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RESEARCH ARTICLE

Optimal singular dividend control with capital injection and affine penalty payment at ruin

Ran Xu

Department of Financial and Actuarial Mathematics, Xi'an Jiaotong-Liverpool University, Suzhou, China. E-mail: ran.xu@xjtlu.edu.cn.

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Abstract

In this paper, we extend the optimal dividend and capital injection problem with affine penalty at ruin in (Xu, R. & Woo, J.K. (2020). *Insurance: Mathematics and Economics* 92: 1–16) to the case with singular dividend payments. The asymptotic relationships between our value function to the one with bounded dividend density are studied, which also help to verify that our value function is a viscosity solution to the associated Hamilton–Jacob–Bellman Quasi-Variational Inequality (HJBQVI). We also show that the value function is the smallest viscosity supersolution within certain functional class. A modified comparison principle is proved to guarantee the uniqueness of the value function as the viscosity solution within the same functional class. Finally, a band-type dividend and capital injection strategy is constructed based on four crucial sets; and the optimality of such band-type strategy is proved by using fixed point argument. Numerical examples of the optimal band-type strategies are provided at the end when the claim size follows exponential and gamma distribution, respectively.

1. Introduction

The optimal dividend and capital injection problem is currently an active research direction in actuarial science and quantitative finance. De Finetti [6] first introduced the optimal dividend problem to the actuarial science literature, where he proposed that the optimal strategy should maximize the expected discounted dividends until the surplus drops below zero (i.e. ruin occurs). Under a discrete risk model, he showed that the optimal strategy should follow the so-called barrier dividend strategy; that is, there exists a non-negative constant barrier such that the excess amount of the surplus above the barrier should be paid out as dividend to the shareholders. The results of such optimization problem under the Cramér–Lundberg model was given by Gerber [8,9], where a band-type dividend strategy is proved to be optimal in general. Later, Azcue and Muler [2] extended the optimal dividend problem under Cramér–Lundberg model to have reinsurance contracts. They obtained the optimal band-type dividend strategy by characterizing the value function as the smallest viscosity solution of the associated Hamilton–Jacobi–Bellman (HJB) equation. Under the same risk model, Albrecher and Thonhauser [1] considered the force of interest in the surplus process; with application of viscosity theory, they proved that the optimal dividend strategy is in general a band-type strategy.

However, the aforementioned optimal dividend strategies obtained in the literature usually causes the almost surely ruin in the optimal dividend problem. The reason is that the dividend optimization framework only considered the maximization of the shareholders' return (in terms of dividends received) without taking into account any related solvency issues. Hence, Thonhauser and Albrecher [16] introduced a component to the objective function that penalizes early ruin of the controlled risk process, such that their value function takes into account both expected dividend payments and time value of ruin. They identified the optimal dividend strategies for both Cramér–Lundberg model and diffusion model, which are barrier strategies for unbounded dividend intensity and threshold strategies for bounded dividend intensity. Loeffen [10] considered such optimal dividend problem with a real-valued terminal payment at ruin under spectrally negative Lévy process, and Loeffen and Renaud [11] further illustrated the optimality of a barrier strategy or the take-the-money-and-run strategy when there exists an affine penalty payment at ruin under the same risk model. On the other hand, in stead of considering penalty payment at ruin, Dickson and Waters [7] introduced the capital injections to the De Finetti's optimal dividend problem under Cramér–Lundberg model, where certain amount of capital will be made by shareholders to protect the insurance company from ruin. Under such framework, ruin never occurs and the optimal dividend strategy was identified by maximizing the difference between dividend paid out and capital injected. The study was extended to have administration costs associated with each capital injection by Scheer and Schmidli [13]. They proved that capital injections are only made when the surplus falls below zero, and showed that the optimal dividend and capital injection strategy is a band-type. The optimal dividend and capital injection problem with transaction costs under diffusion model with regime switching was investigated in Zhu and Yang [22], and Vierkötter and Schmidli [17] further incorporate exponential and linear penalty functions to such optimal control problem under diffusion model. In addition, from the risk management point of view, capital injection problem was also studied in some actuarial papers, see e.g. Nie *et al.* [12], Zhang *et al.* [20], Xu *et al.* [19], etc.

But, in the optimal dividend problem with capital injection, research usually focus on maximizing net profits over an infinite time horizon (i.e. ruin never occurs). Recently, Xu and Woo [18] considered both capital injections and affine penalty payments at ruin for optimal dividend problem under Cramér–Lundberg model with bounded dividend density, where capital injections are made up to the time of ruin (by forcing ruin when surplus drops below zero). The optimality of a band-type strategy for the combination of dividends and capital injections is obtained. Note that Zhao *et al.* [21] studied the optimal periodic dividend and capital injection problem with the case when ruin still can occur, but under spectrally positive Lévy process, their model and method are fundamentally different to our studies. Finally, we remark that in Xu and Woo [18], under assumption of absolutely continuous dividend strategies with bounded dividend rate, there exists a nature boundary condition at infinite that can guarantee the uniqueness of certain viscosity solution to the corresponding HJBQVI. Hence, in this paper, we continue the study by relaxing such assumption on dividend payments; that is, we consider the optimal singular dividend and capital injection problem with affine penalty at ruin. We derive most of our results by finding the asymptotic relationship between the scenario with bounded dividend density and the one with singular dividend payments. In addition, we provide a modified comparison principle such that we can characterize the value function as the unique viscosity solution to the associated HJBQVI within certain functional class. It is noted that the method of using viscosity theory to solve such optimization problem only generates certain abstract optimal solutions, and the numerical analysis are limited in the literature, especially when the claim size distribution is non-exponential (see [3,18]). Therefore, in this paper, we further provide a thorough numerical analysis on the structure of the optimal band-type strategy in various scenarios.

The rest of the paper is organized as follows: the model and some preliminaries are introduced in Section 2; the main results are given in Sections 3–5. To specific, in Section 3, certain characteristics of the value function are derived; the HJBQVI associated with our optimal control problem is derived by utilizing the asymptotic relationship between the value function under bounded dividend density and unbounded case; the uniqueness of certain viscosity solution is proved in Section 4; then, in Section 5, a band-type strategy is proposed based on four crucial sets, where the optimality of such dividend and capital injection strategy is proved at the end. Finally, a comprehensive numerical analysis on the optimal band-type strategy is given in Section 6, followed by some conclusion remarks in Section 7.

2. The model and some preliminaries

Let's consider a complete filtered probability space $(\Omega, \mathcal{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ is the corresponding filtration satisfies the usual condition. Let $U = (U_t)_{t\geq 0}$ be the uncontrolled surplus process of an

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insurance company; and at any time t, it is an $\mathbb{F}$-adapted càdlàg process given by

$$U_t = u + ct - \sum_{i=1}^{N_t} Y_i, \quad t \geq 0,$$

where $N = (N_t)_{t\geq 0}$ is a (homogeneous) Poisson process with intensity $\lambda > 0$. $\{Y_i\}_{i\geq 1}$ are independent and identically distributed positive random variables with common distribution function F and mean $\mu < \infty$, we also assume that F is continuous for the simplicity of the following analysis. Here, the independence between $\{Y_i\}_{i\geq 1}$ and N is assumed. The constant u denotes the initial surplus of the insurance company and c is the premium rate. Note that in the study with capital injection, no positive loading condition is needed. Additionally, we use $\mathbb{P}_u$ and $\mathbb{E}_u$ to denote the probability measure and expectation, respectively, when the initial surplus is u; and for notation simplicity, we suppress the subscript and write as $\mathbb{P}$ and $\mathbb{E}$, respectively, when $u = 0$. In addition, we denote *almost surely* and *almost everywhere* with a.s. and a.e. throughout the paper.

We assume that the insurance company can pay dividend to its shareholders at any time before ruin, and on the other hand, up to ruin time, shareholders can inject capital to the current surplus of the company as well. Let $(L_t^d)_{t\geq 0}$ be the accumulated dividend process for any $d \in \mathcal{D}$, where d is the implemented dividend strategy and $\mathcal{D}$ is the set of all admissible dividend strategies (see Definition 2.1 in the following); and let $(C_t^v)_{t\geq 0}$ denotes the accumulated capital injections until time t, which is given by

$$C_t^v = \sum_{i=1}^{\infty} \zeta_i 1_{\{\omega_i < t\}}, \quad t \geq 0, \tag{2.1}$$

where $\{\omega_i\}_{i\geq 1}$ is a sequence of random time points at which capital injections are made and $\{\zeta_i\}_{i\geq 1}$ are the injected capital amounts. The superscript $v = (\omega_1, \omega_2, \ldots; \zeta_1, \zeta_2, \ldots) \in \mathcal{V}$ denotes the capital injection strategy, where $\mathcal{V}$ is the corresponding admissible set. We further assume that there exists fixed and proportional transaction costs associated with each capital injection. Then, we use $\theta = (d, v)$ to denote a combined dividend and capital injection strategy with Θ be the corresponding admissible set. Hence, the controlled risk process U_t^θ at time t is given by

$$U_t^\theta = U_t - L_t^d + C_t^v, \quad t \geq 0.$$

Let $\tau^\theta := \inf\{t > 0, U_t^\theta < 0\}$ denotes the time of ruin under such controlled risk process, where $\inf \emptyset = \infty$ is assumed as usual. The following definition of the admissible dividend and capital injection strategy is borrowed from Xu and Woo [18].

Definition 2.1. *A strategy $\theta = (d, v) \in \Theta$ is said to be admissible if:*

(i) $\{L_t^d\}_{t\geq 0}$ *is a non-decreasing, $\mathbb{F}$-adapted càglàd process with $L_0^d = 0$, such that dividend payment will not cause ruin or immediate capital injection.*
(ii) $\{\omega_i\}_{i\geq 1}$ *is a sequence of stopping times with respect to filtration $\mathbb{F}$, and $0 \leq \omega_1 < \omega_2 < \cdots$ a.s.;*
(iii) ζ_i *is non-negative and measurable with respect to $\mathcal{F}_{\omega_i}$ for $i = 1, 2, \ldots$;*
(iv) $\mathbb{P}(\lim_{i\to\infty} \omega_i \leq T) = 0$ *for all $T \geq 0$.*

The càglàd assumptions for L_t^d and C_t^v imply that a jump of $U_t^w - U_{t-}^w$ is solely due to a claim (or the jump of the uncontrolled process U), and a jump $U_{t+}^w - U_t^w$ is due to lump sum dividend payment or capital injection but not simultaneously.

We further consider a penalty function $\pi : (-\infty, 0) \to (-\infty, 0]$, which indicates the penalty paid by insurance company when ruin occurs. We are interested in the affine penalty case, where $\pi(y) = K + \Phi y$ with $\Phi \in (0, 1]$ and $K < 0$ (see e.g. [18]). Note that $\Phi \in (0, 1]$ means that the deficit at ruin should be paid at least partial by shareholders, and $K < 0$ means that early ruin is penalized. Meanwhile, for $k > 0$

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and $\phi \geq 1$, we use k and $\phi - 1$ to denote the fixed and proportional transactions costs associated with each capital injection payment. Then, the performance function under an admissible strategy $\theta \in \Theta$ is given by

$$V_\theta(x) = \mathbb{E}_x \Bigg[\int_0^{\tau^\theta} e^{-\delta t}\, \mathrm{d}L_t^d - \sum_{i=1}^{\infty} e^{-\delta \omega_i}(k + \phi\zeta_i) 1_{\{\omega_i < \tau^\theta\}} + e^{-\delta\tau^\theta} \pi(U_{\tau^\theta}^\theta) 1_{\{\tau^\theta < \infty\}} \Bigg], \quad x \in [0, \infty), \tag{2.2}$$

where δ is the discounting factor. The corresponding value function is then defined as

$$V(x) = \sup_{\theta\in\Theta} V_\theta(x), \quad x \in [0, \infty). \tag{2.3}$$

In this paper, we aim at studying the value function and obtain the optimal strategy θ^* (if exists), such that $V_{\theta^*}(x) = V(x)$ for all $x \geq 0$. Note that, since ruin will occur immediately when surplus is below zero, then we directly have $V_\theta(x) = V(x) = \pi(x)$ for any $\theta \in \Theta$ and $x < 0$. Then, we extend the value function V to be defined on $\mathbb{R}$, such that $V(x) = \pi(x)$ for $x \in (-\infty, 0)$.

To proceed, we provide some preliminaries on certain characteristics of the value function in the following lemmas.

Lemma 2.1. *The extended value function $V : \mathbb{R} \to \mathbb{R}$ is increasing and locally Lipschitz in $[0, \infty)$ and upper semi-continuous at 0 with*

$$x - y \leq V(x) - V(y) \leq \phi(x - y) + k, \quad 0 \leq y < x, \tag{2.4}$$

and admits the following linear upper and lower bounds for $x \geq 0$,

$$x + \frac{c + \lambda(K - \Phi\mu)}{\lambda + \delta} \leq V(x) \leq x + \frac{c}{\delta}. \tag{2.5}$$

Proof. For any $0 \leq y < x$, we consider an ϵ-optimal strategy θ_ϵ for initial surplus x such that $V(x) \leq V_{\theta_\epsilon}(x) + \epsilon$; then for initial surplus y, consider the admissible strategy θ_y with initial capital injection $x - y$ followed by applying strategy θ_ϵ; hence we obtain

$$V(y) \geq V_{\theta_y} = V_{\theta_\epsilon}(x) - \phi(x - y) - k \geq V(x) - \phi(x - y) - k - \epsilon.$$

For the other inequality in (2.4), the proof is similar by consider an ϵ-optimal strategy for initial surplus y and an admissible strategy for initial surplus y with immediate dividend payment $x - y$. The increasing property is a direct consequence of (2.4). In addition, for $x \geq 0$, we consider a special dividend strategy d where the initial surplus is paid as lump sum dividend at time zero and premium income are paid continuously as dividend for all $t \geq 0$, i.e. $L_t^d = x + ct$; hence, we obtain an upper bound $\int_0^\infty e^{-\delta t} \mathrm{d}L_t^d$ for the performance function under any admissible strategy in Θ, that is

$$\int_0^\infty e^{-\delta t}\, \mathrm{d}L_t^d = x + \frac{c}{\delta} < \infty,$$

which is exactly the right-hand side of (2.5). The linear lower bound can be obtained by considering the admissible strategy θ with dividend strategy follows $L_t^d = x + ct$ and no capital injection, then ruin will occur at the arrival time of the first claim, hence

$$V(x) \geq V_\theta(x) = \mathbb{E}_x \left[\int_0^{T_1} e^{-\delta t}\, \mathrm{d}L_t^d + e^{-\delta T_1} \pi(-Y_1) \right] = x + \frac{c + \lambda(K - \Phi\mu)}{\lambda + \delta},$$

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where T_1 and Y_1 are the arrival time and amount of the first claim, respectively; $K < 0$ and $\Phi \in [0, 1]$ are the aforementioned parameters in the penalty function π. To prove the locally Lipschitz continuity, we consider initial surplus $y \geq 0$ and any $\epsilon > 0$, let θ_x denotes the ϵ-optimal strategy for any $x > y$, i.e. $V_{\theta_x}(x) \geq V(x) - \epsilon$. Then, consider another strategy θ_y with initial surplus y that pays no dividends and no capital injections if $U_t^{\theta_y} < x$ and follows strategy θ_x if $U_t^{\theta_y}$ reaches x; then, θ_y is obviously an admissible strategy. Hence, we have

$$V(y) \geq V_{\theta_y}(y) \geq V_{\theta_x}(x)e^{-(\lambda+\delta)((x-y)/c)} \geq (V(x) - \epsilon)e^{-(\lambda+\delta)((x-y)/c)},$$

then, we obtain

$$V(x) - V(y) \leq (e^{(\lambda+\delta)((x-y)/c)} - 1)V(x).$$

Finally, the upper semi-continuity at 0 can be derived by including the take-the-money-and-run strategy at 0 into our admissible set, which is reasonable since it should have a higher expected return to run the business rather than simply declare to ruin when surplus is at 0 (see e.g. [18]). Hence, we have $V(0) \geq \pi(0)$. □

We further introduce the capital injection operator $\mathcal{M}$ as follows:

$$\mathcal{M}\varphi(x) := \sup_{y \geq 0}\{\varphi(x+y) - (k + \phi y)\}, \quad x \geq 0, \tag{2.6}$$

with k and $\phi - 1$ be the fixed and proportional transaction costs associated with each capital injection. Obviously, $\mathcal{M}V(x)$ indicates the value function after an immediate capital injection. Below two lemmas illustrate some useful properties of the operator $\mathcal{M}$.

Lemma 2.2. *Let $\varphi(x)$ be an increasing, locally Lipschitz, and upper bounded by linear function $x + m$ for all $x \in [0, \infty)$ and a constant $m > 0$. Then, $\mathcal{M}\varphi(x)$, as a function of x defined in (2.6), is increasing, Lipschitz continuous and linearly bounded.*

Proof. We only prove that $\mathcal{M}\varphi(x)$ is linearly bounded here, for the proof of increasing, Lipschitz continuous can refer to Xu and Woo [18] Lemma 5.1. Note that,

$$\begin{aligned}\mathcal{M}\varphi(x) &= \sup_{y \geq 0}\{\varphi(x+y) - (k + \phi y)\}\\ &\leq \sup_{y \geq 0}\{x + y + m - (k + \phi y)\}\\ &= x + m - k + \sup_{y \geq 0}\{(1-\phi)y\} = x + m - k,\end{aligned}$$

where the last equation holds since $\phi \geq 1$. □

Lemma 2.3. *(i) The capital injection operator $\mathcal{M}$ is convex such that for $h \in [0, 1]$,*

$$\mathcal{M}(hf + (1-h)g) \leq h\mathcal{M}f + (1-h)\mathcal{M}g.$$

(ii) For $h > 0$,

$$\mathcal{M}(-hf + (1+h)g) \geq -h\mathcal{M}f + (1+h)\mathcal{M}g,$$

given that the right-hand side is well-defined.

Proof. The proof follows easily from the sup manipulations (see e.g. [15]), i.e.

$$\sup_x(f(x) + g(x)) \leq \sup_x(f(x)) + \sup_x(g(x)),$$

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and

$$\sup_x(f(x)+g(x)) \geq \sup_x(f(x)) + \inf_x(g(x)).$$

Hence, for any $x \geq 0$,

$$\begin{aligned}\mathcal{M}(hf+(1-h)g)(x) &= \sup_{y\geq 0}\{hf(x+y)+(1-h)g(x+y)-\phi y-k\}\\ &= \sup_{y\geq 0}\{h(f(x+y)-\phi y-k)+(1-h)(g(x+y)-\phi y-k)\}\\ &\leq h\sup_{y\geq 0}\{f(x+y)-\phi y-k\}+(1-h)\sup_{y\geq 0}\{g(x+y)-\phi y-k\}\\ &= h\mathcal{M}f(x)+(1-h)\mathcal{M}g(x).\end{aligned}$$

Similarly,

$$\begin{aligned}\mathcal{M}(-hf+(1+h)g) &= \sup_{y\geq 0}\{-hf(x+y)+(1+h)g(x+y)-\phi y-k\}\\ &= \sup_{y\geq 0}\{-h(f(x+y)-\phi y-k)+(1+h)(g(x+y)-\phi y-k)\}\\ &\geq -h\sup_{y\geq 0}\{f(x+y)-\phi y-k\}+(1+h)\sup_{y\geq 0}\{g(x+y)-\phi y-k\}\\ &= -h\mathcal{M}f(x)+(1+h)\mathcal{M}g(x).\end{aligned}$$

□

3. HJBQVI and viscosity solution

In this section, we first analyze the asymptotic relationships between the value function (V) with singular dividend payments in this paper and the one with bounded dividend intensity (V^b, where b denote the ceiling dividend rate) studied in Xu and Woo [18].

Proposition 3.1. *Let V be the value function given in (2.3), and let $(V^n)_{n\in\mathbb{N}}$ denotes the sequence of value functions analogy to (2.3) but with absolutely continuous dividend density bounded by ceiling dividend rate n. Then, we have*

$$\lim_{n\to\infty} V^n(x) = V(x), \quad \text{for } x \in [0,\infty), \tag{3.1}$$

$$\lim_{n\to\infty} \mathcal{M}V^n(x) = \mathcal{M}V(x), \quad \text{for } x \in [0,\infty). \tag{3.2}$$

Proof. For the proof of (3.1), we follow the analysis in Schmidli [14] Lemma 2.38. Note that we work under the assumptions that dividend payments and capital injections cannot occur simultaneously, and any dividend payment should not result in capital injection and vice versa. Since $\Theta_1^r \subset \Theta_2^r \subset \Theta$, where Θ_1^r and Θ_2^r are the corresponding sets of admissible strategies with ceiling dividend rates n_1 and n_2 with $n_1 < n_2$, respectively (see e.g. [18]). Then, one has $V^n(x)$ is an increasing sequence of n and $\limsup_{n\to\infty} V^n(x) \leq V(x)$. Next, to show that $V(x) \leq \liminf_{n\to\infty} V^n(x)$, we consider, for each $\epsilon > 0$, a dividend strategy d_j with pure jump of size that is greater or equal to ϵ, and combine with an admissible capital injection strategy v such that $V_{(d_j,v)}(x) \geq V(x) - 2\epsilon$. On the other hand, we construct another dividend strategy $\tilde{d}_j$ with absolutely continuous dividend density that is bounded by n. To be specific, under strategy $\tilde{d}_j$, dividends will start to be paid at rate n when a lump sum dividend payment occurs in strategy d_j until the accumulated amount of dividend meet with the lump sum in strategy d_j. Hence,

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with sufficiently large n, the difference between the performance function of these two strategies is bounded by ϵ. Therefore, we have $V(x) - V_{(\tilde{d},v)}(x) \leq 3\epsilon$. Then, by letting $\epsilon \to 0$, we obtain that $V(x) \leq \liminf_{n\to\infty} V^n(x)$. Then, (3.1) holds true.

For (3.2), since $V^n(x)$ is increasing in n and absolutely continuous with respective to x, then there must exist $n^* > n$ and $y^* \geq 0$ such that

$$\sup_{y\geq 0}\{V^n(x+y) - (k+\phi y)\} \leq V^{n^*}(x+y^*) - (k+\phi y^*),$$

let $n, n^* \to \infty$, one arrives at

$$\begin{aligned}\lim_{n\to\infty}&\sup_{y\geq 0}\{V^n(x+y) - (k+\phi y)\}\\ &\leq V(x+y^*) - (k+\phi y^*) \leq \sup_{y\geq 0}\{V(x+y) - (k+\phi y)\}.\end{aligned}$$

Meanwhile, for each $y' \geq 0$,

$$\sup_{y\geq 0}\{V^n(x+y) - (k+\phi y)\} \geq V^n(x+y') - (k+\phi y')$$

then, let $n \to \infty$, one arrives at

$$\lim_{n\to\infty}\sup_{y\geq 0}\{V^n(x+y) - (k+\phi y)\} \geq V(x+y') - (k+\phi y'),$$

since y' is arbitrary, one has

$$\lim_{n\to\infty}\sup_{y\geq 0}\{V^n(x+y) - (k+\phi y)\} \geq \sup_{y\geq 0}\{V(x+y) - (k+\phi y)\}.$$

Then, one completes the proof. □

According to Xu and Woo [18], it is obvious that the HJBQVI associated with the present optimal singular dividend and capital injection problem with affined penalty payment at ruin has the following form,

$$\text{HJBQVI}: \quad \begin{cases}\max\{(\mathcal{A}_\pi - \delta)\varphi(x), 1-\varphi'(x), \mathcal{M}\varphi(x) - \varphi(x)\} = 0, & x \geq 0,\\ \varphi(x) = \pi(x), & x < 0,\end{cases} \tag{3.3}$$

where the operator $\mathcal{A}_\pi$ is defined for any continuously differentiable function h on $[0,\infty)$:

$$\mathcal{A}_\pi h(x) = ch'(x) - \lambda h(x) + \lambda\int_0^x h(x-y)\,\mathrm{d}F(y) + \lambda\int_x^\infty \pi(x-y)\,\mathrm{d}F(y). \tag{3.4}$$

Since the value function given in (2.3) is, in general, not continuously differentiable on $[0,\infty)$, we shall proceed our analysis with the method of viscosity theory (see [5]). The following is the definition of viscosity solution fitting to our HJBQVI given in (3.3).

Definition 3.1 (*Viscosity Solution*)**.** *A function φ is a viscosity subsolution (supersolution) of (3.3) at $x \in [0,\infty)$ if it is locally Lipschitz and for any continuously differentiable function h on $(0,\infty)$ with $\varphi \leq (\geq) h$ and $\varphi(x) = h(x)$, then*

$$\max\{(\mathcal{A}_\pi - \delta)h(x), 1-h'(x), \mathcal{M}h(x) - h(x)\} \geq (\leq)\ 0.$$

We say φ is a viscosity solution of (3.3) if it is both a viscosity subsolution and supersolution of (3.3) at any $x \in [0,\infty)$, and $\varphi(x) = \pi(x)$ for $x < 0$.

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Before we move to next proposition, we introduce a functional class $\mathcal{LB}^\pi(\mathbb{R})$, such that for any $f \in \mathcal{LB}^\pi(\mathbb{R})$, the following conditions are satisfied:

(i) $f : \mathbb{R} \to \mathbb{R}$ is locally Lipschitz continuous on $[0, \infty)$.
(ii) $f(x) = \pi(x)$ for $x < 0$.
(iii) For any $0 \leq y < x$, there exists constants $k > 0$ and $\phi \geq 1$ such that $x - y \leq f(x) - f(y) \leq k + \phi(x - y)$.
(iv) There exists constant $l > 0$ such that $f(x) \leq x + l$ for all $x \in [0, \infty)$.

It is obvious that the value function belongs to this class.

Proposition 3.2. *The value function $V(x)$ defined in (2.3) is a viscosity solution of (3.3).*

Proof. Note that according to Lemma 5.2 in Xu and Woo [18] and Proposition 3.1, one directly has $V(x) \geq \mathcal{M}V(x)$ for $x \geq 0$. And for $x < 0$, by definition of the value function, $V(x) = \pi(x)$.

(i) V **is subsolution:** For any $x \in [0, \infty)$ and $h \in C^1(0, \infty)$ such that $h \geq V$ on $[0, \infty)$ and $h(x) = V(x)$, we need to show that

$$\max\{(\mathcal{A}_\pi - \delta)h(x), 1 - h'(x), \mathcal{M}h(x) - h(x)\} \geq 0. \tag{3.5}$$

When $h(x) = V(x) = \mathcal{M}V(x) = \mathcal{M}h(x)$, (3.5) holds trivially, hence we focus on the case $V(x) > \mathcal{M}V(x)$. If $1 - h'(x) \geq 0$, then (3.5) holds true obviously. Finally, if $1 - h'(x) < 0$, we consider a sequence $d_n \uparrow \infty$ as $n \to \infty$, which corresponds to the ceiling rate for the value function V^{d_n} with bounded dividend density Xu and Woo [18] Eq. (3.1), such that there exists an associated sequence of functions $h_n \in C^1(0, \infty)$ with h_n converges to h uniformly on compact sets and $h'_n(x) \to h'(x)$ when $n \to \infty$, and $h_n \geq V^{d_n}$ on $[0, \infty)$ with $h_n(x) = V^{d_n}(x)$. Then according to Xu and Woo [18] Proposition 5.1 and Proposition 3.1, for sufficiently large n, we have $V^{d_n}(x) > \mathcal{M}V^{d_n}(x)$, then one must have,

$$\sup_{d \in [0, d_n]} \{(\mathcal{A}_\pi - \delta)h_n(x) + (1 - h'_n(x))d\} \geq 0. \tag{3.6}$$

Since $1 - h'(x) < 0$, there exists a sufficiently large $\bar{n}$ such that for all $n > \bar{n}$, one has $1 - h'_n(x) < 0$, and (3.6) becomes $(\mathcal{A}_\pi - \delta)h_n(x) \geq 0$, then, by letting $n \to \infty$, one arrives at $(\mathcal{A}_\pi - \delta)h(x) \geq 0$. Hence, (3.5) holds.

(ii) V **is supsolution:** For any $x \in [0, \infty)$, we have $V(x) \geq \mathcal{M}V(x)$. Then, it remains to show that for any $h \in C^1(0, \infty)$ with $h \leq V$ on $[0, \infty)$ and $h(x) = V(x)$, one has

$$\max\{(\mathcal{A}_\pi - \delta)h(x), 1 - h'(x)\} \leq 0.$$

Similarly, we consider a sequence $d_n \uparrow \infty$ as $n \to \infty$ such that there exists an associated sequence of test function $h_n \in C^1(0, \infty)$ with $h_n \leq V^{d_n}$ on $[0, \infty)$, $h_n(x) = V^{d_n}(x)$, and h_n converges uniformly to h on compact sets, $h'_n(x) \to h'(x)$ when $n \to \infty$. Then, from Xu and Woo [18] Proposition 5.1, one has

$$\sup_{d \in [0, d_n]} \{(\mathcal{A}_\pi - \delta)h_n(x) + (1 - h'_n(x))d\} \leq 0, \quad \text{for all } n. \tag{3.7}$$

Then, one has $1 - h'_n(x) \leq 0$ for sufficiently large n, therefore $1 - h'(x) \leq 0$. In addition, for sufficiently large n when $1 - h'_n(x) \leq 0$, (3.7) reduces to $(\mathcal{A}_\pi - \delta)h_n(x) \leq 0$, and by letting $n \to \infty$, one has $(\mathcal{A}_\pi - \delta)h(x) \leq 0$. Then, we complete the proof.

□

4. Characterization of the value function

In this section, we further characterize the value function as a unique viscosity solution of (3.3) that belongs to the functional class $\mathcal{LB}^{\pi}(\mathbb{R})$. In particular, we first show in the following proposition that V is the smallest viscosity supersolution of (3.3) that belongs to $\mathcal{LB}^{\pi}(\mathbb{R})$.

Proposition 4.1. *The value function V defined in (2.3) is the smallest viscosity supersolution of the HJBQVI (3.3).*

Proof. Let $\bar{h} \in \mathcal{LB}^{\pi}(\mathbb{R})$ be a viscosity supersolution of (3.3). According to Lemma A.1, there exists a sequence of continuously differentiable functions h_n on $\mathbb{R}$ satisfy the condition (iv) in $\mathcal{LB}^{\pi}(\mathbb{R})$ with $h_n(x) = \pi(x)$ for $x < 0$, such that $h_n \le \bar{h}$ on $[0, \infty)$, and when $h_n(x) = \bar{h}(x)$ we have $(\mathcal{A}_{\pi} - \delta)h_n(x) \le 0$ and $h_n'(x) \ge 1$ for $x \ge 0$. In addition, h_n converges uniformly to $\bar{h}$ on compact sets and $h_n'(x)$ converges to $\bar{h}'(x)$ a.e. Then, let us consider the controlled process U^{θ} with an arbitrary admissible strategy $\theta = (d, v)$. Denote the cumulative dividend process as

$$L_t^d = \int_0^t \mathrm{d}\tilde{L}_t^d + \sum_{L_{s+}^d \neq L_s^d} (L_{s+}^d - L_s^d),$$

where $\tilde{L}_t^d$ denotes the continuous part of the dividend process, and $L_{s+}^d - L_s^d$ denotes the corresponding jump components. In addition, let capital injection strategy be an impulse strategy $v = (\omega_1, \omega_2, \ldots; \zeta_1, \zeta_2, \ldots)$. We apply Itô's formula within the interval $[\omega_i^+ \wedge \tau^{\theta}, \omega_{i+1} \wedge \tau^{\theta})$, then we arrive at

$$\begin{aligned}
&e^{-\delta(\omega_{i+1}\wedge\tau^{\theta})} h_n(X^{\theta}_{\omega_{i+1}\wedge\tau^{\theta}}) - e^{-\delta(\omega_i^+\wedge\tau^{\theta})} h_n(X^{\theta}_{\omega_i^+\wedge\tau^{\theta}}) \\
&\quad = c\int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s} h_n'(X^{\theta}_{s^-})\,\mathrm{d}s - \int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s} h_n'(X^{\theta}_{s^-})\,\mathrm{d}\tilde{L}_s^d - \delta \int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s} h_n(X^{\theta}_{s^-})\,\mathrm{d}s \\
&\qquad + \sum_{\substack{X_s^{\theta} \neq X_{s^-}^{\theta} \\ s\in(\omega_i^+\wedge\tau^{\theta},\, \omega_{i+1}\wedge\tau^{\theta}]}} e^{-\delta s}(h_n(X_s^{\theta}) - h_n(X_{s^-}^{\theta})) \\
&\qquad + \sum_{\substack{X_{s^+}^{\theta} \neq X_s^{\theta} \\ s\in[\omega_i^+\wedge\tau^{\theta},\, \omega_{i+1}\wedge\tau^{\theta})}} e^{-\delta s}(h_n(X_{s^+}^{\theta}) - h_n(X_s^{\theta})).
\end{aligned}$$

Note that within the interval $[\omega_i^+ \wedge \tau^{\theta}, \omega_{i+1} \wedge \tau^{\theta})$, the jumps of $X_{s^+}^{\theta} - X_s^{\theta}$ is equal to the jumps of L_s^d, that is $X_{s^+}^{\theta} - X_s^{\theta} = -(L_{s^+}^d - L_s^d)$, then with the fact that $h_n'(\cdot) \ge 1$, we have

$$\begin{aligned}
&-\int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s} h_n'(X^{\theta}_{s^-})\,\mathrm{d}\tilde{L}_s^d + \sum_{\substack{X_{s+}^{\theta} \neq X_s^{\theta} \\ s\in[\omega_i^+\wedge\tau^{\theta},\, \omega_{i+1}\wedge\tau^{\theta})}} e^{-\delta s}(h_n(X_{s+}^{\theta}) - h_n(X_s^{\theta})) \\
&\quad = -\int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s} h_n'(X^{\theta}_{s^-})\,\mathrm{d}\tilde{L}_s^d - \sum_{\substack{L_{s+}^u \neq L_s^d \\ s\in[\omega_i^+\wedge\tau^{\theta},\, \omega_{i+1}\wedge\tau^{\theta})}} e^{-\delta s}\left(\int_0^{L_{s+}^d - L_s^d} h_n'(X_{s^-}^{\theta} - y)\,\mathrm{d}y\right) \\
&\quad \le -\int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s}\,\mathrm{d}\tilde{L}_s^d - \sum_{\substack{L_{s+}^d \neq L_s^d \\ s\in[\omega_i^+\wedge\tau^{\theta},\, \omega_{i+1}\wedge\tau^{\theta})}} e^{-\delta s}(L_{s+}^d - L_s^d) \\
&\quad = -\int_{\omega_i^+\wedge\tau^{\theta}}^{\omega_{i+1}\wedge\tau^{\theta}} e^{-\delta s}\,\mathrm{d}L_s^d. \qquad (4.1)
\end{aligned}$$

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On the other hand, for the jumps $X_s^\theta - X_{s-}^\theta$ which only related to the arrival of claims, we define

$$M_t = \sum_{\substack{X_s^\theta \neq X_{s-}^\theta \\ s \le t}} e^{-\delta s}(h_n(X_s^\theta) - h_n(X_{s-}^\theta)) - \lambda \int_0^t e^{-\delta s} \int_0^\infty (h_n(X_{s-}^\theta - y) - h_n(X_s^\theta))\, \mathrm{d}F(y)\, \mathrm{d}s,$$

which is obviously a zero mean martingale; then, one can obtain that

$$e^{-\delta(\omega_{i+1}\wedge\tau^\theta)} h_n(X^\theta_{\omega_{i+1}\wedge\tau^\theta}) - e^{-\delta(\omega_i^+\wedge\tau^\theta)} h_n(X^\theta_{\omega_i^+\wedge\tau^\theta}) \le -\int_{\omega_i^+\wedge\tau^\theta}^{\omega_{i+1}\wedge\tau^\theta} e^{-\delta s}\, \mathrm{d}L_s^d + \int_{\omega_i^+\wedge\tau^\theta}^{\omega_{i+1}\wedge\tau^\theta} e^{-\delta s}(\mathcal{A} - \delta) h_n(X_{s-}^\theta)\, \mathrm{d}s + (M_{\omega_{i+1}\wedge\tau^\theta} - M_{\omega_i^+\wedge\tau^\theta}),$$

where

$$\begin{aligned}(\mathcal{A} - \delta) h_n(x) &= c h_n'(x) - (\lambda + \delta) h_n(x) + \lambda \int_0^\infty h_n(x-y)\, \mathrm{d}F(y) \\ &= c h_n'(x) - (\lambda + \delta) h_n(x) + \lambda \int_0^x h_n(x-y)\, \mathrm{d}F(y) + \lambda \int_x^\infty \pi(x-y)\, \mathrm{d}F(y) \\ &= (\mathcal{A}_\pi - \delta) h_n(x).\end{aligned}$$

By taking expectation on both sides of the above inequality, one arrives at

$$\begin{aligned}&\mathbb{E}_x[e^{-\delta(\omega_{i+1}\wedge\tau^\theta)} h_n(X^\theta_{\omega_{i+1}\wedge\tau^\theta})] - \mathbb{E}_x[e^{-\delta(\omega_i^+\wedge\tau^\theta)} h_n(X^\theta_{\omega_i^+\wedge\tau^\theta})] \\ &\quad \le -\mathbb{E}_x\left[\int_{\omega_i^+\wedge\tau^\theta}^{\omega_{i+1}\wedge\tau^\theta} e^{-\delta s}\, \mathrm{d}L_s^d\right] + \mathbb{E}_x\left[\int_{\omega_i^+\wedge\tau^\theta}^{\omega_{i+1}\wedge\tau^\theta} e^{-\delta s}(\mathcal{A}_\pi - \delta) h_n(X_{s-}^\theta)\, \mathrm{d}s\right].\end{aligned} \tag{4.2}$$

Summing both sides of (4.2) from $i = 0$ to $i = m$, it follows that

$$\begin{aligned}&h_n(x) + \sum_{i=1}^m \mathbb{E}_x[e^{-\delta(\omega_i\wedge\tau^\theta)}(h_n(X^\theta_{\omega_i^+\wedge\tau^\theta}) - h_n(X^\theta_{\omega_i\wedge\tau^\theta}))] - \mathbb{E}_x[e^{-\delta(\omega_{m+1}\wedge\tau^\theta)} h_n(X^\theta_{\omega_{m+1}\wedge\tau^\theta})] \\ &\quad \ge \mathbb{E}_x\left[\int_0^{\omega_{m+1}\wedge\tau^\theta} e^{-\delta s}\, \mathrm{d}L_s^d\right] - \mathbb{E}_x\left[\int_0^{\omega_{m+1}\wedge\tau^\theta} e^{-\delta s}(\mathcal{A}_\pi - \delta) h_n(X_{s-}^\theta)\, \mathrm{d}s\right].\end{aligned} \tag{4.3}$$

Note that when there is a capital injection before τ^θ, the following equation holds

$$X^\theta_{\omega_i^+} = X^\theta_{\omega_i} + \zeta_i. \tag{4.4}$$

Hence when $\omega_i < \tau^\theta$, from (4.4) and (2.6) one has

$$h_n(X^\theta_{\omega_i^+}) = h_n(X^\theta_{\omega_i} + \zeta_i) \le \mathcal{M} h_n\left(X^\theta_{\omega_i}\right) + k + \phi\zeta_i,$$

which yields

$$h_n(X^\theta_{\omega_i^+}) - h_n(X^\theta_{\omega_i}) \le \mathcal{M} h_n(X^\theta_{\omega_i}) - h_n(X^\theta_{\omega_i}) + k + \phi\zeta_i.$$

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But, when $\omega_i \geq \tau^\theta$, we obtain $h_n(X^\theta_{\omega_i^+\wedge\tau^\theta}) = h_n(X^\theta_{\omega_i\wedge\tau^\theta}) = \pi(X^\theta_{\omega_i\wedge\tau^\theta})$. Hence, it follows that (4.3) may be expressed as

$$\begin{aligned} h_n(x) &+ \sum_{i=1}^{m} \mathbb{E}_x[e^{-\delta\omega_i}(\mathcal{M}h_n(X^\theta_{\omega_i}) - h_n(X^\theta_{\omega_i}))1_{\{\omega_i<\tau^\theta\}}] \\ &\geq \mathbb{E}_x\left[\int_0^{\omega_{m+1}\wedge\tau^\theta} e^{-\delta s}\,\mathrm{d}L^d_s + e^{-\delta(\omega_{m+1}\wedge\tau^\theta)}h_n(X^\theta_{\omega_{m+1}\wedge\tau^\theta}) - \sum_{i=1}^{m} e^{-\delta\omega_i}(k+\phi\zeta_i)1_{\{\omega_i<\tau^\theta\}}\right] \\ &\quad - \mathbb{E}_x\left[\int_0^{\omega_{m+1}\wedge\tau^\theta} e^{-\delta s}(\mathcal{A}_\pi - \delta)h_n(X^\theta_{s-})\,\mathrm{d}s\right]. \end{aligned} \tag{4.5}$$

Next, we show that

$$\begin{aligned} \lim_{m,n\to\infty} \mathbb{E}_x&\left[\int_0^{\omega_{m+1}\wedge\tau^\theta} e^{-\delta s}(\mathcal{A}_\pi-\delta)h_n(X^\theta_{s-})\,\mathrm{d}s\right] \\ &= \mathbb{E}_x\left[\int_0^{\tau^\theta} e^{-\delta s}(\mathcal{A}_\pi-\delta)\bar{h}(X^\theta_{s-})\,\mathrm{d}s\right] \leq 0, \end{aligned} \tag{4.6}$$

where the second inequality holds true since $\bar{h}$ is a viscosity supersolution of (3.3) for $x \geq 0$; note that we also have $\bar{h}'(x) \geq 1$ a.e. In addition, since h_n converges to $\bar{h}$ uniformly on compact sets and $h_n'(x)$ converges to $\bar{h}'(x)$ a.e.; then, we have

$$e^{-\delta s}(\mathcal{A}_\pi-\delta)h_n(X^\theta_{s-}) \overset{n\to\infty}{\longrightarrow} e^{-\delta s}(\mathcal{A}_\pi-\delta)\bar{h}(X^\theta_{s-}) \quad \text{a.e.}$$

In addition, with a similar analysis as in Azcue and Muler [2] Lemma A.2, we have for $x \geq 0$,

$$1 \leq \bar{h}'(x) \leq \frac{\lambda+\delta}{c}\bar{h}(x) - \frac{\lambda}{c}\int_0^x \bar{h}(x-y)\,\mathrm{d}F(y) - \frac{\lambda}{c}M(x) \leq \frac{\lambda\bar{F}(x)+\delta}{c}\bar{h}(x) - \frac{\lambda}{c}M(x) \quad \text{a.e.,}$$

where

$$M(x) = \int_x^\infty \Pi(x-y)\,\mathrm{d}F(y) < 0,$$

and

$$1 \leq h_n'(x) \leq \frac{\lambda\bar{F}(x)+\delta}{c}h_n(x) - \frac{\lambda}{c}M(x).$$

Hence,

$$\begin{aligned} &e^{-\delta s}|(\mathcal{A}_\pi-\delta)\bar{h}(X^\theta_{s-}) - (\mathcal{A}_\pi-\delta)h_n(X^\theta_{s-})| \\ &\leq e^{-\delta s}\left(c\bar{h}'(X^\theta_{s-}) + (\lambda+\delta)\bar{h}(X^\theta_{s-}) + \lambda\int_0^{X^\theta_{s-}} \bar{h}(X^\theta_{s-}-y)\,\mathrm{d}F(y) + \lambda|M(X^\theta_{s-})|\right) \\ &\quad + e^{-\delta s}\left(ch_n'(X^\theta_{s-}) + (\lambda+\delta)h_n(X^\theta_{s-}) + \lambda\int_0^{X^\theta_{s-}} h_n(X^\theta_{s-}-y)\,\mathrm{d}F(y) + \lambda|M(X^\theta_{s-})|\right) \\ &\leq e^{-\delta s}(\lambda+2\delta)(\bar{h}(X^\theta_{s-}) + h_n(X^\theta_{s-})) + 4\lambda e^{-\delta s}|M(X^\theta_{s-})| \\ &\leq 2e^{-\delta s}(\lambda+2\delta)(x+cs+N) + 4\lambda e^{-\delta s}M \end{aligned}$$

for sufficiently large N and M. Therefore, $e^{-\delta s}|(\mathcal{A}_\pi-\delta)\bar{h}(X^\theta_{s-}) - (\mathcal{A}_\pi-\delta)h_n(X^\theta_{s-})|$ is bounded by a positive integrable function as shown above, then (4.6) holds true by dominated convergence theorem.

Finally, with the help of monotone and bounded convergence theorem, we let $m, n \to \infty$ on both sides of (4.5), and utilize the uniformly convergence of h_n to $\bar{h}$ and $\bar{h}(x) = \pi(x)$ for $x < 0$, and (4.6), we arrive at

$$\begin{aligned}&\bar{h}(x) + \sum_{i=1}^{\infty} \mathbb{E}_x[e^{-\delta\omega_i}(\mathcal{M}\bar{h}(X^\theta_{\omega_i}) - \bar{h}(X^\theta_{\omega_i}))1_{\{\omega_i<\tau^\theta\}}] \\ &\geq \mathbb{E}_x\left[\int_0^{\tau^\theta} e^{-\delta s}\,\mathrm{d}L^d_s + e^{-\delta\tau^\theta}\pi(X^\theta_{\tau^\theta})1_{\{\tau^\theta<\infty\}} - \sum_{i=1}^{\infty} e^{-\delta\omega_i}(k+\phi\zeta_i)1_{\{\omega_i<\tau^\theta\}}\right]. \end{aligned} \tag{4.7}$$

In addition, since $\mathcal{M}\bar{h}(x) - \bar{h}(x) \leq 0$ for all $x \in [0,\infty)$ and the strategy θ is arbitrary, we get

$$\bar{h}(x) \geq V(x).$$

□

As discussed in Xu and Woo [18], the uniqueness can be obtained with the known boundary condition at infinity when the dividend payment is restricted to the class of absolutely continuous strategy with bounded dividend density. However, when we extend to the singular dividend payment, more efforts are needed to the show the uniqueness. Hence in the following, we provide a modified comparison principle, with which we can show that V is the unique viscosity solution of (3.3) within the class $\mathcal{LB}^\pi(\mathbb{R})$.

Lemma 4.1. *Let ξ be a subsolution and η a supersolution of (3.3). Assume that there is a function $w \in C^1(0,\infty)$ and positive function κ such that*

$$\begin{cases}\max\{(\mathcal{A}_\pi - \delta)w(x), 1 - w'(x), \mathcal{M}w(x) - w(x)\} \leq -\kappa(x), & x \geq 0,\\ w(x) = \pi(x), & x < 0.\end{cases} \tag{4.8}$$

Define

$$\xi_m := \left(1 + \frac{1}{m}\right)\xi - \frac{1}{m}w, \quad \eta_m := \left(1 - \frac{1}{m}\right)\eta + \frac{1}{m}w.$$

Then, ξ_m is a subsolution of

$$\begin{cases}\max\{(\mathcal{A}_\pi - \delta)\varphi(x), 1 - \varphi'(x), \mathcal{M}\varphi(x) - \varphi(x)\} - \dfrac{\kappa(x)}{m} = 0, & x \geq 0,\\ \varphi(x) = \pi(x), & x < 0.\end{cases}$$

And η_m is a supersolution of

$$\begin{cases}\max\{(\mathcal{A}_\pi - \delta)\varphi(x), 1 - \varphi'(x), \mathcal{M}\varphi(x) - \varphi(x)\} + \dfrac{\kappa(x)}{m} = 0, & x \geq 0,\\ \varphi(x) = \pi(x), & x < 0.\end{cases} \tag{4.9}$$

Proof. Since ξ is a subsolution of (3.3), then according to Definition 3.1, for any continuously differentiable function h with $h \geq \xi$ and $h(x) = \xi(x)$, we have

$$\max\{(\mathcal{A}_\pi - \delta)h(x), 1 - h'(x), \mathcal{M}h(x) - h(x)\} \geq 0.$$

Then, we construct the continuously differentiable function $h_m = (1 + 1/m)h - (1//m)w$ such that $h_m \geq \xi_m$ and at x where $h_m(x) = \xi_m(x)$, with the help of Lemma 2.3 we must have

$$\begin{aligned}
&\max\{(\mathcal{A}_\pi - \delta)h_m(x), 1 - h_m'(x), \mathcal{M}h_m(x) - h_m(x)\} \\
&= \max\left\{\left(1 + \frac{1}{m}\right)(\mathcal{A}_\pi - \delta)h(x) - \frac{1}{m}(\mathcal{A}_\pi - \delta)w(x),\right. \\
&\quad \left.\left(1 + \frac{1}{m}\right)(1 - h'(x)) - \frac{1}{m}(1 - w'(x)), \mathcal{M}h_m(x) - h_m(x)\right\} \\
&\geq \max\left\{\left(1 + \frac{1}{m}\right)(\mathcal{A}_\pi - \delta)h(x) - \frac{1}{m}(\mathcal{A}_\pi - \delta)w(x),\right. \\
&\quad \left(1 + \frac{1}{m}\right)(1 - h'(x)) - \frac{1}{m}(1 - w'(x)), \\
&\quad \left.\left(1 + \frac{1}{m}\right)(\mathcal{M}h(x) - h(x)) - \frac{1}{m}(\mathcal{M}w(x) - w(x))\right\} \geq \frac{\kappa(x)}{m}.
\end{aligned}$$

The proof for η_m is similar, we omit the detail here. □

Remark 4.1. A thorough discussion on how to find a suitable function ω in the following comparison result can refer to Seydel [15] Example 2.2. In general, ω can be chosen from the class of functions with the form $\omega_1 x^p + \omega_2$ for $p > 1$ and $x \geq 0$.

Proposition 4.2 (*Comparison principle*)**.** *Let $\xi \in \mathcal{LB}^\pi(\mathbb{R})$ be a subsolution and $\eta \in \mathcal{LB}^\pi(\mathbb{R})$ be a supersolution of (3.3). Assume that there is a function w as introduced in Lemma 4.1 with* $\lim_{x\to\infty} w(x)/x = \infty$. *If $\xi(0) \leq \eta(0)$, then $\xi(x) \leq \eta(x)$ for all $x \in [0, \infty)$.*

Proof. The proof follows the method used in Albrecher and Thonhauser [1], see also Azcue and Muler [2]; however, difficulties raised from the capital injection part in the HJBQVI, which is resolved by utilizing the method discussed in Seydel [15]. Let η_m for $m \in \mathbb{N}$ as defined in Lemma 4.1. Then, it is sufficient to show that $\xi \leq \eta_m$ for all m large. For any fixed $m \in \mathbb{N}$, let $0 < M := \sup_{x\geq 0}\{\xi(x) - \eta_m(x)\} < \infty$, and $x^* := \operatorname{argmax}_{x\geq 0}\{\xi(x) - \eta_m(x)\}$. Since $\xi(x)$ is linearly bounded and $\eta_m(x)$ is increasing as polynomial function with degree $p > 1$, then we can find a sufficient large B such that $\xi(x) - \eta_m(x) \leq 0$ for $x > B$. Furthermore, since ξ and η_m are locally Lipschitz continuous, there exists a constant $n > 0$ such that

$$\frac{\xi(y) - \xi(x)}{y - x} \leq n, \quad \frac{\eta_m(y) - \eta_m(x)}{y - x} \leq n, \quad \text{for } 0 \leq x \leq y \leq B. \tag{4.10}$$

Then, we consider a set A as

$$A = \{(x, y) \,|\, 0 \leq x \leq y \leq B\},$$

we define an auxiliary function

$$H_\epsilon(x, y) := \xi(x) - \eta_m(y) - \frac{\epsilon}{2}(x - y)^2 - \frac{2n}{\epsilon^2(y - x) + \epsilon},$$

and let $M_\epsilon := \sup_{(x,y)\in A} H_\epsilon(x, y)$ with the maximizer (x_ϵ, y_ϵ). Then, it is obvious that

$$M_\epsilon \geq H_\epsilon(x^*, x^*) = M - \frac{2n}{\epsilon},$$

which is positive for sufficient large ϵ, then we arrive at

$$\liminf_{\epsilon\to\infty} M_\epsilon \geq M > 0.$$

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Note that we shall prove that the maximizer (x_ϵ, y_ϵ) is not on the boundary of set A in order to retain the differentiability at x_ϵ and y_ϵ. We postpone the proof to Lemma A.2 in the Appendix. Next, we introduce the other two auxiliary functions,

$$u(x) = \eta_m(y_\epsilon) + \frac{\epsilon}{2}(x - y_\epsilon)^2 + \frac{2n}{\epsilon^2(y_\epsilon - x) + \epsilon} + H_\epsilon(x_\epsilon, y_\epsilon), \tag{4.11}$$

$$v(y) = \xi(x_\epsilon) - \frac{\epsilon}{2}(x_\epsilon - y)^2 - \frac{2n}{\epsilon^2(y - x_\epsilon) + \epsilon} - H_\epsilon(x_\epsilon, y_\epsilon). \tag{4.12}$$

Note that u and v are continuously differentiable, and $\xi(x) - u(x) = H_\epsilon(x, y_\epsilon) - H_\epsilon(x_\epsilon, y_\epsilon) \leq 0$, which reaches the maximum 0 at x_ϵ, i.e. $\xi(x_\epsilon) = u(x_\epsilon)$. Similarly, $\eta_m(y) - v(y) = H_\epsilon(x_\epsilon, y_\epsilon) - H_\epsilon(x_\epsilon, y) \geq 0$ and reaches the minimum at y_ϵ, i.e. $\eta_m(y_\epsilon) = v(y_\epsilon)$. Since ξ is a subsolution of (3.3) and η_m is a supersolution of (4.9), we have at the points x_ϵ and y_ϵ

$$\max\{(\mathcal{A}_\pi - \delta)(\xi, u)(x_\epsilon), 1 - u'(x_\epsilon), \mathcal{M}\xi(x_\epsilon) - \xi(x_\epsilon)\} \geq 0,$$
$$\max\{(\mathcal{A}_\pi - \delta)(\eta_m, v)(y_\epsilon), 1 - v'(y_\epsilon), \mathcal{M}\eta_m(y_\epsilon) - \eta_m(y_\epsilon)\} \leq -\frac{\kappa}{m},$$

where $\kappa = \kappa(y_\epsilon) > 0$, $\kappa(\cdot)$ is the positive function introduced in Lemma 4.1, and

$$(\mathcal{A}_\pi - \delta)(\xi, u)(x_\epsilon) = cu'(x_\epsilon) - (\lambda + \delta)\xi(x_\epsilon) + \lambda\int_0^{x_\epsilon} \xi(x_\epsilon - y)\,\mathrm{d}F(y) + \lambda\int_{x_\epsilon}^{\infty} \pi(x_\epsilon - y)\,\mathrm{d}F(y),$$

and

$$(\mathcal{A}_\pi - \delta)(\eta_m, v)(y_\epsilon) = cv'(y_\epsilon) - (\lambda + \delta)\eta_m(y_\epsilon) + \lambda\int_0^{y_\epsilon} \eta_m(y_\epsilon - z)\,\mathrm{d}F(z) + \lambda\int_{y_\epsilon}^{\infty} \pi(y_\epsilon - z)\,\mathrm{d}F(z),$$

which are the operators used in an equivalent formulation of viscosity solution comparing to Definition 3.1, (see e.g. [2] Remark 3.3).

By the definition of u and v, we have

$$u'(x_\epsilon) = v'(y_\epsilon) = \epsilon(x_\epsilon - y_\epsilon) + \frac{2n}{(\epsilon(y_\epsilon - x_\epsilon) + 1)^2}.$$

On the other hand, since

$$H_\epsilon(x_\epsilon, x_\epsilon) + H_\epsilon(y_\epsilon, y_\epsilon) \leq 2H_\epsilon(x_\epsilon, y_\epsilon),$$

one has

$$\begin{aligned}&\xi(x_\epsilon) - \eta_m(x_\epsilon) + \xi(y_\epsilon) - \eta_m(y_\epsilon) - \frac{4n}{\epsilon}\\ &\quad\leq 2\left(\xi(x_\epsilon) - \eta_m(y_\epsilon) - \frac{\epsilon}{2}(x_\epsilon - y_\epsilon)^2 - \frac{2n}{\epsilon^2(y_\epsilon - x_\epsilon) + \epsilon}\right).\end{aligned}$$

Rearranging the above inequality and using (4.10), one arrives at

$$\begin{aligned}&\epsilon(x_\epsilon - y_\epsilon)^2 \leq \xi(x_\epsilon) - \xi(y_\epsilon) + \eta_m(x_\epsilon) - \eta_m(y_\epsilon) + \frac{4n(y_\epsilon - x_\epsilon)}{\epsilon(y_\epsilon - x_\epsilon) + 1}\\ \Rightarrow\quad &\epsilon(x_\epsilon - y_\epsilon)^2 \leq 2n|x_\epsilon - y_\epsilon| + 4n(y_\epsilon - x_\epsilon)\\ \Rightarrow\quad &|x_\epsilon - y_\epsilon|\left(1 - \frac{4n}{\epsilon}\right) \leq \frac{2n}{\epsilon}.\end{aligned}$$

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Then, we have for ϵ sufficiently large such that $4n/\epsilon < 1$,

$$0 \le |x_\epsilon - y_\epsilon| \left(1 - \frac{4n}{\epsilon}\right) \le \frac{2n}{\epsilon}. \tag{4.13}$$

Hence, let $(\epsilon_n)_{n\ge1}$ be an increasing sequence such that $(x_{\epsilon_n}, y_{\epsilon_n}) \to (\tilde{x}, \tilde{y})$ when $\epsilon_n \to \infty$, then according to (4.13), we must have $\tilde{x} = \tilde{y}$.

Case 1: Assume $\mathcal{M}\xi(x_\epsilon) - \xi(x_\epsilon) \ge 0$. Since $\mathcal{M}\eta_m(y_\epsilon) - \eta_m(y_\epsilon) \le -\kappa/m$, select $\nu > 0$ and $\hat{y} \ge 0$ such that $\xi(\tilde{x} + \hat{y}) - k - \phi\hat{y} + \nu > \mathcal{M}\xi(\tilde{x})$, then we have

$$\begin{aligned}
M &\le \liminf_{\epsilon\to\infty} M_\epsilon \\
&= \liminf_{\epsilon\to\infty} \left(\xi(x_\epsilon) - \eta_m(y_\epsilon) - \frac{\epsilon}{2}(x_\epsilon - y_\epsilon)^2 - \frac{2n}{\epsilon^2(y_\epsilon - x_\epsilon) + \epsilon}\right) \\
&\le \liminf_{\epsilon\to\infty} \left(\mathcal{M}\xi(x_\epsilon) - \mathcal{M}\eta_m(y_\epsilon) - \frac{\kappa}{m} - \frac{\epsilon}{2}(x_\epsilon - y_\epsilon)^2 - \frac{2n}{\epsilon^2(y_\epsilon - x_\epsilon) + \epsilon}\right) \\
&= \mathcal{M}\xi(\tilde{x}) - \mathcal{M}\eta_m(\tilde{x}) - \frac{\kappa}{m} \\
&< \xi(\tilde{x} + \hat{y}) - k - \phi\hat{y} + \nu - \eta_m(\tilde{x} + \hat{y}) + k + \phi\hat{y} - \frac{\kappa}{m} \\
&= \xi(\tilde{x} + \hat{y}) - \eta_m(\tilde{x} + \hat{y}) + \nu - \frac{\kappa}{m} \\
&\le M + \nu - \frac{\kappa}{m},
\end{aligned}$$

which is a contradiction when ν is sufficiently small.

Case 2: Assume $\mathcal{M}\xi(x_\epsilon) - \xi(x_\epsilon) < 0$, we must have

$$\begin{cases} (\mathcal{A}_\pi - \delta)(\xi, u)(x_\epsilon) \ge 0, \\ (\mathcal{A}_\pi - \delta)(\eta_m, v)(y_\epsilon) \le -\dfrac{\kappa}{m}. \end{cases}$$

Then in the following, we derive contradiction from inequality

$$(\mathcal{A}_\pi - \delta)(\xi, u)(x_\epsilon) > (\mathcal{A}_\pi - \delta)(\eta_m, v)(y_\epsilon).$$

By noting that $u'(x_\epsilon) = v'(y_\epsilon)$, one has

$$\begin{aligned}
(\lambda + \delta)(\xi(x_\epsilon) - \eta_m(y_\epsilon)) < \lambda \int_0^{x_\epsilon} \xi(x_\epsilon - z)\,\mathrm{d}F(z) - \lambda \int_0^{y_\epsilon} \eta_m(y_\epsilon - z)\,\mathrm{d}F(z) \\
+ \lambda \int_{x_\epsilon}^\infty \pi(x_\epsilon - z)\,\mathrm{d}F(z) - \lambda \int_{y_\epsilon}^\infty \pi(y_\epsilon - z)\,\mathrm{d}F(z).
\end{aligned}$$

Similarly, consider the sequence $(\epsilon_n)_{n\ge1}$ such that $(x_{\epsilon_n}, y_{\epsilon_n}) \to (\tilde{x}, \tilde{x})$ as $\epsilon_n \to \infty$, then one arrives at

$$(\lambda + \delta)(\xi(\tilde{x}) - \eta_m(\tilde{x})) < \lambda \int_0^{\tilde{x}} [\xi(\tilde{x} - z) - \eta_m(\tilde{x} - z)]\,\mathrm{d}F(z) \le \lambda M \int_0^{\tilde{x}} \mathrm{d}F(z),$$

then,

$$M \le \liminf_{\epsilon\to\infty} M_\epsilon \le \lim_{n\to\infty} M_{\epsilon_n} = \xi(\tilde{x}) - \eta_m(\tilde{x}) < \frac{\lambda}{\lambda + \delta} M,$$

which is a contradiction. Finally, we complete the proof by noting that the above derivations are still valid when $m \to \infty$. □

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According to Proposition 3.2 and the above comparison principle, we are able to characterize the value function V as the viscosity solution of (3.3) with the smallest value at 0, that is we define

$$V(0) = \inf\{u(0) \mid u \text{ is a viscosity solution to the HJBQVI and } u \in \mathcal{LB}^{\pi}(\mathbb{R})\}.$$

Proposition 4.3. *The value function we characterized above is the unique viscosity solution of the HJBQVI (3.3) within the class $\mathcal{LB}^{\pi}(\mathbb{R})$.*

Proof. Let $h \in \mathcal{LB}^{\pi}(\mathbb{R})$ and $g \in \mathcal{LB}^{\pi}(\mathbb{R})$ be two viscosity solutions of (3.3) with smallest value at zero. On the one hand, let h be the subsolution and g be the supersolution of (3.3) with $h(0) \le g(0)$, then according to Proposition 4.2 and Definition 3.1, we have $h \le g$. On the other hand, let g be the subsolution, h be the supersolution and $g(0) \le h(0)$, then we arrive at $g \le h$. Hence, we have $h = g$. □

Note that, Proposition 4.1 directly provides a verification result for the optimal strategy.

Corollary 4.1. *Consider an admissible strategy $\theta \in \Theta$ and the associated performance function V_θ such that $V_\theta \in \mathcal{LB}^{\pi}(\mathbb{R})$, and V_θ is a supersolution of the HJBQVI (3.3), then $V_\theta = V$ and θ is in turn an optimal strategy.*

Proof. According to Proposition 4.1, we have $V_\theta \ge V$; however, since θ is an admissible strategy, by definition of the value function, $V_\theta \le V$. Hence, we have $V = V_\theta$. □

5. Construction of the optimal strategy

In this section, we discuss the general structure of the optimal strategy for the optimal singular dividend and capital injection problem with affine penalty payment at ruin. It has been showed in the literature that the candidate optimal strategy is in band-type, which can be represented using several abstract sets (see e.g. [1,3,18]). To be specific, the general optimal strategy can be described through the following four sets:

$$\begin{aligned}
\mathcal{D}_1 &= \{x \in (0,\infty) : (\mathcal{A}^*_\pi - \delta)V(x) < 0 \text{ and } V'(x) = 1\},\\
\mathcal{D}_2 &= \{x \in [0,\infty) : (\mathcal{A}^*_\pi - \delta)V(x) = 0\},\\
\mathcal{N} &= \{x \in (\mathcal{D}_1 \cup \mathcal{D}_2)^c : V(x) > \mathcal{M}V(x)\},\\
\mathcal{C} &= \{x \in (\mathcal{D}_1 \cup \mathcal{D}_2)^c : V(x) = \mathcal{M}V(x)\},
\end{aligned}$$

where

$$(\mathcal{A}^*_\pi - \delta)V(x) = c - (\lambda+\delta)V(x) + \lambda\int_0^x V(x-y)\,\mathrm{d}F(y) + \lambda\int_x^\infty \pi(x-y)\,\mathrm{d}F(y) = 0. \tag{5.1}$$

Note that $\mathcal{C}$ represents the area where immediate capital injection is optimal, and the set $\mathcal{N}$ represents the area with no dividend payments and capital injections. $\mathcal{D}_1$ and $\mathcal{D}_2$ represent the areas with lump sum and continuous dividend payments at the rate equal to premium rate. To begin, we first state a local version of Proposition 4.1, and introduce an auxiliary function $U_y(x)$ for $y > 0$ in the following, and then provide some technical lemmas on the relationship between the U_y and value function V.

Lemma 5.1. *For some $\hat{x} > 0$, if either $(\mathcal{A}^*_\pi - \delta)V(\hat{x}) = 0$ or $V'(\hat{x}) = 1$, and $\bar{h}(x) \in \mathcal{LB}^{\pi}(\mathbb{R})$ is a viscosity supersolution of (3.3) for $x \in [0,\hat{x})$, then we have $\bar{h}(x) \ge V(x)$ for all $x \in [0,\hat{x}]$. Furthermore, let $\Theta_{\hat{x}}$ be the set of admissible strategies such that the controlled surplus process $U^\theta_t \le \hat{x}$ for all $t \ge 0$, and let $\theta \in \Theta_{\hat{x}}$ be an admissible strategy such that the performance function $V_\theta(x) \in \mathcal{LB}^{\pi}(\mathbb{R})$, and is a viscosity supersolution of (3.3); then $V_\theta(x) = V(x)$ for all $x \in [0,\hat{x}]$.*

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Proof. The proof can refer to Proposition 5.7 and Theorem 5.8 in Azcue and Muler [2], where our capital injection and penalty payment at ruin make no difference in the analysis. □

For any $y > 0$, define

$$U_y(x) = \begin{cases} V(x), & x \le y, \\ x - y + V(y), & x > y. \end{cases} \tag{5.2}$$

Lemma 5.2. *Consider $\tilde{x} > 0$ such that $(\mathcal{A}^*_\pi - \delta)V(\tilde{x}) < 0$ or $V'(\tilde{x}) = 1$. Then for any $y < \tilde{x}$, if U_y is a viscosity supersolution of (3.3) in $(y, \tilde{x}]$, then $U_y(x) = V(x)$ for all $x \in [0, \tilde{x}]$.*

Proof. We follow the proof in Proposition 5.10 of Azcue and Muler [2]. First, we show that $U_y(x) \ge V(x)$ for $x \in [0, \tilde{x}]$. According to the definition of $U_y(x)$ in (5.2) and Lemma 5.1, we only need to show that U_y is a viscosity supersolution of (3.3) at $y < \tilde{x}$. Note that, $U'_y(y^+) = 1$, and according to Azcue and Muler [4] Definition 3.2 (see also [2] Remark 3.5), and the fact that $V'(x) \ge 1$, then there exists a test function (say φ) such that U_y is a viscosity supersolution only when $U'_y(y^-) = V'(y^-) = 1$, then $\varphi'(y) = 1$, and

$$\begin{aligned}
&(\mathcal{A}_\pi - \delta)(\varphi, U_y)(y) \\
&\quad = c - (\lambda + \delta)U_y(y) + \lambda \int_0^y U_y(y - z)\,\mathrm{d}F(z) + \lambda \int_y^\infty \pi(y - z)\,\mathrm{d}F(z) \\
&\quad = c - (\lambda + \delta)V(y) + \lambda \int_0^y V(y - z)\,\mathrm{d}F(z) + \lambda \int_y^\infty \pi(y - z)\,\mathrm{d}F(z) \le 0,
\end{aligned}$$

since V is a viscosity supersolution of (3.3) at y. And

$$\begin{aligned}
&\mathcal{M}U_y(x) - U_y(x) \\
&\quad = \sup_{z \ge x}\{U_y(z) - k - \phi(z - x)\} - U_y(x) \\
&\quad = \sup_{z \ge x}\{z - y + V(y) - k - \phi(z - x)\} - U_y(x) \\
&\quad = x - y + V(y) - k - V(y) - x + y = -k < 0.
\end{aligned}$$

Hence, U_y is a viscosity supersolution of (3.3) at y. Next, we show that $U_y(x) \le V(x)$ for $x \in (y, \tilde{x}]$. Consider any $\epsilon > 0$, and an ϵ-optimal strategy θ such that $V(y) \le V_\theta(y) + \epsilon$. Then, for initial surplus $x > y$, consider another strategy θ_x, where the amount of $x - y$ is payout as dividend immediately and follow strategy θ thereafter; hence, θ_x is an admissible strategy as well. Then, we have for any $\epsilon > 0$ and $x > y$,

$$U_y(x) - \epsilon = V(y) + x - y - \epsilon \le V_\theta(y) + x - y = V_{\theta_x}(x) \le V(x),$$

by letting $\epsilon \to 0$, we arrive at $U_y(x) \le V(x)$ for $x > y$. Hence, $U_y(x) = V(x)$ for $x \in [0, \tilde{x}]$. □

Lemma 5.3. *For any $y > 0$ if $U_y(x)$ as defined in (5.2) is a viscosity supersolution of (3.3) for $x \in (y, \infty)$, then $U_y(x) = V(x)$ for all $x \ge 0$.*

Proof. The proof is similar to the proof of Lemma 5.2, we omit the detail here. □

Finally, we provide the topological structures of the above-defined abstract sets, and propose the candidate optimal band-type strategy.

Proposition 5.1. *(i) $\mathcal{D}_2$ is closed.*

(ii) $\mathcal{D}_1$ is left-open, and the lower limit of any connected components of $\mathcal{D}_1$ belongs to $\mathcal{D}_2$. There exists x^* which is large enough satisfying $(x^*, \infty) \subset \mathcal{D}_1$.

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(iii) $\mathscr{C}$ *is closed.*
(iv) $\mathscr{N}$ *is right-open, and the connected components of* $\mathscr{N}$ *is bounded, the upper limit of any connected components of* $\mathscr{N}$ *is in* $\mathscr{D}_2$*.*

Proof. The proof follows the similar steps in Azcue and Muler [2], Albrecher and Thonhauser [1] and Xu and Woo [18].

(i) Given that the claim size distribution F is continuous, $(\mathcal{A}_\pi^* - \delta)V(\cdot)$ is also continuous, therefore $\mathscr{D}_2$ is closed.
(ii) Consider any $\hat{x} \in \mathscr{D}_1$, then we have $(\mathcal{A}_\pi^* - \delta)V(\hat{x}) < 0$ and $V'(\hat{x}) = 1$. Let us consider the auxiliary function defined in (5.2) $U_{\hat{x}-h}$ for each small $h > 0$, then we have for any $x \in (\hat{x} - h, \hat{x})$

$$\begin{aligned}
&(\mathcal{A}_\pi^* - \delta)U_{\hat{x}-h}(x)\\
&\quad \le c - (\lambda+\delta)U_{\hat{x}-h}(\hat{x}-h) + \lambda\int_0^{\hat{x}} U_{\hat{x}-h}(\hat{x}-y)\,\mathrm{d}F(y) + \lambda\int_{\hat{x}}^{\infty}\pi(\hat{x}-y)\,\mathrm{d}F(y)\\
&\quad \le c - (\lambda+\delta)V(\hat{x}-h) + \lambda\int_0^{\hat{x}} V(x-y)\,\mathrm{d}F(y) + \lambda\int_{\hat{x}}^{\infty}\pi(x-y)\,\mathrm{d}F(y)\\
&\quad = (\mathcal{A}_\pi^* - \delta)V(\hat{x}) + (\lambda+\delta)(V(\hat{x}) - V(\hat{x}-h)),
\end{aligned}$$

where the second last inequality holds true since $V(\hat{x}-y) - V(\hat{x}-h) \ge (\hat{x}-y) - (\hat{x}-h)$ for $y \in (0, h)$ from Lemma 2.1. Then, since V is continuous, there must exist a sufficient small $\hat{h} > 0$ such that $(\mathcal{A}_\pi^* - \delta)U_{\hat{x}-\hat{h}}(x) < 0$ for all $x \in (\hat{x}-\hat{h}, \hat{x})$. In addition,

$$\begin{aligned}
&\mathcal{M}U_{\hat{x}-\hat{h}}(x) - U_{\hat{x}-\hat{x}}(x)\\
&\quad = \sup_{y\ge x}\{U_{\hat{x}-\hat{h}}(y) - k - \phi(y-x)\} - U_{\hat{x}-\hat{h}}(x)\\
&\quad = \sup_{y\ge x}\{y - (\hat{x}-\hat{h}) + V(\hat{x}-\hat{h}) - k - \phi(y-x)\} - U_{\hat{x}-\hat{h}}(x)\\
&\quad = x - (\hat{x}-\hat{h}) + V(\hat{x}-\hat{h}) - k - V(\hat{x}-\hat{h}) - x + (\hat{x}-\hat{h}) = -k < 0,
\end{aligned}$$

for all $x \in (\hat{x}-\hat{h}, \hat{x})$. Therefore, $U_{\hat{x}-\hat{h}}$ is a viscosity supersolution of (3.3) in $(\hat{x}-\hat{h}, \hat{x})$. Hence, according to Lemma 5.2, we have $U_{\hat{x}-\hat{h}} = V$ in $[0, \hat{x})$, therefore $(\hat{x}-\hat{h}, \hat{x}) \in \mathscr{D}_1$, i.e. $\mathscr{D}_1$ is left-open. To prove that the lower limit of any connected components of $\mathscr{D}_1$ is in $\mathscr{D}_2$, one need to show for any sufficiently small $h > 0$ such that $(\hat{x}, \hat{x}+h) \subset \mathscr{D}_1$ and $\hat{x} \notin \mathscr{D}_1$, then $\hat{x} \in \mathscr{D}_2$. Note that if $\hat{x} \notin \mathscr{D}_2$, it must in $\mathscr{N}$. However, since $V(x)$ is a viscosity supersolution of (3.3), if $\hat{x} \in \mathscr{N}$, we have $(\mathcal{A}_\pi^* - \delta)V(\hat{x}) < 0$ and $V'(\hat{x}) > 1$ given the derivative exists. Then, assume that there exists a sequence $x_n \uparrow \hat{x}$ such that $V'(x_n)$ exists; we can show that the sequence is not in $\mathscr{D}_2$ since it is closed; and the sequence is also not in $\mathscr{D}_1$, since if so, according to Lemma A.3 we must have $(\hat{x}-h_0, \hat{x}) \subset \mathscr{D}_1$ for some $h_0 > 0$ and in turn $V'(x) = 1$ for all $x \in (\hat{x}-h_0, \hat{x})$ which is a contradiction. Therefore, the sequence x_n must in $\mathscr{N}$ as well and there exists $h' > 0$ such that $(\hat{x}-h', \hat{x}) \subset \mathscr{N}$. In other words, one obtain that $V'(\hat{x}^-) > 1$ and $V'(\hat{x}^+) = 1$. Since V is a viscosity subsolution of (3.3) and $\hat{x} \notin \mathscr{C}$, then the test function say φ with $\varphi'(\hat{x}) = 1$ should gives the following inequality

$$\begin{aligned}
&c\varphi'(\hat{x}) - (\lambda+\delta)V(\hat{x}) + \lambda\int_0^{x} V(x-y)\,\mathrm{d}F(y) + \lambda\int_x^{\infty}\pi(x-y)\,\mathrm{d}F(y)\\
&\quad = (\mathcal{A}_\pi^* - \delta)V(\hat{x}) \ge 0,
\end{aligned}$$

which is a contradiction. Hence, one arrive at $\hat{x} \in \mathscr{D}_2$.
Finally, we show that there exists x^* large enough such that $(x^*, \infty) \subset \mathscr{D}_1$. According to Lemma 5.3, we only need to show that for sufficiently large $x^* > 0$, the auxiliary function $U_{x^*}(x)$ is a

viscosity supersolution of (3.3) for $x \in (x^*, \infty)$. Note that $U'_{x^*}(x) = 1$ for $x \in (x^*, \infty)$ by definition and $\mathcal{M}U_{x^*}(x) \le U_{x^*}(x)$ holds true obviously, then we only to need show that $(\mathcal{A}_\pi - \delta)U_{x^*}(x) \le 0$ for $x > x^*$. Since $U'_{x^*}(x^*) = 1$, then

$$
\begin{aligned}
&(\mathcal{A}_\pi - \delta)U_{x^*}(x) \\
&\quad = c - (\lambda + \delta)U_{x^*}(x) + \lambda \int_0^x U_{x^*}(x-z)\,\mathrm{d}F(z) + \lambda \int_x^\infty \pi(x-z)\,\mathrm{d}F(z) \\
&\quad \le c - (\lambda + \delta)(x - x^* + V(x^*)) + \lambda \int_0^x (x - z - x^* + V(x^*))\,\mathrm{d}F(z) \\
&\quad \le c - \delta(x - x^* + V(x^*)),
\end{aligned}
$$

by noting that for each $x^* \ge 0$, $c - \delta(x - x^* + V(x^*))$ is a decreasing function of x; hence, when $V(x^*) \ge c/\delta$, we arrive at $(\mathcal{A}_\pi - \delta)U_{x^*}(x) \le 0$ for $x > x^*$. Then, from (2.5), we have the lower bound $V(x^*) \ge x^* + (c + \lambda(K - \Phi\mu))/(\lambda + \delta)$, therefore, the result follows by choosing $x^* = c/\delta - (c + \lambda(K - \Phi\mu))/(\lambda + \delta)$.

(iii) From Lemmas 2.1 and 2.2, we have that $V(x) - \mathcal{M}V(x)$ is continuous, hence $\mathscr{C}$ is closed.

(iv) Consider $x_0 \in \mathcal{N}$ and a sequence $x_n \downarrow x_0$, since $\mathscr{D}_2$ and $\mathscr{C}$ are closed sets, the sequence is not in $\mathscr{D}_2$ and $\mathscr{C}$. Assume that $x_n \in \mathscr{D}_1$, since the lower limit of any connected component of $\mathscr{D}_1$ belongs to $\mathscr{D}_2$, there must exist a subsequence $x'_n \in \mathscr{D}_2$ such that $x'_n \downarrow x_0$, which is a contradiction. Hence, $x_n \in \mathcal{N}$ and $\mathcal{N}$ is right-open. The connected components of $\mathcal{N}$ is bounded follows obviously from the fact that there is no $\hat{x}$ sufficient large such that $[\hat{x}, \infty) \subset \mathcal{N}$. On the other hand, since $\mathscr{D}_1$ is left-open, then the upper limit of any connected component of $\mathcal{N}$ is in $\mathscr{D}_2$.

□

In the following, we define a band-type dividend and capital injection strategy based on the above-mentioned crucial sets; the optimality of such band-type strategy is proved in Proposition 5.2.

Definition 5.1. *We define a band-type dividend and capital injection strategy associated with the partition of* $[0, \infty) = \mathscr{D}_1 \cup \mathscr{D}_2 \cup \mathcal{N} \cup \mathscr{C}$ *as follows: If the surplus level is in the set* $\mathscr{D}_1$ *pay a lump-sum dividend immediately, such that the ending surplus level is at the lower limit of current connected compound of* $\mathscr{D}_1$ *(which is in* $\mathscr{D}_2$ *as we showed in Proposition 5.1). If the surplus level is in* $\mathscr{D}_2$ *pay out the incoming premium directly as dividend until the arrival of the next claim. If current surplus is in* $\mathcal{N}$*, no action is taken. And, if current surplus is in* $\mathscr{C}$*, an immediate capital injection is implemented, where the injection amount is determined according to the capital injection operator given in (2.6).*

Proposition 5.2. *The band-type dividend and capital injection strategy given in Definition 5.1 is the optimal strategy among all admissible ones.*

Proof. We denote such band-type strategy as θ_b, hence, we want to show that $V(x) = V_{\theta_b}(x)$ for all $x \ge 0$. The proof follows the fixed point argument. Let's consider a complete metric space $\mathbb{M}$ of continuous functions $f : [0, \infty) \to \mathbb{R}$ satisfying that

$$f(x) = x - x^* + f(x^*), \quad \text{for } x \ge x^*,$$

with the metric being the supremum norm $d(\cdot, \cdot)$ defined as

$$d(f_1, f_2) = \sup_{x \ge 0} |f_1(x) - f_2(x)|.$$

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We define an operator $\mathcal{T}:\mathbb{M}\to\mathbb{M}$ as

$$\mathcal{T}(f)(x)=\mathbb{E}_x\left[\int_0^{T_1} e^{-\delta t}\,\mathrm{d}L_t^{\theta_b}-\sum_{i=1}^{\infty} e^{-\delta\omega^{\theta_b}}(k+\phi\zeta_i^{\theta_b})1_{\{\omega^{\theta_b}<T_1\}}+e^{-\delta T_1}f(X_{T_1}^{\theta_b})\right],$$

where T_1 is the arrival time of the first claim. Note that, according to Proposition 5.1(ii), there exists $x^*\in\mathscr{D}_2$ such that $(x^*,\infty)\subset\mathscr{D}_1$. Since $V'(x)=1$ for $x\in\mathscr{D}_1$, then we have $V(x)=x-x^*+V(x^*)$ for $x\in(x^*,\infty)$, hence $V\in\mathbb{M}$. It is also obvious that $|\mathcal{T}f_1(x)-\mathcal{T}f_2(x)|\le(\lambda/(\lambda+\delta))d(f_1,f_2)$, hence, $\mathcal{T}$ is a contraction mapping with modulus less than 1, i.e. $\mathcal{T}$ admits a unique fixed point. According to Definition 5.1, θ_b is a stationary strategy, then we have $\mathcal{T}V_{\theta_b}=V_{\theta_b}$. Finally, we show that the value function V is also a fixed point of $\mathcal{T}$, i.e. $\mathcal{T}V=V$. For $x\in\mathscr{D}_2$, we have

$$\begin{aligned}\mathcal{T}V(x)&=\mathbb{E}_x\left[\int_0^{T_1} e^{-\delta t}c\,\mathrm{d}t+e^{-\delta T_1}V(X_{T_1}^{\theta_b})\right]\\&=\frac{c}{\lambda+\delta}+\int_0^{\infty}\lambda e^{-(\lambda+\delta)t}\left\{\int_0^x V(x-z)\,\mathrm{d}F(z)+\int_x^{\infty}\pi(x-z)\,\mathrm{d}F(z)\right\}\mathrm{d}t\\&=\frac{c}{\lambda+\delta}+\frac{\lambda}{\lambda+\delta}\left\{\int_0^x V(x-z)\,\mathrm{d}F(z)+\int_x^{\infty}\pi(x-z)\,\mathrm{d}F(z)\right\}\\&=V(x),\end{aligned}\tag{5.3}$$

where the last equation holds since we have $(\mathcal{A}_\pi^*-\delta)V(x)=0$ for $x\in\mathscr{D}_2$.

For $x\in\mathscr{D}_1$, let $\hat{x}=\inf\{y:(y,x]\subset\mathscr{D}_1\}$, then $\hat{x}\in\mathscr{D}_2$, and from 5.3 we have

$$\mathcal{T}V(x)=x-\hat{x}+\mathcal{T}V(\hat{x})=V(x).$$

For $x\in\mathscr{N}$, let us consider $\bar{x}=\min\{y>x,y\notin\mathscr{N}\}$, then $\bar{x}\in\mathscr{D}_2$. Let $s=(\bar{x}-x)/c$ and $\bar{y}(t)=x+ct$, one has $\bar{y}(t)\in\mathscr{N}$ for $t<s$. Then, we have

$$\begin{aligned}\mathcal{T}V(x)&=\mathbb{E}_x[e^{-\delta s}V(\bar{x})1_{\{T_1>s\}}]+\mathbb{E}_x[e^{-\delta T_1}V(\bar{y}(T_1)-Y_1)1_{\{T_1\le s\}}]\\&=e^{-(\lambda+\delta)s}V(\bar{x})+\int_0^s\lambda e^{-(\lambda+\delta)t}\left\{\int_0^{\bar{y}(t)}V(\bar{y}(t)-z)\,\mathrm{d}F(z)+\int_{\bar{y}(t)}^{\infty}\pi(\bar{y}(t)-z)\,\mathrm{d}F(z)\right\}\mathrm{d}t\\&=e^{-(\lambda+\delta)s}V(\bar{x})+\int_0^s(-e^{-(\lambda+\delta)t}V(\bar{y}(t)))'\,\mathrm{d}t\\&=V(x),\end{aligned}$$

where the second last equation holds true since $V(x)$ (as an viscosity solution) is an a.e. (or weak) solution of

$$ch'(x)-(\lambda+\delta)h(x)+\lambda\int_0^x h(x-z)\,\mathrm{d}F(z)+\lambda\int_x^{\infty}\pi(x-z)\,\mathrm{d}F(z)=0,$$

for $x\in\mathscr{N}$, see e.g. Xu and Woo [18].

Finally, for $x\in\mathscr{C}$, we have immediate capital injection that bring the surplus to $\mathscr{N}$, then

$$\mathcal{T}V(x)=\mathcal{T}V(y^*)-k-\phi(y^*-x)=V(y^*)-k-\phi(y^*-x)=\mathcal{M}V(x)=V(x),$$

where $y^*\in\mathscr{N}$ satisfying $y^*=\operatorname{argmax}_y\{y\ge x|V(y)-k-\phi(y-x)\}$ and for $x\in\mathscr{C}$ we have $\mathcal{M}V(x)=V(x)$.

Then, we have $V=V_{\theta_b}$, and θ_b is the optimal strategy. □

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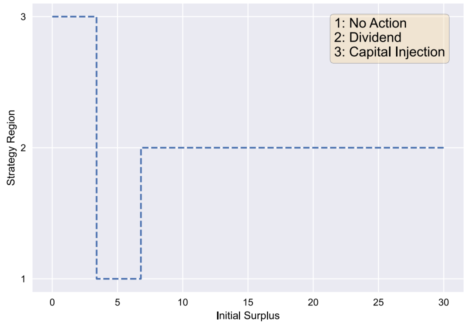
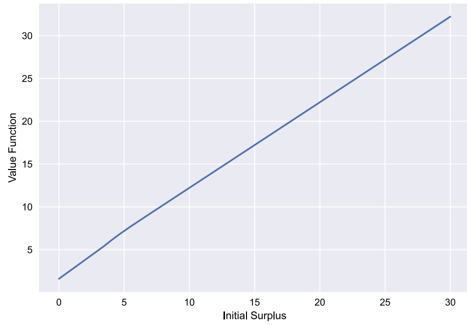
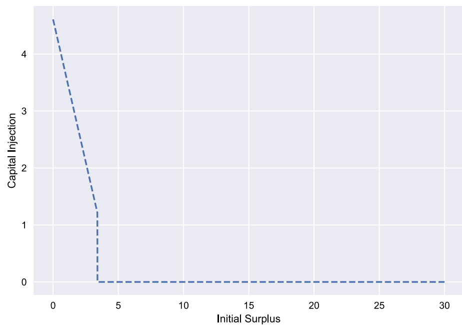


(i) Optimal strategy



(ii) Value function



(iii) Optimal capital injection amount

Figure 1. *Exponential claim size distribution: Benchmark example.*

6. Numerical illustration

To further illustrate the explicit form of the optimal dividend and capital injection strategy, we provide some numerical examples in this section. We first assume that the claim size follows exponential distribution, where the resulting optimal strategy is in the form of band-type strategy with one dividend barrier. Then, we further assume the claim size follows gamma distribution, under which the optimal band-type strategy could have a more complicated structure based on the values of transaction costs for capital injection and penalty payments at ruin.

6.1. Exponential distribution

Example 6.1. We first consider the case when claim size follows exponential distribution with probability density function $f(x) = \beta e^{-\beta x}$, where $\beta = 1$; we assume that the Poisson intensity $\lambda = 1$ and the premium rate $c = 1.5$. We set a benchmark example for the analysis, where the parameters associated with transaction costs and penalty function are given as $K = -5$, $\Phi = 0.7$, $k = 0.1$ and $\phi = 1.1$. The discount factor $\delta = 0.05$. The numerical results of the value function, the optimal band-type strategy and optimal capital injection amount versus initial surplus for this benchmark example are given in Figure 1.

Figure 1(i) illustrates the structure of the optimal band-type strategy, where we use 1,2,3 to denote the "no action," "paying dividend" (lump sum or continuously at premium rate) and "capital injection" range, respectively. In this benchmark example, the optimal dividend strategy is 1-barrier strategy at surplus level 6.791; when surplus level locates within the range $[0, 3.392]$, the optimal strategy is to

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Table 1. *Varying fixed transaction costs k.*

k	0.001	0.01	***0.1***	0.5	1	2	4
$\underline{x}$	4.472	4.190	3.392	2.205	1.460	1.460	–
$\bar{x}$	4.608	4.608	4.608	4.608	4.607	4.607	–
$\hat{x}$	6.792	6.791	6.791	6.806	6.831	6.831	6.777
$\bar{x}-\underline{x}$	0.136	0.418	1.215	2.403	3.148	3.148	–

Table 2. *Varying proportional transaction costs ϕ.*

ϕ	1.01	1.05	***1.1***	1.2	1.5	1.8	2
$\underline{x}$	4.041	3.766	3.392	2.845	1.801	1.117	0.757
$\bar{x}$	5.700	5.189	4.608	3.854	2.557	1.756	1.346
$\hat{x}$	7.686	6.796	6.791	6.791	6.791	6.791	6.791
$\bar{x}-\underline{x}$	1.659	1.423	1.215	1.009	0.756	0.640	0.588

Table 3. *Varying fixed penalty payments at ruin K.*

K	***−5***	−4	−2	−1	−0.5	−0.1	−0.01
$\underline{x}$	3.392	3.161	2.637	2.337	2.175	2.039	2.008
$\bar{x}$	4.608	4.377	3.852	3.552	3.390	3.255	3.223
$\hat{x}$	6.791	6.563	6.051	5.697	5.535	5.443	5.412
$\bar{x}-\underline{x}$	1.215	1.215	1.215	1.215	1.215	1.215	1.215

Table 4. *Varying proportional penalty payments at ruin Φ.*

Φ	0.01	***0.1***	0.3	0.5	0.7	0.8	1
$\underline{x}$	3.235	3.256	3.302	3.348	3.392	3.415	3.458
$\bar{x}$	4.450	4.471	4.518	4.563	4.608	4.630	4.674
$\hat{x}$	6.644	6.671	6.699	6.745	6.791	6.774	6.868
$\bar{x}-\underline{x}$	1.215	1.215	1.215	1.215	1.215	1.215	1.215

inject capital and bring the surplus level to 4.608. Figure 1(iii) shows the corresponding amount of capital injection for each surplus level. And the resulting value function for the benchmark example is calculated and illustrated in Figure 1(ii).

Example 6.2. In this example, we show how the transaction costs and penalty payments will influence the optimal dividend and capital injection strategy. We numerically calculate the optimal band-type strategy by varying k, ϕ, K and Φ from the benchmark example, respectively (the value for the benchmark example is highlight in bold italics). The results are given in Tables 1–4. Note that we use $\hat{x}$ to denote the optimal dividend barrier, $\underline{x}$ to denote the up level of capital injection region (the bottom level of capital injection region, if exists, is always 0 in exponential case) and use $\bar{x}$ to denote the surplus level after capital injection.

In Table 1, we change the fixed transaction costs k for the capital injection from almost zero (0.001) to a considerable large value 4. It is showed in the table that k has critical effect on the up level of

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capital injection region but somewhat independent to the dividend barrier and surplus level after capital injection (if exists); when k is small, it is optimal to allow capital injection for large region above zero. But when k increases the capital injection region will shrink; and when k is sufficient large (equal to 4 or above in our example), it is no longer optimal to allow any capital injections; then, the band-type strategy reduces to barrier strategy. Table 2 gives the corresponding optimal band-type strategies for different values of proportional transaction costs ϕ. Similarly, ϕ has very limited influence on dividend barrier, but the capital injection region including the optimal surplus level after capital injection highly dependent on the value of ϕ. For a small value of proportional transaction costs (say 1%), it is optimal to allow capital injection in a broad range above zero $[0, 4.401]$, and the optimal capital inject amount is 1.659. If we increase the proportional transaction costs to 100% (i.e. $\phi = 2$), then the optimal capital injection region shrinks to $[0, 0.757]$ and the optimal capital injection amount also decreases from previous 1.659 to only 0.588. On the other hand, Tables 3 and 4 illustrate the optimal strategies when varying the fixed (K) and proportional (Φ) penalty payments at ruin. It is obvious that penalty payment has no effect on the optimal capital injection amount ($\bar{x} - \underline{x}$). In addition, we can observe from the tables that the up level of capital injection region ($\underline{x}$) is increasing with respect to $|K|$ and Φ, which means that a higher requirement of penalty payment when ruin occurs will result in a larger region of capital injection above zero, in order to reduce the possibility of ruin when surplus level is low. The fixed penalty also influences the final optimal dividend barrier, where smaller fixed penalty will generate lower dividend barrier (i.e. more aggressive dividend strategy); but the influence from proportional penalty is rather limited.

6.2. Gamma distribution

It is interesting to further consider the optimal band-type strategy when the claim size follows gamma distribution. According to the numerical results in Azcue and Muler [3] and Xu and Woo [18], the optimal dividend strategy is often in the form of 2-barrier strategy in certain scenarios. Hence, in this subsection, we numerically investigate the optimal dividend and capital injection strategy with different values of penalty payments and transaction costs for capital injection under gamma distributed claim sizes.

Example 6.3. We assume claim size follows gamma distribution with probability density function $f(x) = xe^{-x}$ (i.e. $Gamma(2,1)$). Similar to the examples in exponential distribution, we set a benchmark example with $\lambda = 10$, $c = 21.5$, $\delta = 0.1$, $k = 0.1$, $\phi = 1.05$, $K = -2$ and $\Phi = 0.1$. The numerical results of the value function, the optimal band-type strategy and optimal capital injection amount versus initial surplus are given in Figure 2.

Figure 2(i) shows that the optimal strategy for the benchmark example under gamma distribution has two optimal dividend barriers with one at $x = 0$ and the other at $x = 6.464$. The optimal capital injection region $[0.229, 1.980]$ is located between the two dividend barriers. This optimal band-type strategy tells that when the surplus level x is in the set $\{0, 6.464\}$, it is optimal to pay dividend at a constant rate 21.5 (the premium rate). And the amount of $x - 0$ and $x - 6.464$ should be paid out immediately as dividend if $x \in (0, 0.229)$ and $x \in (6.464, \infty)$, respectively; if $x \in [0.229, 1.980]$, it is optimal to inject capital and bring the surplus level to 4.513; finally, if $x \in (1.980, 6.464)$, no action is needed. Figure 2(ii) and (iii) illustrates the corresponding value function and optimal capital injection amount, respectively.

Example 6.4. Similar to the exponential case, we investigate the optimal band-type strategies by varying respectively k, ϕ, K and Φ from the benchmark example under gamma distribution. The results are summarized in Tables 5–8. Note that, in the following tables, we use $\hat{x}_1$ and $\hat{x}_2$ (if it is 2-barrier dividend strategy) to denote the “first” and “second” dividend barrier, whenever there is only 1 barrier in the final optimal strategy we keep $\hat{x}_1$ empty and use $\hat{x}_2$ to denote the dividend barrier. In addition, we use y_1 to denote the surplus level when it is optimal to change from lump sum dividend payment to capital injection or no action (if the final optimal band-type strategy does not have capital injection

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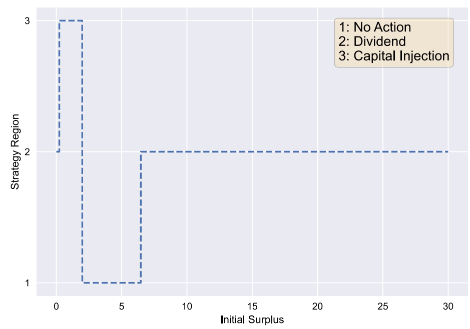
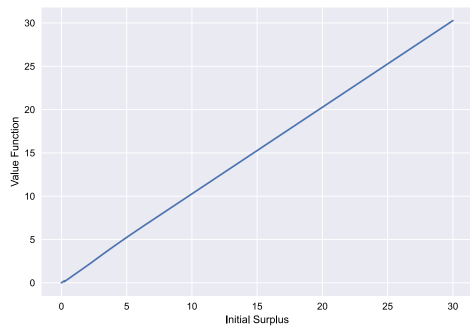
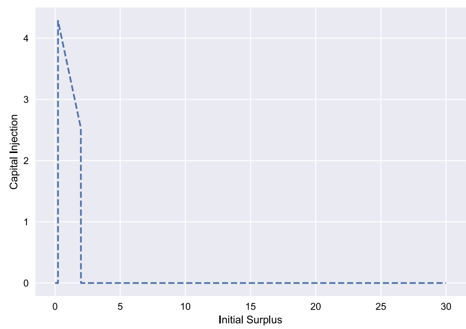


(i) Optimal strategy



(ii) Value function



(iii) Optimal capital injection amount

Figure 2. *Gamma claim size distribution: Benchmark example.*

Table 5. *Varying fixed transaction costs k.*

k	0.001	0.01	***0.1***	0.5	1	2
$\hat{x}_1$	0.000	0.000	0.000	0.000	0.000	0.000
y_1	0.229	0.229	0.229	0.229	0.229	0.229
$\underline{x}$	3.796	3.293	1.980	0.358	–	–
$\bar{x}$	4.049	4.091	4.513	6.659	–	–
$\hat{x}_2$	5.870	5.922	6.464	9.946	12.688	12.688
$\bar{x} - \underline{x}$	0.254	0.798	2.533	6.301	–	–

region). The up level of capital injection region and the surplus level after capital injection are denoted by $\underline{x}$ and $\bar{x}$, respectively.

It is quite interesting to observe that under gamma distribution we have different types of optimal band-type strategy for different value of transaction costs and penalty payments. In particular, from Table 5, we observe that the fixed transaction costs k is independent of first dividend barrier and first lump sum dividend payment region, but a higher value of k generates higher level for second dividend barrier. Similar to the exponential case, when k is sufficient large, it is non-optimal to allow capital injection; however in the gamma case, the value of k has a more significant effect on the ending surplus level after capital injection (if exists) comparing to exponential case. The increasing value of $\bar{x}$ when k increases may also explain or contribute to the increasing trend in second dividend barrier. The proportional

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Table 6. *Varying proportional transaction costs ϕ.*

ϕ	1.001	1.01	***1.05***	1.1	1.2	1.5
$\hat{x}_1$	0.000	0.000	0.000	0.000	0.000	0.000
y_1	0.229	0.229	0.229	0.229	0.229	0.229
$\underline{x}$	3.228	2.815	1.980	1.322	–	–
$\bar{x}$	5.941	5.451	4.513	3.894	–	–
$\hat{x}_2$	5.984	5.856	6.464	8.014	12.688	12.688
$\bar{x} - \underline{x}$	2.714	2.636	2.533	2.572	–	–

Table 7. *Varying fixed penalty payments at ruin K.*

K	−5	−3	**−2**	−1	−0.1	−0.01
$\hat{x}_1$	–	–	0.000	0.000	0.000	0.000
y_1	–	–	0.229	0.848	1.387	1.447
$\underline{x}$	6.385	5.054	1.980	1.727	1.702	1.715
$\bar{x}$	9.302	7.972	4.513	4.405	4.524	4.552
$\hat{x}_2$	14.757	13.367	6.464	6.610	15.679	15.421
$\bar{x} - \underline{x}$	2.916	2.918	2.533	2.678	2.822	2.836

Table 8. *Varying proportional penalty payments at ruin Φ.*

Φ	0.001	0.01	0.05	***0.1***	0.2	0.5
$\hat{x}_1$	0.000	0.000	0.000	0.000	–	–
y_1	0.772	0.731	0.538	0.229	–	–
$\underline{x}$	1.997	1.989	1.966	1.980	4.729	5.189
$\bar{x}$	4.556	4.546	4.513	4.513	7.647	8.105
$\hat{x}_2$	6.555	6.539	6.481	6.464	13.024	13.482
$\bar{x} - \underline{x}$	2.559	2.557	2.547	2.533	2.918	2.916

transaction costs ϕ plays a similar role as k. When the proportion is large than 20%, it will be non-optimal to allow any capital injection above zero, which is more sensitive comparing to the exponential case where capital injection region still exists even when the proportion is 100%. Furthermore, Tables 5 and 6 also show that transaction costs of capital injection has no effect on the 2-barrier structure for dividend in the optimal band-type strategy.

However, the fixed and proportional penalty payment do have effect on the optimal dividend structure. In particular, from Table 7, we observe that when the fixed penalty $|K|$ is large (say $K = -5$), the optimal band-type strategy has only 1 dividend barrier at $x = 14.757$; and when $|K|$ decreases to 2 or even smaller, the optimal strategy will have 2-dividend barriers $\{0, 6.464\}$. We also observe that the first lump sum dividend payment region $(\hat{x}_1, y_1)$ is broadened; and because of the 2-barrier structure for dividend, the capital injection region (i.e. $(y_1, \underline{x})$) is also sensitive to the value of K. The influence to the optimal band-type strategy from proportional penalty Φ is similar. According to Table 8, when the proportional penalty is small (say $\Phi = 0.001$), it is optimal to have 2-barrier for dividend payment that is pay lump sum dividend or at premium when $x \in [0, 0.772) \cap [6.555, \infty)$. However, when Φ increases to 20% or above, the first dividend region will diminished to none resulting in 1-barrier structure for dividend in the final optimal band-type strategy.

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7. Conclusion

This paper aims at extending the optimal dividend and capital injection problem in Xu and Woo [18] to the case with singular dividend payments. The asymptotic relationships between the value function (as well as the post capital injection value function) of these two scenarios are given. Viscosity theory is applied to show that the value function is the smallest viscosity supersolution of the corresponding HJBQVI within certain functional class. The uniqueness of such viscosity solution can be proved by showing a modified comparison principle, where constructing strict viscosity supersolution is applied in the proof in order to resolve the capital injection perturbation to the standard proof of such comparison principle. Finally, a band-type dividend and capital injection strategy is proposed based on four crucial sets and their topological structures. The optimality of such band-type strategy are given by applying the fixed point argument. Finally, some numerical examples are presented when the claim size follows exponential and gamma distribution, respectively. It is observed from the numerical results that under exponential distribution, the optimal band-type strategy is, in general, a combination of 1-barrier dividend structure with one capital injection region and no action region, which may reduce to just barrier dividend strategy when the fixed transaction cost is sufficient large. Under the gamma distribution, the scenarios are more complicated, where 1-barrier and 2-barrier dividend structure are both possible; and the optimality of certain dividend structure depends on the value of penalty payments.

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Competing interests. The authors declare no conflict of interest.

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Appendix A

Lemma A.1. *Let $g(x) \in \mathcal{LB}^{\pi}(\mathbb{R})$ be an viscosity supersolution of (3.3) which is upper semi-continuous at 0. Then, we can find a sequence of continuously differentiable function h_n on $\mathbb{R}$ with $h_n(x) = \pi(x)$ for $x < 0$ such that*

(a) h_n satisfies the growth condition (iv) of $\mathcal{LB}^{\pi}(\mathbb{R})$ class.
(b) $h_n'(x) \geq 1$ for $x \geq 0$.
(c) $h_n \leq g$ on $[0, \infty)$.
(d) h_n converges to g uniformly on compact sets and $h_n'(x)$ converges to $g'(x)$ a.e.

Proof. The proof follows the same steps in the proof of Xu and Woo [18] Lemma 6.1 and Azcue and Muler [2] Lemma A.2. □

Lemma A.2. *The maximizer (x_ϵ, y_ϵ) defined in Proposition 4.2 cannot be obtained on the boundary of A.*

Proof. The proof is an analogy to the proof of Albrecher and Thonhauser [1] Lemma 2.5 and Azcue and Muler [2] Proposition 4.2. First of all, by assumption, $\xi(0) \leq \eta(0)$, then for m sufficiently large, we have

$$H_\epsilon(0,0) = \xi(0) - \eta_m(0) - \frac{2n}{\epsilon} < 0,$$

and

$$H_\epsilon(x,B) = \xi(x) - \eta_m(B) - \frac{\epsilon}{2}(x-B)^2 - \frac{2n}{\epsilon^2(B-x)+\epsilon} \leq \xi(B) - \eta_m(B) < 0.$$

In addition, we show that the maximizer is not on the boundary when $x = y$. Note that for all $x > 0$,

$$\begin{aligned} &\limsup_{h \downarrow 0} \frac{H_\epsilon(x,x) - H_\epsilon(x-h,x)}{h} \\ &\quad = \limsup_{h \downarrow 0} \frac{\xi(x) - \xi(x-h) - 2n/\epsilon + (\epsilon/2)h^2 + 2n/(\epsilon^2 h + \epsilon)}{h} \\ &\quad \leq \limsup_{h \downarrow 0} \left(n - \frac{2n}{\epsilon h + 1} + \frac{\epsilon h}{2} \right) = -n < 0. \end{aligned}$$

where the last inequality holds true because of (4.10). On the other hand, since $H_\epsilon(0,0) < 0$, then by continuity of H_ϵ we have that for some $\delta_\epsilon > 0$, $H_\epsilon(0,y) < 0$ for all $y \in [0, \delta_\epsilon]$. Lastly, for $y \in (\delta_\epsilon, \infty)$,

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one has for ϵ sufficiently large,

$$\limsup_{h\downarrow 0}\frac{H_\epsilon(0,y)-H_\epsilon(h,y)}{h}$$
$$=\limsup_{h\downarrow 0}\frac{\xi(0)-\xi(h)+\frac{\epsilon}{2}h^2-\epsilon hy+\frac{2nh}{(\epsilon y+1)(\epsilon(y-h)+1)}}{h}$$
$$\leq \limsup_{h\downarrow 0}\left(-1+\frac{\epsilon}{2}h-\epsilon y+\frac{2n}{(\epsilon y+1)(\epsilon(y-h)+1)}\right)$$
$$=-1-\epsilon y+\frac{2n}{(\epsilon y+1)^2}<0.$$

Therefore, we finish the proof that the maximizer (x_ϵ, y_ϵ) is not obtained on the boundary of A. □

Lemma A.3. *For some $\tilde{x}>0$ such that $(\mathcal{A}^*_\pi-\delta)V(\tilde{x})<0$, and there exists a sequence $x_n\in\mathscr{D}_1$ such that $V'(x_n)$ exists and $x_n\uparrow\tilde{x}$. Then, there exists $\epsilon>0$ such that $(\tilde{x}-\epsilon,\tilde{x})\subset\mathscr{D}_1$.*

Proof. Since $(\mathcal{A}^*_\pi-\delta)V(\cdot)$ is a continuous function, there must exist $h_0>0$ and $\epsilon>0$ such that

$$(\mathcal{A}^*_\pi-\delta)V(x)<-2\epsilon,\qquad \text{for } x\in[\tilde{x}-2h_0,\tilde{x}]$$

Let us assume that

$$h_0<\frac{\epsilon}{(\lambda+\delta)(k_V+k_U)},$$

where k_V and k_U be the maximum Lipschitz constants for V and $U_y, y\geq 0$ on $(0,\tilde{x}]$, respectively. Next, for the sequence $x_n\in\mathscr{D}_1$ and sufficient large n such that $x_n\in[\tilde{x}-h_0,\tilde{x}]$, we consider the auxiliary function $U_{x_n-h_0}(x)$ defined in (5.2); according to Proposition 5.2, we have $U_{x_n-h_0}(x)=V(x)$ for $x\in[x_n-h_0,x_n]$ if we can show that $U_{x_n-h_0}$ is a viscosity supersolution of (3.3) in $(x_n-h_0,x_n]$. Note that for $x\in(x_n-h_0,x_n]$

$$(\mathcal{A}^*_\pi-\delta)U_{x_n-h_0}(x)-(\mathcal{A}^*_\pi-\delta)V(x)$$
$$=(\lambda+\delta)(V(x)-U_{x_n-h_0}(x))+\lambda\int_0^{x-(x-h_0)}(U_{x_n-h_0}(x-y)-V(x-y))\,\mathrm{d}F(y)$$
$$\leq(\lambda+\delta)(k_V+k_V)h_0<\epsilon.$$

Hence, we have $(\mathcal{A}^*_\pi-\delta)U_{x_n-h_0}(x)<(\mathcal{A}^*_\pi-\delta)V(x)+\epsilon<-\epsilon$.

On the other hand, $U_{x_n-h_0}$ is a viscosity supersolution of (3.3) if for any test function φ we have

$$c\varphi'(x)-(\lambda+\delta)U_{x_n-h_0}(x)+\lambda\int_0^x U_{x_n-h_0}(x-y)\,\mathrm{d}F(y)+\lambda\int_x^\infty\pi(x-y)\,\mathrm{d}F(y)\leq 0,$$

and

$$\mathcal{M}U_{x_n-h_0}(x)\leq U_{x_n-h_0}(x).$$

Note that $\varphi'(x)\leq U'_{x_n-h_0}(x^+)=1$, then we arrive at

$$c\varphi'(x)-(\lambda+\delta)U_{x_n-h_0}(x)+\lambda\int_0^x U_{x_n-h_0}(x-y)\,\mathrm{d}F(y)+\lambda\int_x^\infty\pi(x-y)\,\mathrm{d}F(y)$$
$$\leq c-(\lambda+\delta)U_{x_n-h_0}(x)+\lambda\int_0^x U_{x_n-h_0}(x-y)\,\mathrm{d}F(y)+\lambda\int_x^\infty\pi(x-y)\,\mathrm{d}F(y)$$
$$=(\mathcal{A}^*_\pi-\delta)U_{x_n-h_0}(x)<-\epsilon.$$

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$\mathcal{M}U_{x_n-h_0}(x) \le U_{x_n-h_0}(x)$ follows directly from the proof in Proposition 5.1. Therefore, we have obtained that $[x_n - h_0, x_n] \subset \mathscr{D}_1$ for sufficiently large n; then,

$$(\tilde{x} - h_0, \tilde{x}) \subset \bigcup_{n\in\mathbb{N}} [x_n - h_0, x_n] \subset \mathscr{D}_1.$$

□

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