b]}[x_1'(t) - \beta'(t)] := x_1'(a) - \beta'(a) > 0$$

and

$$x_1''(a) - \beta''(a) \leqslant 0.$$

By (3.13), Definition 2.1 and condition (iv), we have the contradiction

$$\begin{aligned}
\beta'(a) &< x_1'(a) \\
&= [B - g(w_1(a, x_1'(a))) + w_1(a, x_1'(a))] + [x_1''(a)]^p \\
&\leqslant B - g(\beta'(a)) + \beta'(a) + [\beta''(a)]^p \\
&\leqslant \beta'(a).
\end{aligned}$$

If $t_1 = b$, we have

$$\max_{t\in[a,b]}[x_1'(t) - \beta'(t)] := x_1'(b) - \beta'(b) > 0$$

and

$$x_1''(b) - \beta''(b) \geqslant 0.$$

By (3.14), Definition 2.1 and condition (iv), we have the contradiction

$$\begin{aligned}
\beta'(b) &< x_1'(b) \\
&= [C - \phi(w_0(b, x_1(b)), w_1(b, x_1'(b)), x_1''(b)) + w_1(b, x_1'(b))] \\
&\leqslant C - \phi(\beta(b), \beta'(b), \beta''(b)) + \beta'(b) \\
&\leqslant \beta'(b).
\end{aligned}$$

Thus,

$$x_1'(t) \leqslant \beta'(t), \quad t \in [a, b].$$

Using an analogous technique, we obtain that $\alpha'(t) \leqslant x_1'(t)$ for every $t \in [a, b]$. From

$$\alpha(a) \leqslant A \leqslant \beta(a),$$

and by integration we have

$$\alpha(t) \leqslant x_1(t) \leqslant \beta(t), \quad t \in [a, b].$$

Therefore, $x_1(t)$ is in fact a solution of BVP (1.1), (1.2). □

In the case of nonlinear boundary conditions (1.3) a similar existence result to Theorem 3.1 can be obtained for problem (1.1), (1.3).

4. Example

Example 4.1. Consider the boundary-value problem

$$x''' = -x(x')^2 - t^2(x'')^3, \tag{4.1}$$

$$x(0) = 0, \tag{4.2}$$

$$(x'(0))^3 - (x''(0))^p = 1, \tag{4.3}$$

$$-\frac{4}{\pi}\tan^{-1} x(1) + 2x'(1) + (x''(1))^3 = 1, \tag{4.4}$$

where p is an odd number.

Let

$$\begin{aligned} f(t, x, y, z) &= -xy^2 - t^2 z^3, \\ g(y) &= y^3, \\ \phi(x, y, z) &= -\frac{4}{\pi}\tan^{-1} x + 2y + z^3. \end{aligned}$$

Define

$$\alpha(t) = -t, \quad \beta(t) = t, \quad t \in [0, 1],$$

then $\alpha(t)$, $\beta(t)$ are lower and upper solutions of BVP (4.1)–(4.4). Furthermore, we find that f satisfies the sign-type Nagumo condition (N_+^*) in

$$D = \{(t, x, y, z) \in [0, 1] \times \mathbb{R}^3 : -t \leqslant x \leqslant t, \ -1 \leqslant y \leqslant 1\}$$

with $\Phi(z) = 2$. It is easy to prove that all the conditions of Theorem 3.1 are satisfied. Therefore, from Theorem 3.1, there exists a solution $x(t)$ for BVP (4.1)–(4.4) such that

$$-t \leqslant x(t) \leqslant t, \quad -1 \leqslant x'(t) \leqslant 1, \quad t \in [0, 1].$$

It is clear that the results of [**5**] do not apply to Example 4.1. It shows that the result in this paper is new and valuable.

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