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ASYMPTOTIC BEHAVIOUR OF THE FIRST POSITIONS OF UNIFORM PARKING FUNCTIONS

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Abstract

In this paper we study the asymptotic behaviour of a random uniform parking function π_n of size n. We show that the first k_n places $\pi_n(1), \ldots, \pi_n(k_n)$ of π_n are asymptotically independent and identically distributed (i.i.d.) and uniform on $\{1, 2, \ldots, n\}$, for the total variation distance when $k_n = \mathrm{o}(\sqrt{n})$, and for the Kolmogorov distance when $k_n = \mathrm{o}(n)$, improving results of Diaconis and Hicks. Moreover, we give bounds for the rate of convergence, as well as limit theorems for certain statistics such as the sum or the maximum of the first k_n parking places. The main tool is a reformulation using conditioned random walks.

Keywords: Random walks; total variation distance; Cayley trees

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Secondary 60G50; 60B10

1. Introduction

A *parking function* of size n is a function $\pi_n : [\![1, n]\!] \to [\![1, n]\!]$ such that, if $\pi_n'(1) \le \cdots \le \pi_n'(n)$ is the non-decreasing rearrangement of $(\pi_n(1), \ldots, \pi_n(n))$, then $\pi_n'(i) \le i$ for all i. Konheim and Weiss [16] first introduced parking functions, in the context of information storage, to study hashing functions, and they have shown that there are $(n+1)^{n-1}$ parking functions of size n. Since then, parking functions have become a subject of interest in the fields of combinatorics, probability, group theory, and computer science. More precisely, parking functions are linked to the enumerative theory of trees and forests [8], to coalescent processes [6, 7], to the analysis of set partitions [20], hyperplane arrangements [19, 21], polytopes [9, 22], and sandpile groups [10]. Finally, the study of probabilistic properties of parking functions has recently attracted some interest [11, 15, 25]. We refer to [24] for an extensive survey. Our initial interest in parking functions comes from the study of minimal factorisations of cycles [3].

For all $n \ge 1$, consider a random parking function $(\pi_n(i))_{1 \le i \le n}$ chosen uniformly among all the $(n+1)^{n-1}$ possible parking functions of size n. For all $1 \le k \le n$, let

$$d_{TV}(k, n) := \frac{1}{2} \sum_{i_1, \ldots, i_k = 1}^{n} \left| \mathbb{P}(\pi_n(1) = i_1, \ldots, \pi_n(k) = i_k) - \frac{1}{n^k} \right| \tag{1}$$

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denote the total variation distance between $(\pi_n(1), \dots, \pi_n(k))$ and $(U_n(1), \dots, U_n(k))$, where $(U_n(i))_{1\le i\le n}$ are independent and identically distributed (i.i.d.) and uniform in $[\![1, n]\!]$. Diaconis and Hicks [11, Corollary 6] have shown that $d_{TV}(1, n)$ tends to 0 as n tends to infinity, and conjectured that for any fixed k, $d_{TV}(k, n)$ should be $\mathrm{O}(k/\sqrt{n})$. In the same paper they studied the Kolmogorov distance

$$d_K(k, n) := \max_{1\le i_1\cdots i_k\le n} \left| \mathbb{P}(\pi_n(1) \le i_1, \dots, \pi_n(k) \le i_k) - \frac{i_1 \cdots i_k}{n^k} \right|, \tag{2}$$

and have shown that [11, Theorem 3] for $1 \le k \le n$,

$$d_K(k, n) = \mathrm{O}\left(k\sqrt{\frac{\log n}{n}} + \frac{k^2}{n} \right).$$

They also discuss the growth threshold of k at which d_K no longer converges towards 0, and find that for k of order n the convergence fails. We prove a stronger version of Diaconis and Hicks' conjecture when k is allowed to grow with n at rate at most $\sqrt{n}$. Moreover, the Kolmogorov distance converges towards 0 when $k = \mathrm{o}(n)$, as shown in the following result.

Theorem 1. (i) *If $k_n = \mathrm{o}(\sqrt{n})$, then*

$$d_{TV}(k_n, n) = \mathrm{O}\left(\frac{k_n}{\sqrt{n}} \right). \tag{3}$$

(ii) *If $k_n = \mathrm{o}(n)$ and $\sqrt{n} = \mathrm{o}(k_n)$, then*

$$d_K(k_n, n) = \mathrm{O}\left(\frac{\sqrt{n}}{k_n} + \left(\frac{k_n}{n} \right)^{0.19} \right). \tag{4}$$

Remark 1. In Theorem 1(ii), $\sqrt{n}$ is assumed to be $\mathrm{o}(k_n)$. Since the function $k \mapsto d_K(k, n)$ is non-decreasing for fixed n, the distance $d_K(k_n, n)$ still tends towards 0 as long as $k_n = \mathrm{o}(n)$. Thus sequence $a_n = n$ satisfies $d_K(k_n, a_n) \to 0$ if $k_n = \mathrm{o}(a_n)$ and $d_K(k_n, a_n) \not\to 0$ if $a_n = \mathrm{O}(k_n)$. It would be very interesting to identify such a sequence for d_{TV} instead of d_K, and, in particular, to see if $d_{TV}(k_n, n) \to 0$ when $k_n = \mathrm{o}(n)$.

The main idea in proving Theorem 1 is to express the law of π_n in terms of a conditioned random walk (Proposition 2 below) and to control its moments, uniformly in time (Proposition 4). Such uniform estimates on conditioned random walks are delicate to establish, and we believe them to be of independent interest. As an application, we obtain limit theorems for the maximum and the sum of the first k_n parking places. Namely we obtain the following corollary (whose proof is postponed to the last section).

Corollary 1. (i) *If $k_n = \mathrm{o}(\sqrt{n})$ and $k_n \to \infty$, then the convergence*

$$\sqrt{\frac{12}{k_n}} \left(\frac{\pi_n(1) + \cdots + \pi_n(k_n)}{n} - \frac{k_n}{2} \right) \longrightarrow \mathcal{N}(0, 1)$$

holds in distribution, where $\mathcal{N}(0, 1)$ is a standard normal distribution.

(ii) *If $k_n = \mathrm{o}(n)$ and $k_n \to \infty$, then the convergence*

$$k_n \left(1 - \frac{1}{n} \max\{\pi_n(1), \dots, \pi_n(k_n)\} \right) \longrightarrow \mathcal{E}(1)$$

holds in distribution, where $\mathcal{E}(1)$ is an exponential distribution with mean 1.

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Remark 2. The complete sum $\pi_n(1) + \cdots + \pi_n(n)$ has been studied, and converges, after renormalisation, towards a more complicated distribution involving zeros of the Airy function (see [11, Theorem 14]).

When $k_n \sim cn$ we obtain the following limit theorem for the first k_n parking places. The proof uses other techniques and Proposition 2 (or rather its proof).

Proposition 1. *If $k_n \sim cn$ with $c \in (0, 1]$, then for all $a \in \mathbb{N}$ there exists an integer-valued random variable S^*_a such that $0 \leq S^*_a \leq a$ almost surely and*

$$\mathbb{P}(n - \max\{\pi_n(1), \ldots, \pi_n(k_n)\} \geq a) \longrightarrow \mathbb{E}\big[(1-c)^{a-S^*_a}\big].$$

In Section 2 we use a bijection between parking functions and Cayley trees and use it to reformulate the law of π_n in terms of conditioned random walks. Then in Section 3 we bound the moments of a conditioned random walk in order to control the probability mass function of π_n and prove Theorem 1(i). In Section 4 we prove (ii) using arguments developed in the previous sections. The last section is devoted to the proof of Corollary 1 and Proposition 1.

In the following, C denotes a constant which may vary from line to line.

2. Bijection between parking functions and Cayley trees

Here the goal is to use the bijection found by Chassaing and Marckert [8] between parking functions of size n and *Cayley trees* with $n+1$ vertices (i.e. acyclic connected graphs with $n+1$ vertices labelled from 0 to n). This bijection will allow us to express the joint distribution of the first positions of a uniform parking function in terms of random walks. To this end, we start with some notation and definitions. Let $\mathfrak{C}_{n+1}$ be the set of Cayley trees with $n+1$ vertices labelled from 0 to n, where the vertex labelled 0 is distinguished from the others (we call it the *root* of the tree). Also, let P_n be the set of parking functions of size n. We consider the *breadth-first search* on a tree $t \in \mathfrak{C}_{n+1}$ by ordering the children of each vertex of t in increasing order of their labels (thus t is viewed as a plane tree) and then taking the regular breadth-first search associated with the plane order (see [8] for a detailed definition and see Figure 1 for an example). For $t \in \mathfrak{C}_{n+1}$ and $1 \leq i \leq n$, define $r(i, t)$ to be the rank for the breadth-first search of the parent of the vertex labelled i in t. The bijection of Chassaing and Marckert is described in the following theorem.

Theorem 2. (Chassaing and Marckert.) *The map*

$$t \mapsto (r(1, t), \ldots, r(n, t)) \tag{5}$$

is a bijection between $\mathfrak{C}_{n+1}$ and P_n.

Remark 3. Chassaing and Louchard [7] described a similar bijection using what they call the *standard* order instead of breadth-first search.

Let $(X_i)_{i\geq 1}$ be i.i.d. random variables distributed as a Poisson distribution of parameter 1. For all $n \geq 0$ we set $S_n := \sum_{i=1}^{n} (X_i - 1)$ and, for all $a \in \mathbb{Z}$, $\tau_a := \min\{n \geq 1 : S_n = a\}$ the first time that the random walk $(S_n)_n$ reaches a. Consider the probability measure $\mathbb{P}_n := \mathbb{P}(\cdot|\tau_{-1} = n+1)$ and set $\mathbb{E}_n := \mathbb{E}[\,\cdot\,|\tau_{-1} = n+1]$. It is well known that a Bienaymé–Galton–Watson tree with a critical Poisson reproduction law conditioned on having n vertices has the same distribution, when we uniformly randomly label the vertices from 1 to n, as a uniform Cayley tree with n

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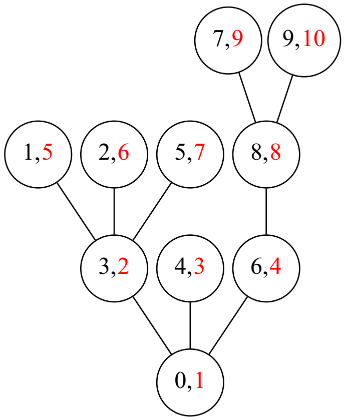

FIGURE 1. Example of a Cayley tree t with 10 vertices. For every vertex, its label is represented in black on the left and its rank for the breadth-first search is in red on the right. Here, for example, we have $r(5, t) = 2$. The parking function associated with this tree by (5) is (2, 2, 1, 1, 2, 1, 8, 4, 8).

vertices (see e.g. [14, Example10.2]). From this, Chassaing and Marckert deduce the following corollary.

Corollary 2. (Chassaing and Marckert.) *Let T_{n+1} be a random Cayley tree in $\mathfrak{C}_{n+1}$ with uniform distribution. The random vector $(\#\{1 \le j \le n : r(j, T_{n+1}) = i\})_{1 \le i \le n+1}$ has the same distribution as $(X_i)_{1 \le i \le n+1}$ under $\mathbb{P}_n$.*

We are now able to state and prove the main result of this section.

Proposition 2. *Fix $1 \le k \le n$ and $1 \le i_1, \ldots, i_k \le n$. Let $j_1 < \cdots < j_r$ be such that $\{i_1, \ldots, i_k\} = \{j_1, \ldots, j_r\}$ (so the vector $(j_1, \ldots, j_r)$ is the vector $(i_1, \ldots, i_k)$ where repeated elements have been reduced to a single one, and r is the number of distinct elements of $(i_1, \ldots, i_k)$). Define $m_s = \#\{u : i_u = j_s\}$ for all $1 \le s \le r$. Then*

$$\mathbb{P}(\pi_n(1) = i_1, \ldots, \pi_n(k) = i_k) = \frac{(n-k)!}{n!} \mathbb{E}_n\left[\prod_{s=1}^{r} (X_{j_s})_{m_s}\right], \tag{6}$$

where $(x)_m := x(x-1)\cdots(x-m+1)$.

Proof. Let T_{n+1} be a random Cayley tree in $\mathfrak{C}_{n+1}$ with uniform distribution. Let $\mathfrak{S}(k, n)$ denote the set of all injections between $[\![1, k]\!]$ and $[\![1, n]\!]$. We have

$$\begin{aligned}
\mathbb{P}(\pi_n(1) = i_1, \ldots, \pi_n(k) = i_k) &= \frac{(n-k)!}{n!} \sum_{\sigma \in \mathfrak{S}(k,n)} \mathbb{P}(\pi_n(\sigma(1)) = i_1, \ldots, \pi_n(\sigma(k)) = i_k) \\
&= \frac{(n-k)!}{n!} \mathbb{E}\left[\sum_{\sigma \in \mathfrak{S}(k,n)} \mathbb{1}_{r(\sigma(1), T_{n+1}) = i_1, \ldots, r(\sigma(k), T_{n+1}) = i_k}\right] \\
&= \frac{(n-k)!}{n!} \mathbb{E}\left[\prod_{s=1}^{r} (\#\{1 \le j \le n : r(j, T_{n+1}) = j_s\})_{m_s}\right] \\
&= \frac{(n-k)!}{n!} \mathbb{E}\left[\prod_{s=1}^{r} (X_{j_s})_{m_s}\right].
\end{aligned}$$

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The first equality comes from the fact that any permutation of a parking function is still a parking function, thus any permutation induces a bijection in P_n. The second equality comes from Theorem 2 and the last from Corollary 2. This completes the proof. □

3. Convergence for the total variation distance

In this section we suppose that $k_n = \mathrm{o}(\sqrt{n})$. We will write k instead of k_n to ease notation, but keep in mind that k depends on n. The goal of this section is to show item (i) of Theorem 1.

3.1. Probability that the parking places are distinct

The first step is to reduce the problem to distinct parking places; in this case equation (6) becomes easier. To this end we introduce the set of distinct indices

$$D_n := \{(u_1, \dots, u_k) \in [\![1, n]\!]^k : i \neq j \Rightarrow u_i \neq u_j\}.$$

We also introduce the set

$$G_n := \{(u_1, \dots, u_k) \in [\![1, n]\!]^k : \mathbb{P}(\pi_n(1) = u_1, \dots, \pi_n(k) = u_k) \geq (n-k)!/n!\}$$

and the quantity

$$\delta(k, n) := \sum_{\substack{(i_1,\dots,i_k) \\ \in D_n \cap G_n}} \left[\mathbb{P}(\pi_n(1) = i_1, \dots, \pi_n(k) = i_k) - \frac{(n-k)!}{n!}\right]. \tag{7}$$

The next lemma shows that the first k parking places of a uniform parking function are all distinct with high probability. It also shows that if $\delta(k, n)$ is $\mathrm{O}(k/\sqrt{n})$ then so is $d_{TV}(k, n)$. Recall that $(U_n(i))_{1\leq i\leq n}$ are i.i.d. uniformly distributed in $[\![1, n]\!]$.

Lemma 1. *We have*

(i) $\mathbb{P}((U_n(1), \dots, U_n(k)) \in D_n) = 1 + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right)$,

(ii) $\mathbb{P}((\pi_n(1), \dots, \pi_n(k)) \in D_n) = 1 + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right)$,

(iii) $\delta(k, n) = \mathrm{O}\left(\frac{k}{\sqrt{n}}\right) \Longrightarrow d_{TV}(k, n) = \mathrm{O}\left(\frac{k}{\sqrt{n}}\right)$.

Proof. Let μ_n be the law of $(\pi_n(1), \dots, \pi_n(k))$ and ν_n the law of $(U_n(1), \dots, U_n(k))$, with support on the same finite space $E_n := [\![1, n]\!]^k$. First we check (i). By Markov's inequality,

$$\nu_n(D_n^{\mathrm{c}}) = \mathbb{P}\left(\sum_{r<s} \mathbb{1}_{U_n(r)=U_n(s)} \geq 1\right) \leq \sum_{r<s} \mathbb{P}(U_n(r) = U_n(s)) = \sum_{r<s} \frac{1}{n} = \frac{1}{n}\frac{k(k-1)}{2}.$$

Since $k = \mathrm{o}(\sqrt{n})$, we have

$$\frac{1}{n}\frac{k(k-1)}{2} = \mathrm{O}\left(\frac{k^2}{n}\right) = \mathrm{O}\left(\frac{k}{\sqrt{n}}\right).$$

Now we check (ii). To do so, we will use the Prüfer encoding (or a slight variant thereof) of a rooted Cayley tree $t \in \mathfrak{C}_n$ into a sequence $(p_1, \dots, p_{n-2}) \in [\![0, n-1]\!]^{n-2}$, which we now

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explain. For $a \in [\![1, n-1]\!]$, define $p(t, a)$ to be the label of the parent of the vertex labelled a in t. Also, define $\ell(t)$ as the biggest leaf label of t, and t^* the tree t obtained after removing the leaf labelled $\ell(t)$ and its adjacent edge. Finally we define the sequence of trees $t_1 := t$, $t_i := t^*_{i-1}$ for $2 \leq i \leq n-2$. The Prüfer encoding of t is then defined as $p_i := p(t_i, \ell(t_i))$. For example, the Prüfer encoding of the tree in Figure 1 is (8, 8, 6, 0, 3, 0, 3, 3). The key property of this encoding is that it is a bijection between the sets $\mathfrak{C}_n$ and $[\![0, n-1]\!]^{n-2}$. Now, let T_{n+1} be a uniform Cayley tree in $\mathfrak{C}_{n+1}$. Theorem 2 implies that $\mu_n(D_n)$ is equal to the probability that the vertices labelled 1 to k in T_{n+1} have distinct parents. Let $(v_1, \ldots, v_k)$ be a random vector with uniform distribution in D_n independent of T_{n+1}. Since the distribution of T_{n+1} is invariant under permutation of the labels, the previous probability is also equal to the probability that the vertices labelled $v_1, \ldots, v_k$ have distinct parents in T_{n+1}. Let $(p_1, \ldots, p_{n-1})$ be the Prüfer encoding of the tree T_{n+1}. We complete this vector with $p_n := 0$ (this comes from the fact that t_{n-2} has two vertices, one of them being the root labelled 0). Since T_{n+1} is uniformly distributed in $\mathfrak{C}_{n+1}$, the vector $(p_1, \ldots, p_{n-1})$ is uniformly distributed in $[\![0, n]\!]^{n-1}$. From the previous discussion and the definition of the Prüfer encoding, we deduce that

$$\mu_n(D_n) = \mathbb{P}((p_{v_1}, \ldots, p_{v_k}) \in D_n).$$

Consider the event $Z_n := \{v_1, \ldots, v_k \neq n\}$. Under this event, it is easy to see that $(p_{v_1}, \ldots, p_{v_k})$ has the same law as k i.i.d. random variables uniformly distributed in $[\![0, n]\!]$. So from (i) we have

$$\mu_n(D_n) \geq \mathbb{P}((p_{v_1}, \ldots, p_{v_k}) \in D_n \mid Z_n)\, \mathbb{P}(Z_n) = \left(1 + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right)\right) \mathbb{P}(Z_n).$$

To conclude, notice that $\mathbb{P}(Z_n) = 1 - k/n$.

Finally we show (iii). Assume that $\delta(k, n) = \mathrm{O}(k/\sqrt{n})$. For all $i_1, \ldots, i_k$, let $\Delta_{i_1,\ldots,i_k}$ denote the quantity $(\mathbb{P}(\pi_n(1) = i_1, \ldots, \pi_n(k) = i_k) - (n-k)!/n!)$. Notice that $n^k(n-k)!/n! - 1 = \mathrm{O}(k/\sqrt{n})$, so

$$2d_{TV}(k, n) = \sum_{i_1,\ldots,i_k=1}^{n} |\Delta_{i_1,\ldots,i_k}| + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right).$$

Let d_n^+ denote the sum of $\Delta_{i_1,\ldots,i_k}$ over the indices in G_n and d_n^- the opposite of the sum over the indices in $E_n \setminus G_n$. We have

$$d_n^+ - d_n^- = \sum_{i_1,\ldots,i_k=1}^{n} \Delta_{i_1,\ldots,i_k} = 1 - \frac{n^k(n-k)!}{n!} = \mathrm{O}\left(\frac{k}{\sqrt{n}}\right).$$

The last two equalities imply that

$$2d_{TV}(k, n) = d_n^+ + d_n^- + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right) = 2d_n^+ + \mathrm{O}\left(\frac{k}{\sqrt{n}}\right).$$

In conclusion we just need to show that d_n^+ is $\mathrm{O}(k/\sqrt{n})$. Notice that

$$d_n^+ = \delta(k, n) + \mu_n(D_n^{\mathrm{c}} \cap G_n) - \nu_n(D_n^{\mathrm{c}} \cap G_n) \times \frac{n^k(n-k)!}{n!}.$$

From (i), (ii), and the assumption on $\delta(k, n)$, we deduce that d_n^+ is indeed $\mathrm{O}(k/\sqrt{n})$. □

To prove (i) of Theorem 1 it remains to show that $\delta(k, n) = \mathrm{O}(k/\sqrt{n})$. This is the goal of the next three sections.

3.2. A monotonicity argument

In this section we bound the terms $\mathbb{E}_n[X_{i_1}\cdots X_{i_k}]$ that appear in (6) when $i_1, \dots, i_k$ are distinct with terms involving $\mathbb{E}_n[S_{i_1}\cdots S_{i_k}]$, since the latter are more manageable. More precisely, the aim of this section is to prove the following result.

Proposition 3. *Fix $i_1, \dots, i_k \in [\![1, n]\!]$ pairwise distinct. We have*

$$i_1 \cdots i_k\, \mathbb{E}_n[X_{i_1}\cdots X_{i_k}] \le \mathbb{E}_n[(S_{i_1}+i_1)\cdots(S_{i_k}+i_k)]. \tag{8}$$

To prove Proposition 3 we first state a really useful lemma which, put in simple terms, says that the steps of the random walk S tend to decrease under $\mathbb{P}_n$.

Lemma 2. *Fix $n \ge k \ge 1$ and $m_1, \dots, m_k \ge 1$. Let $1 \le i_1 < \cdots < i_k \le n$ and $1 \le j_1 < \cdots < j_k \le n$ such that $j_r \le i_r$ for all $1 \le r \le k$. Let $f : \mathbb{N} \times \mathbb{N} \setminus \{0\} \mapsto [0, \infty)$ be a non-negative function such that $f(0, m) = 0$. Then*

$$\mathbb{E}_n[f(X_{i_1}, m_1)\cdots f(X_{i_k}, m_k)] \le \mathbb{E}_n[f(X_{j_1}, m_1)\cdots f(X_{j_k}, m_k)]. \tag{9}$$

Proof of Lemma 2. To ease notation, we define the random vector $\mathbf{X} := (X_1, \dots, X_{n+1})$ and write $\mathbf{x}$ as a shortcut to designate an element $(x_1, \dots, x_{n+1})$ of $\mathbb{N}^{n+1}$. Let $s := \min\{r \ge 1 : j_r < i_r\}$. We only need to treat the case where $i_r = j_r$ for all $r \ne s$ and $j := j_s = i_s - 1$ (the general result can then be obtained by induction). Let $\sigma = (j\, j+1) \in \mathfrak{S}_{n+1}$ be the permutation that transposes j and $j+1$. Let

$$\mathcal{E}_n := \{\mathbf{x} \in \mathbb{N}^{n+1} : (x_1 - 1) + \cdots + (x_t - 1) = -1 \text{ if and only if } t = n+1\}$$

and let

$$\mathcal{E}'_n := \{\mathbf{x} \in \mathcal{E}_n : x_{j+1} > 0 \text{ or } (x_1 - 1) + \cdots + (x_{j-1} - 1) > 0\}.$$

Note that $(x_1, \dots, x_{n+1}) \mapsto (x_{\sigma(1)}, \dots, x_{\sigma(n+1)})$ is a bijection on $\mathcal{E}'_n$. Then

$$\begin{aligned}\mathbb{E}\left[\prod_{r=1}^{k} f(X_{i_r}, m_r)\mathbb{1}_{\tau_{-1}=n+1}\right] &= \sum_{\mathbf{x}\in\mathcal{E}_n}\prod_{r=1}^{k} f(x_{i_r}, m_r)\,\mathbb{P}[\mathbf{X}=\mathbf{x}] \\ &= \sum_{\mathbf{x}\in\mathcal{E}'_n}\prod_{r=1}^{k} f(x_{\sigma(i_r)}, m_r)\,\mathbb{P}[\mathbf{X}=\mathbf{x}] \\ &\quad + \sum_{\mathbf{x}\in\mathcal{E}_n\setminus\mathcal{E}'_n}\prod_{r=1}^{k} f(x_{i_r}, m_r)\,\mathbb{P}[\mathbf{X}=\mathbf{x}].\end{aligned}$$

If $\mathbf{x} \in \mathcal{E}_n \setminus \mathcal{E}'_n$, then $f(x_{j+1}, m_{j+1}) = f(0, m_{j+1}) = 0$; in particular, since f is non-negative,

$$f(x_{i_1}, m_1)\cdots f(x_{i_k}, m_k) \le f(x_{\sigma(i_1)}, m_1)\cdots f(x_{\sigma(i_k)}, m_k).$$

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Finally

$$\mathbb{E}\left[\prod_{r=1}^{k} f(X_{i_r}, m_r)\mathbb{1}_{\tau_{-1}=n+1}\right] \le \sum_{\mathbf{x}\in\mathcal{E}_n}\prod_{r=1}^{k} f(x_{\sigma(i_r)}, m_r)\,\mathbb{P}[\mathbf{X}=\mathbf{x}]$$
$$= \mathbb{E}\left[\prod_{r=1}^{k} f(X_{\sigma(i_r)}, m_r)\mathbb{1}_{\tau_{-1}=n+1}\right]$$
$$= \mathbb{E}\left[\prod_{r=1}^{k} f(X_{j_r}, m_r)\mathbb{1}_{\tau_{-1}=n+1}\right]. \qquad \square$$

Remark 4. In Lemma 2 we can for instance take $f(x, m) = x^m$ or $f(x, m) = (x)_m$. Note that in Lemma 2 the indices $(i_r)_r$ must be pairwise distinct as well as the indices $(j_r)_r$. In the proof of Proposition 3, we extend the result for $f(x, m) = x^m$ to the case where only the $(i_r)_r$ are pairwise distinct.

Proof of Proposition 3. First we show the following inequality. Fix $n \ge k \ge 1$. Let $1 < i_1 < \cdots < i_k \le n$, $1 \le j_1 \le \cdots \le j_k \le n$ be such that $j_r \le i_r$ for all $1 \le r \le k$. Then

$$\mathbb{E}_n[X_{i_1}\cdots X_{i_k}] \le \mathbb{E}_n[X_{j_1}\cdots X_{j_k}]. \tag{10}$$

To show (10) it is actually enough to show the following result. Let $J \subset [\![1, n]\!]$ and $2 \le i \le n$ such that i and $i-1$ do not belong to J. Let $m_j \ge 1$ for $j \in J$ and $m \ge 1$. Then

$$\mathbb{E}_n\left[X_{i-1}^m X_i \prod_{j\in J} X_j^{m_j}\right] \le \mathbb{E}_n\left[X_{i-1}^{m+1}\prod_{j\in J} X_j^{m_j}\right]. \tag{11}$$

Inequality (10) can then be obtained by induction using Lemma 2 and (11). By Young's inequality,

$$X_{i-1}^m X_i \le \frac{m}{m+1}X_{i-1}^{m+1} + \frac{1}{m+1}X_i^{m+1}.$$

Combining this with Lemma 2 gives (11) and concludes the proof of (10). Now, using inequality (10), we obtain

$$i_1\cdots i_k\,\mathbb{E}_n[X_{i_1}\cdots X_{i_k}] \le \sum_{j_r\le i_r}\mathbb{E}_n[X_{j_1}\cdots X_{j_k}] = \mathbb{E}_n[(S_{i_1}+i_1)\cdots(S_{i_k}+i_k)],$$

which concludes the proof of Proposition 3. $\square$

3.3. Bounding the moments of a random walk conditioned to be an excursion

The goal of this section is to find bounds for the moments of the random walk S conditioned to be an excursion. More precisely, the aim of this section is to show the following result.

Proposition 4. *There exists a constant $C > 0$ such that for all $n \ge 2$, $1 \le k \le n-1$ and $d \ge 1$,*

(i) *we have*

$$\mathbb{E}_n\left[S_k^d\right] \le (Cdn)^{d/2}, \tag{12}$$

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(ii) *and*

$$\mathbb{E}_n\big[S_{n-k}^d\big] \le \left(\frac{n}{n-k}\right)^{3/2} (Cdk)^{d/2}, \tag{13}$$

(iii) *as well as*

$$\mathbb{E}_n\big[S_k^d\big] \le \left(\frac{n}{n-k}\right)^{3/2} \big(Cd\sqrt{k}\big)^d. \tag{14}$$

Remark 5. Items (i) and (ii) of Proposition 4 actually hold true for any random walk $(S_n)_{n\geq 0}$ starting from 0 with i.i.d. increments all having the distribution μ whose support is $\mathbb{N}\cup\{-1\}$ and such that μ has mean 0 and finite variance. However, (iii) uses in addition the fact that a Poisson random walk has a sub-exponential tail (see e.g. [23, Example 3]), namely, for all $k \geq 1$ and $x \geq 0$,

$$\mathbb{P}(S_k \ge x) \le \exp\left(-\frac{x^2}{2(k+x)}\right). \tag{15}$$

To prove Proposition 4 we will use the following lemma, whose proof is postponed to the end of this section. This lemma is similar to the cyclic lemma in spirit, but instead of conditioning the walk to be an excursion we only condition it to stay positive.

Lemma 3. *Let $n \geq 1$ and $F : \mathbb{R}^n \to [0, +\infty)$ be invariant under cyclic shifts. Then*

$$\mathbb{E}[F(X_1, \ldots, X_n)\mathbb{1}_{S_1,\ldots,S_n>0}] \le \frac{1}{n}\mathbb{E}[F(X_1, \ldots, X_n)(S_n \wedge n)\mathbb{1}_{S_n>0}]. \tag{16}$$

Proof of Proposition 4. We recall that C denotes a constant which may vary from line to line. For (i), according to [2, equation (32)], the maximum of the excursion of S has a sub-Gaussian tail, namely there exist constants $C, \alpha > 0$ such that, for all $n \geq 1$ and $x \geq 0$,

$$\mathbb{P}_n(M_n \ge x) \le C\mathrm{e}^{-\alpha x^2/n},$$

where $M_n := \max\{S_0, \ldots, S_n\}$ is the maximum of the walk S on $[0, n]$. So we have

$$\begin{aligned}
\mathbb{E}_n\big[n^{-d/2}S_k^d\big] &\le \mathbb{E}_n\big[n^{-d/2}M_n^d\big] \\
&= \int_0^\infty \mathrm{d}x^{d-1}\mathbb{P}_n\big(M_n \ge \sqrt{n}x\big)\,\mathrm{d}x \\
&\le \int_0^\infty C\mathrm{d}x^{d-1}\,\mathrm{e}^{-\alpha x^2}\,\mathrm{d}x \\
&\le C^d d^{d/2}.
\end{aligned}$$

This shows (i).

The following computation is a common starting point to show both (ii) and (iii). Let $H(S_k, x)$ be either the indicator function $\mathbb{1}_{S_k=x}$ or $\mathbb{1}_{S_k\geq x}$ with $x > 0$. Using the fact that $\mathbb{P}(\tau_{-1} = n+1)$ is equivalent to a constant times $n^{-3/2}$ (see e.g. [17, equation (10)]), and then using the Markov property and finally the cyclic lemma (see e.g. [18, Section 6.1]), we get

$$\begin{aligned}
\mathbb{E}_n[H(S_k, x)] &\le C\,n^{3/2}\,\mathbb{E}[H(S_k, x)\,\mathbb{1}_{\tau_{-1}=n+1}] \\
&\le C\,n^{3/2}\,\mathbb{E}[H(S_k, x)\,\mathbb{1}_{S_1,\ldots,S_k\geq 0}\,\mathbb{P}(\tau_{-1-S_k} = n-k)] \\
&\le C\,n^{3/2}\,\mathbb{E}\left[H(S_k, x)\,\mathbb{1}_{S_1,\ldots,S_k\geq 0}\,\frac{1+S_k}{n-k}\,\mathbb{P}(S'_{n-k} = -1-S_k)\right],
\end{aligned}$$

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where S' is independent of S and has the same distribution. Now we use Janson's inequality [13, Lemma 2.1], which states that $\mathbb{P}(S_r = -m) \le Cr^{-1/2}\,\mathrm{e}^{-\lambda m^2/r}$ for all $r \ge 1$ and $m \ge 0$, to get

$$\mathbb{E}_n[H(S_k, x)] \le C\, n^{3/2}\, \mathbb{E}\Bigg[H(S_k, x)\, \mathbb{1}_{S_1,\ldots,S_k\ge 0}\, \frac{1+S_k}{(n-k)^{3/2}}\, \mathrm{e}^{-\lambda(1+S_k)^2/(n-k)}\Bigg]$$
$$\le C\left(\frac{n}{n-k}\right)^{3/2} \mathbb{E}\big[H(S_k, x)\, \mathbb{1}_{S_1,\ldots,S_k\ge 0}\, S_k\, \mathrm{e}^{-\lambda S_k^2/(n-k)}\big]. \quad (17)$$

To prove (ii) we first define $T_k := \max\{0 \le i \le k : S_i = 0\}$. Let $p_k(x)$ denote the probability of the event $\{S_k = x,\ S_1, \ldots, S_k \ge 0\}$. Using the Markov property, we obtain

$$p_k(x) = \sum_{i=0}^{k-1} \mathbb{P}(S_k = x,\ S_1, \ldots, S_k \ge 0,\ T_k = i)$$
$$= \sum_{i=0}^{k-1} \mathbb{P}(S_1, \ldots, S_i \ge 0,\ S_i = 0)\, \mathbb{P}(S_{k-i} = x,\ S_1, \ldots, S_{k-i} > 0).$$

We apply Lemma 3 and the local limit theorem (see e.g. [12, Theorem 4.2.1]), which gives a constant $C > 0$ such that $\mathbb{P}(S_{k-i} = x) \le C(k-i)^{-1/2}$, for every $k > i$ and $x \in \mathbb{Z}$, so that

$$p_k(x) \le \sum_{i=0}^{k-1} \mathbb{P}(S_1, \ldots, S_i \ge 0,\ S_i = 0)\frac{x}{k-i}\mathbb{P}(S_{k-i} = x)$$
$$\le C\, x \sum_{i=0}^{k-1} \mathbb{P}(S_1, \ldots, S_i \ge 0,\ S_i = 0)\frac{1}{(k-i)^{3/2}}.$$

Notice that

$$\mathbb{P}(S_1, \ldots, S_i \ge 0,\ S_i = 0) = e\, \mathbb{P}(\tau_{-1} = i+1) \le \frac{C}{(i+1)^{3/2}}.$$

So, finally we have

$$p_k(x) \le C\, x \sum_{i=0}^{k-1} \frac{1}{((i+1)(k-i))^{3/2}} = \frac{C\, x}{(k+1)^{3/2}} \sum_{i=1}^{k-1}\left(\frac{1}{i+1} + \frac{1}{k-i}\right)^{3/2} \le \frac{C\, x}{k^{3/2}}.$$

Putting the previous inequality in (17) with $H(S_k, x) = \mathbb{1}_{S_k = x}$ and replacing k with $n-k$ gives

$$\mathbb{P}_n(S_{n-k} = x) \le C\left[\frac{n}{k(n-k)}\right]^{3/2} x^2\, \mathrm{e}^{-\lambda x^2/k}.$$

We can now bound the dth moment of S_{n-k}:

$$\mathbb{E}_n\big[S_{n-k}^d\big] \le C\left[\frac{n}{k(n-k)}\right]^{3/2} \int_0^\infty x^{d+2}\, \mathrm{e}^{-\lambda x^2/k}\, \mathrm{d}x$$
$$\le C^d k^{d/2}\left[\frac{n}{n-k}\right]^{3/2} \int_0^\infty x^{d+2}\, \mathrm{e}^{-x^2}\, \mathrm{d}x$$
$$\le C^d k^{d/2}\left[\frac{n}{n-k}\right]^{3/2} d^{d/2}.$$

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This concludes the proof of (ii).

To prove (iii) we follow the same principle. Using the Markov property and Lemma 3, we have

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{S_k\geq x}\,\mathbb{1}_{S_1,\dots,S_k\geq 0}\,S_k] &= \sum_{i=0}^{k-1}\mathbb{E}[\mathbb{1}_{S_k\geq x}\,\mathbb{1}_{S_1,\dots,S_k\geq 0}\,S_k\,\mathbb{1}_{T_k=i}]\\ &= \sum_{i=0}^{k-1}\mathbb{P}(S_1,\,\dots,\,S_i\geq 0,\;S_i=0)\mathbb{E}[\mathbb{1}_{S_{k-i}\geq x}\,\mathbb{1}_{S_1,\dots,S_{k-i}>0}\,S_{k-i}]\\ &\leq \sum_{i=0}^{k-1}\mathbb{P}(S_1,\,\dots,\,S_i\geq 0,\;S_i=0)\frac{1}{k-i}\mathbb{E}\big[\mathbb{1}_{S_{k-i}\geq x}\,S_{k-i}^2\big].\end{aligned}$$

Then we apply the Cauchy–Schwarz inequality:

$$\mathbb{E}\big[\mathbb{1}_{S_{k-i}\geq x}\,S_{k-i}^2\big]\leq \mathbb{P}(S_{k-i}\geq x)^{1/2}\mathbb{E}\big[S_{k-i}^4\big]^{1/2}\leq C(k-i)\mathbb{P}(S_{k-i}\geq x)^{1/2}.$$

The last inequality comes from an explicit computation of the fourth central moment of a Poisson distribution. We combine the last inequality with (15) to get

$$\mathbb{E}\big[\mathbb{1}_{S_{k-i}\geq x}\,S_{k-i}^2\big]\leq C(k-i)\;\exp\bigg(-\frac{x^2}{4(k-i+x)}\bigg)\leq C(k-i)\;\exp\bigg(-\frac{x^2}{4(k+x)}\bigg).$$

Putting everything together, we obtain

$$\begin{aligned}\mathbb{E}[\mathbb{1}_{S_k\geq x}\,\mathbb{1}_{S_1,\dots,S_k\geq 0}\,S_k] &\leq C\exp\bigg(-\frac{x^2}{4(k+x)}\bigg)\sum_{i=0}^{k-1}\mathbb{P}(S_1,\,\dots,\,S_i\geq 0,\;S_i=0)\\ &\leq C\exp\bigg(-\frac{x^2}{4(k+x)}\bigg)\sum_{i=0}^{k-1}e\,\mathbb{P}(\tau_{-1}=i+1)\\ &\leq C\exp\bigg(-\frac{x^2}{4(k+x)}\bigg).\end{aligned}$$

We combine the last inequality with (17) to get

$$\begin{aligned}\mathbb{E}_n\big[k^{-d/2}S_k^d\big] &= \int_0^\infty \mathrm{d}x^{d-1}\mathbb{P}_n\big(S_k\geq\sqrt{k}x\big)\,\mathrm{d}x\\ &\leq C\bigg[\frac{n}{n-k}\bigg]^{3/2}\int_0^\infty \mathrm{d}x^{d-1}\;\exp\bigg(-\frac{kx^2}{4(k+x\sqrt{k})}\bigg)\,\mathrm{d}x.\end{aligned}$$

We cut the last integral into two parts and obtain

$$\int_0^\infty \mathrm{d}x^{d-1}\;\exp\bigg(-\frac{kx^2}{4(k+x\sqrt{k})}\bigg)\,\mathrm{d}x\leq\int_0^{\sqrt{k}}\mathrm{d}x^{d-1}\,\mathrm{e}^{-x^2/8}\,\mathrm{d}x+\int_{\sqrt{k}}^\infty \mathrm{d}x^{d-1}\,\mathrm{e}^{-x/8}\,\mathrm{d}x.$$

Noticing that the last two integrals are both smaller than $C^d d^d$, this concludes the proof of (iii). □

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We now prove Lemma 3.

Proof of Lemma 3. For $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ and $i \in [\![0, n-1]\!]$, let x^i denote the ith cyclic permutation of x, namely $x^i := (x_{1+i}, \dots, x_n, x_1, \dots, x_i)$. Consider the set

$$A_n := \left\{ (x_1, \dots, x_n) \in \mathbb{N}^n : \text{for all } k \in [\![1, n]\!], \sum_{i=1}^{k} (x_i - 1) > 0 \right\}$$

and write $X := (X_1, \dots, X_n)$. Then

$$\begin{aligned}
\mathbb{E}[F(X_1, \dots, X_n) \mathbb{1}_{S_1,\dots,S_n>0}] &= \mathbb{E}[F(X) \mathbb{1}_{X \in A_n}] \\
&= \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{E}[F(X^i) \mathbb{1}_{X^i \in A_n}] \qquad (18) \\
&= \frac{1}{n} \mathbb{E}\left[F(X) \sum_{i=0}^{n-1} \mathbb{1}_{X^i \in A_n} \right] \\
&\leq \frac{1}{n} \mathbb{E}[F(X)(S_n \wedge n) \mathbb{1}_{S_n>0}].
\end{aligned}$$

The inequality comes from the fact that the number of cyclic shifts of X such that $X^i \in A_n$ is almost surely bounded by $(S_n \wedge n)\mathbb{1}_{S_n>0}$. □

3.4. Proof of Theorem 1(i)

Recall we want to show that $d_{TV}(k, n) = \mathrm{O}(k/\sqrt{n})$. In Section 3.1 we have shown that it is enough to show $\delta(k, n) = \mathrm{O}(k/\sqrt{n})$, where the definition of $\delta(k, n)$ is given by (7). Thanks to Proposition 2, this quantity can be rewritten as

$$\delta(k, n) = \frac{n^k (n-k)!}{n!} \int_{\Lambda_k} [\mathbb{E}_n[X_{\lceil nt_1 \rceil} \cdots X_{\lceil nt_k \rceil}] - 1] \, \mathrm{d}t_1 \cdots \mathrm{d}t_k,$$

where

$$\Lambda_k := \{(t_1, \dots, t_k) \in (0, 1]^k : (\lceil nt_1 \rceil, \dots, \lceil nt_k \rceil) \in D_n \cap G_n\}.$$

As was already mentioned in Section 3.1, $n^k(n-k)!/n! = 1 + \mathrm{O}(k/\sqrt{n})$, so for our purpose it is sufficient to bound the integral

$$I(k, n) := \int_{\Lambda_k} [\mathbb{E}_n[X_{\lceil nt_1 \rceil} \cdots X_{\lceil nt_k \rceil}] - 1] \, \mathrm{d}t_1 \cdots \mathrm{d}t_k.$$

Using inequality (8), we obtain

$$I(k, n) \leq \int_{[0,1]^k} \left[\mathbb{E}_n \left[\left(\frac{S_{\lceil nt_1 \rceil}}{\lceil nt_1 \rceil} + 1 \right) \cdots \left(\frac{S_{\lceil nt_k \rceil}}{\lceil nt_1 \rceil} + 1 \right) \right] - 1 \right] \mathrm{d}t_1 \cdots \mathrm{d}t_k. \qquad (19)$$

Notice that (i) and (iii) of Proposition 4 imply that for all $0 \leq k \leq n$

$$\mathbb{E}_n \left[S_k^d \right] \leq \left(Cd\sqrt{k} \right)^d. \qquad (20)$$

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Indeed, Proposition 4(i) covers the case $k \geq n/2$ of equation (20) and (iii) covers the case $k < n/2$ (notice that the cases $k = 0$ and $k = n$ are trivial, since, conditionally on $\tau_{-1} = n + 1$, $S_0 = S_n = 0$). Hölder's inequality shows that for $0 \leq i_1, \ldots, i_d \leq n$

$$\mathbb{E}_n[S_{i_1} \cdots S_{i_d}] \leq (Cd)^d \sqrt{i_1 \cdots i_d}. \tag{21}$$

Expanding the products in (19) and using (21) gives

$$I(k, n) \leq \sum_{d=1}^{k} \binom{k}{d} (Cd)^d n^{-d/2} \int_{[0,1]^d} \frac{\mathrm{d}t_1 \cdots \mathrm{d}t_d}{(t_1 \cdots t_d)^{1/2}}.$$

Notice that $t \mapsto t^{-1/2}$ is integrable, so

$$I(k, n) \leq \sum_{d=1}^{k} \binom{k}{d} (Cd)^d n^{-d/2}.$$

Using the bound $\binom{k}{d} \leq (ke/d)^d$,

$$I(k, n) \leq \sum_{d=1}^{k} \left(Ce \frac{k}{\sqrt{n}} \right)^d.$$

Since $k = \mathrm{o}(\sqrt{n})$, we conclude that $I(k, n) = \mathrm{O}(k/\sqrt{n})$.

4. Convergence for the Kolmogorov distance

In this section we suppose that $k_n = \mathrm{o}(n)$ and $\sqrt{n} = \mathrm{o}(k_n)$. We will write k instead of k_n to ease notation, but keep in mind that k depends on n. The goal of this section is to show Theorem 1(ii). The following lemma allows us to replace the cumulative probability in (2) with the term $\mathbb{E}_n[(S_{i_1} + i_1) \cdots (S_{i_n} + i_n)]$, which is more manageable.

Lemma 4. *There is a constant $C > 0$ such that*

$$d_K(k, n) \leq \frac{1}{n^k} \max_{1 \leq i_1 \cdots i_k \leq n} |\mathbb{E}_n[(S_{i_1} + i_1) \cdots (S_{i_k} + i_k)] - i_1 \cdots i_k| + \frac{Ck}{n}. \tag{22}$$

Before proving this lemma we show how it implies Theorem 1(ii). We will also need the following simple lemma, which extends [4, equation (27.5)].

Lemma 5. *Let $r \geq 1$, $w_1, \ldots, w_r$, and $z_1, \ldots, z_r$ be complex numbers of modulus smaller than or equal to $a > 0$ and $b > 0$, respectively. Then*

$$\left| \prod_{i=1}^{r} w_i - \prod_{i=1}^{r} z_i \right| \leq \sum_{i=1}^{r} |w_i - z_i| a^{r-i} b^{i-1}. \tag{23}$$

Proof of Lemma 5. The result readily follows from the identity

$$\prod_{i=1}^{r} w_i - \prod_{i=1}^{r} z_i = (w_1 - z_1) \prod_{i=2}^{r} w_i + z_1 \left(\prod_{i=2}^{r} w_i - \prod_{i=2}^{r} z_i \right).$$

□

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Proof of Theorem 1(ii). Let $1/2 < \alpha < 1$. We define a sequence of intervals $I_1, \ldots, I_{M+3}$ (depending on n) in the following way:

$$I_1 := [1, n-k), \quad I_2 := [n-k, n-k^{\alpha}), \quad I_3 := \left[n-k^{\alpha}, n-k^{\alpha^2}\right), \quad \ldots$$
$$I_{M+1} := \left[n-k^{\alpha^{M-1}}, n-k^{\alpha^M}\right), \quad I_{M+2} := \left[n-k^{\alpha^M}, n-n/k\right), \quad I_{M+3} := [n-n/k, n],$$

where M is the biggest integer such that $n-k^{\alpha^M} \le n-n/k$. Let $1 \le i_1, \ldots, i_k \le n$. Using Lemma 5 (with $a=b=1$ and noticing that $(S_i+i)/n$ under $\mathbb{P}_n$ is almost surely smaller than 1 for every i), we decompose the quantity $\mathbb{E}_n[(S_{i_1}+i_1)\cdots(S_{i_k}+i_k) - i_1\cdots i_k]$ depending on those intervals to which the i_j belong, so that

$$\mathbb{E}_n\left[\left(\frac{S_{i_1}+i_1}{n}\right)\cdots\left(\frac{S_{i_k}+i_k}{n}\right) - \frac{i_1}{n}\cdots\frac{i_k}{n}\right] \le \sum_{m=1}^{M+3}\mathbb{E}_n\left[\prod_{i_j\in I_m}\frac{S_{i_j}+i_j}{n} - \prod_{i_j\in I_m}\frac{i_j}{n}\right]. \tag{24}$$

Fix $m \in \{1, \ldots, M+3\}$ and let $\iota_1, \ldots, \iota_{r_m}$ denote the i_j that belong to I_m. If $m=1$, then by Lemma 5 and Proposition 4(i),

$$\begin{aligned}\mathbb{E}_n\left[\prod_{i_j\in I_1}\frac{S_{i_j}+i_j}{n} - \prod_{i_j\in I_1}\frac{i_j}{n}\right] &\le \frac{1}{n}\sum_{j=1}^{r_1}\mathbb{E}_n[S_{\iota_j}]\left(\frac{n-k}{n}\right)^{j-1}\\ &\le \frac{C\sqrt{n}}{n}\frac{n}{k}\\ &= \frac{C\sqrt{n}}{k}.\end{aligned}$$

If $2 \le m \le M+2$, we follow the same principle but we use Proposition 4(ii) instead:

$$\begin{aligned}\mathbb{E}_n\left[\prod_{i_j\in I_m}\frac{S_{i_j}+i_j}{n} - \prod_{i_j\in I_m}\frac{i_j}{n}\right] &\le \frac{1}{n}\sum_{j=1}^{r_m}\mathbb{E}_n[S_{\iota_j}]\left(\frac{n-k^{\alpha^{m-1}}}{n}\right)^{j-1}\\ &\le \frac{Ck^{\alpha^{m-2}/2}}{n}\frac{n}{k^{\alpha^{m-1}}}\\ &= \frac{C}{k^{(\alpha-1/2)\alpha^{m-2}}}.\end{aligned}$$

The previous computation works in the case $m=M+2$ because the maximality of M implies that $n-n/k \le n-k^{\alpha^{M+1}}$. Finally, if $m=M+3$,

$$\begin{aligned}\mathbb{E}_n\left[\prod_{i_j\in I_{M+3}}\frac{S_{i_j}+i_j}{n} - \prod_{i_j\in I_{M+3}}\frac{i_j}{n}\right] &\le \frac{1}{n}\sum_{j=1}^{r_{M+3}}\mathbb{E}_n[S_{\iota_j}]\\ &\le r_{M+3}\frac{C}{n}\left(\frac{n}{k}\right)^{1/2}\\ &\le C\left(\frac{k}{n}\right)^{1/2}.\end{aligned}$$

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Notice that $k^{\alpha^M} \geq n/k$, so for all $0 \leq m \leq M$, $k^{\alpha^{M-m}} \geq (n/k)^{\alpha^{-m}}$. Summing the preceding bounds for $2 \leq m \leq M+2$ gives

$$\begin{aligned}
\sum_{m=2}^{M+2} \mathbb{E}_n\left[\prod_{i_j \in I_m} \frac{S_{i_j}+i_j}{n} - \prod_{i_j \in I_m} \frac{i_j}{n}\right] &\leq C \sum_{m=0}^{M} \frac{1}{k^{(\alpha-1/2)\alpha^m}} \\
&= C \sum_{m=0}^{M} \frac{1}{k^{(\alpha-1/2)\alpha^{M-m}}} \\
&\leq C \sum_{m=0}^{M} \left(\frac{k}{n}\right)^{(\alpha-1/2)\alpha^{-m}} \\
&\leq C\left(\frac{k}{n}\right)^{(\alpha-1/2)} + C \sum_{m=1}^{M} \left(\frac{k}{n}\right)^{-(\alpha-1/2)e \ln(\alpha) m} \\
&\leq C\left(\frac{k}{n}\right)^{(\alpha-1/2)} + C\left(\frac{k}{n}\right)^{-(\alpha-1/2)e \ln(\alpha)},
\end{aligned}$$

where we used the fact that $-e \ln(\alpha) m \leq 1/\alpha^m$ for all $m \geq 1$. If we take α such that $\alpha - 1/2 = \exp(-\mathrm{e}^{-1}) - 1/2 \simeq 0.1922$, then the last quantity is $\mathrm{O}(k/n)^{0.19}$. Putting everything together, we finally obtain

$$\frac{1}{n^k} \max_{1 \leq i_1 \cdots i_k \leq n} \left|\mathbb{E}_n\left[\prod_{r=1}^{k}(S_{i_r}+i_r)\right] - \prod_{r=1}^{k} i_r\right| \leq C\left[\frac{\sqrt{n}}{k} + \left(\frac{k}{n}\right)^{0.19} + \left(\frac{k}{n}\right)^{1/2}\right].$$

Combining the last display with Lemma 4 gives the desired result. □

Now we prove Lemma 4.

Proof of Lemma 4. Define the empirical distribution function of π_n, namely, for all $i \in \{1, \ldots, n\}$,

$$F_n(i) := \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{\pi_n(j) \leq i}.$$

As suggested in [11], we can use a result of Bobkov [5, Theorem 1.1]. The sequence $(\pi_n(1), \ldots, \pi_n(n))$ is an exchangeable extension of $(\pi_n(1), \ldots, \pi_n(k))$, meaning that the distribution of $(\pi_n(1), \ldots, \pi_n(n))$ stays the same after any permutation. So by Theorem 1.1 of [5] we have

$$\max_{1 \leq i_1 \cdots i_k \leq n} |\mathbb{P}(\pi_n(1) \leq i_1, \ldots, \pi_n(k) \leq i_k) - \mathbb{E}[F_n(i_1) \cdots F_n(i_k)]| \leq C\frac{k}{n},$$

where C is a universal constant. Using Proposition 2 and Corollary 2, we find

$$\mathbb{E}[F_n(i_1) \cdots F_n(i_k)] = \frac{1}{n^k}\mathbb{E}_n[(S_{i_1}+i_1) \cdots (S_{i_k}+i_k)].$$

Indeed, Proposition 2 implies that $(F_n(1), \ldots, F_n(n))$ and $(G_n(1), \ldots, G_n(n))$ have the same distribution, with

$$G_n(i) := \frac{1}{n} \sum_{j=1}^{n} \mathbb{1}_{r(j, T_{n+1}) \leq i},$$

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where T_{n+1} is a uniform random tree of $\mathfrak{C}_{n+1}$. Then Corollary 2 shows that

$$G_n(i) = \frac{1}{n}\#\{1 \le j \le n : r(j, T_{n+1}) \le i\} \overset{(d)}{=} \frac{1}{n}(S_i + i)$$

jointly for $i \in \{1, \ldots, n\}$. This concludes the proof. □

5. Sum and maximum of the first parking places

We begin this section with the proof of Corollary 1. Then we finish by proving Proposition 1.

Proof of Corollary 1. (i) Recall that $(U_n(i))_{1\le i\le n}$ are i.i.d. uniformly distributed in $[\![1, n]\!]$. By the central limit theorem, the convergence

$$\sqrt{\frac{12}{k_n}}\left(\frac{U_n(1) + \cdots + U_n(k_n)}{n} - \frac{k_n}{2}\right) \longrightarrow \mathcal{N}(0, 1)$$

holds in distribution. Using the first item of Theorem 1, we deduce that the total variation distance between the distributions of $\sum_{i=1}^{k_n} U_n(i)$ and $\sum_{i=1}^{k_n} \pi_n(i)$ tends to 0. Thus the above convergence still holds when $U_n(i)$ is replaced by $\pi_n(i)$.

(ii) Let $x > 0$. Then

$$\mathbb{P}\left[k_n\left(1 - \frac{1}{n}\max\{U_n(1), \ldots, U_n(k_n)\}\right) \ge x\right] = 0 \vee \frac{1}{n}\left\lfloor n\left(1 - \frac{x}{k_n}\right)\right\rfloor^{k_n} \xrightarrow[n\to\infty]{} \mathrm{e}^{-x}.$$

Using Theorem 1(ii), we deduce that the above convergence still holds when $U_n(i)$ is replaced by $\pi_n(i)$. □

Proof of Proposition 1. In this proof we write k instead of k_n to ease notation. For every $a \ge 0$:

$$\mathbb{P}(\pi_n(1), \ldots, \pi_n(k) \le n - a) = \frac{(n-k)!}{n!}\mathbb{E}_n[(S_{n-a} + n - a)_k]. \tag{25}$$

Indeed, following the same computation as in the proof of Proposition 2, we have

$$\begin{aligned}
\mathbb{P}(\pi_n(1), \ldots, \pi_n(k) \le n - a) &= \frac{(n-k)!}{n!}\sum_{\sigma\in\mathfrak{S}(k,n)} \mathbb{P}(\pi_n(\sigma(1)), \ldots, \pi_n(\sigma(k)) \le n - a) \\
&= \frac{(n-k)!}{n!}\mathbb{E}\left[\sum_{\sigma\in\mathfrak{S}(k,n)} \mathbb{1}_{r(\sigma(1),T_{n+1}),\ldots,r(\sigma(k),T_{n+1})\le n-a}\right] \\
&= \frac{(n-k)!}{n!}\mathbb{E}[(X_1 + \cdots + X_{n-a})_k],
\end{aligned}$$

which leads to (25) since $X_1 + \cdots + X_{n-a} = S_{n-a} + n - a$. Let τ_n be a Bienaymé–Galton–Watson tree with a critical Poisson offspring distribution μ conditioned on having n vertices, and define S^n to be the associated Ł ukasiewicz path. More precisely, if $v_1, \ldots, v_n$ are the vertices of τ_n ordered according to the lexicographic order (see e.g. [17, Section 1.1]), then for all $0 \le k \le n$,

$$S^n_k := \#\{e : e \text{ is an edge adjacent to a vertex } v_i \text{ with } i \le k\} - k.$$

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In the previous definition S_k^n is deduced from the first k vertices $v_1, \ldots, v_k$, but it is possible to see S_k^n in terms of the last $n-k$ vertices $v_{k+1}, \ldots, v_n$:

$$S_k^n = n-1-k-\#\{e : e \text{ is an edge between two vertices } v_i \text{ and } v_j \text{ with } i, j > k\}.$$

It is known that S^{n+1} and S under $\mathbb{P}_n$ have the same distribution. Thus equality (25) can be rewritten in the following way:

$$\mathbb{P}(\pi_n(1), \ldots, \pi_n(k) \leq n-a) = \frac{(n-k)!}{n!} \mathbb{E}\left[\left(S_{n-a}^{n+1} + n - a\right)_k\right]. \tag{26}$$

Let τ^* be the so-called Kesten's tree associated with μ (see e.g. [1, Section 2.3]). Let $\preceq$ denote the lexicographic order on the set of vertices of τ^*. It is always possible to find a unique infinite sequence $u_1, u_2, \ldots$ of distinct vertices of τ^* such that for all $i \geq 1$, $\{u : u \text{ is a vertex of } \tau^* \text{ such that } u_i \preceq u\} = \{u_1, \ldots, u_i\}$. In other words, $u_1, u_2, \ldots$ are the last vertices of τ^* for the lexicographic order, which, necessarily, lie on the right of the infinite spine. Similarly to the Łukasiewicz path we can define the quantity

$$S_a^* := a - \#\{e : e \text{ is an edge between two vertices } u_i \text{ and } u_j \text{ with } i, j \leq a+1\}.$$

It is known that τ_n converges in distribution, for the local topology, towards τ^* (see e.g. [1, Section 3.3.5]). Making use of Skorokhod's representation theorem, suppose that the latter convergence holds almost surely. Thus S_{n-a}^{n+1} converges almost surely towards S_a^*. Consequently, the convergence

$$\begin{aligned}\frac{(n-k)!}{n!}\left(S_{n-a}^{n+1}+n-a\right)_k &= \frac{(n-k)!}{\left(n-k+S_{n-a}^{n+1}-a\right)!}\frac{\left(n+S_{n-a}^{n+1}-a\right)!}{n!} \\ &\sim (n-cn)^{a-S_a^*}\frac{1}{n^{a-S_a^*}} \\ &\longrightarrow (1-c)^{a-S_a^*}\end{aligned}$$

holds almost surely. Since the above sequence is bounded by 1, we deduce that the convergence of the expectation holds, which concludes the proof. □

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