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CHARACTERIZATION OF NON-LINEAR TRANSFORMATIONS POSSESSING KERNELS

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1. Introduction. Recently, in collaboration with Martin [**10**] and Sundaresan [**11**], I obtained a characterization of certain classes of non-linear functionals defined on spaces of measurable functions (see also [**12**]). The functionals in question had the form

$$F(x) = \int_T (\varphi \circ x)\, d\mu = \int_T \varphi(x(t))\, d\mu(t) \tag{1.1}$$

with a continuous "kernel" $\varphi: R \to R$, or

$$F(x,y) = \int_{S\times T} (\varphi \circ (x,y))\, d\mu \otimes \nu = \int_{S\times T} \varphi(x(s), y(t))\, d\mu(s)\, d\nu(t) \tag{1.2}$$

with a separately continuous kernel $\varphi: R^2 \to R$. There are direct applications of this work to the theory of generalized random processes in probability (see [**8**]) and to the theory of fading memory in continuum mechanics [**3**]. However, the main motivation for these studies was an interest in possible application to the functional analytic study of non-linear differential equations. From the standpoint of this latter application it would also be desirable to characterize the broader class of functionals having the form

$$F(x) = \int_T \varphi(x(t), t)\, d\mu(t), \tag{1.3}$$

where the kernel $\varphi: R \times T \to R$ satisfies "Carathéodory conditions". This can be readily understood if we recall that the existence theory for $\dot{x}(t) = \varphi(x(t), t)$, with φ a function satisfying Carathéodory conditions, is very close to that for $\dot{x}(t) = \varphi(x(t))$ with $\varphi: R \to R$ continuous (see, e.g., [**2**]).

In the present paper we obtain an abstract characterization for functionals having the form (1.3), a characterization which is of the kind obtained earlier for functionals having the form (1.1). In addition, we characterize corresponding transformations from $L^p(T)$ to $C(S)$, where $C(S)$ is the space of continuous functions on a compact Hausdorff space. Our proofs utilize some results appearing in Krasnosel′skiĭ's important summary [**9**] of work on transformations of the type $x \to \varphi \circ x$. For some work on a problem analogous to ours for

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functionals on the space of continuous functions on a compact metric space, see [**1**].†

2. Throughout this paper, $T = (T, \Sigma, \mu)$ is a complete measure space, R is the real line with Lebesgue measure, and $M(T)$ denotes the space of extended real-valued measurable functions on T.

Definition. A real-valued function $\varphi: R \times T \to R$ is said to be of *Carathéodory type* for T and we write $\varphi \in \mathrm{Car}(T)$ if it satisfies the following conditions,

(1) $\varphi(\cdot, t): R \to R$ is continuous for almost all $t \in T$,

(2) $\varphi(c, \cdot): T \to R$ is measurable for all $c \in R$.

One can extend this definition in an obvious way to functions $\varphi: R^m \times T \to R^n$. We remark that $\mathrm{Car}(T)$ is a subspace of the vector space $M(R \times T)$.

If x is an extended real-valued measurable function on T and φ is in $\mathrm{Car}(T)$, then the function $\varphi \circ x$ defined by

$$(\varphi \circ x)(t) = \varphi(x(t), t),$$

is also a measurable function on T. This is obviously true when x is a measurable function whose range is a finite set. In the general case, x is the limit everywhere of a sequence of functions x_n of the above type. Hence by continuity of φ in its first argument, $\varphi \circ x$, as the pointwise limit almost everywhere of the measurable functions $\varphi \circ x_n$, is measurable. Thus for each $\varphi \in \mathrm{Car}(T)$, the mapping $x \to \varphi \circ x$ is a mapping of $M(T)$ into itself.

It is useful to single out certain subspaces of the vector space $\mathrm{Car}(T)$ in terms of their mapping properties.

Definition. Given the number p, $1 \leqq p \leqq \infty$, a function φ of Carathéodory type for T is said to be in the *Carathéodory p-class*, and we write $\varphi \in \mathrm{Car}^p(T)$ if φ maps $L^p(T)$ into $L^1(T)$. That is, φ is in $\mathrm{Car}^p(T)$ if

$$\varphi \circ x \in L^1(T) \quad \text{for all } x \in L^p(T).$$

Remark. For the case of a non-atomic σ-finite measure space it is known [**9**, p. 27] that φ is in $\mathrm{Car}^p(T)$, $1 \leqq p < \infty$, if and only if

$$|\varphi(x, t)| \leqq a(t) + b|x|^p$$

for some $a \in L^1(T)$.

THEOREM 1. *Let $T = (T, \Sigma, \mu)$ be a finite or σ-finite measure space. Let F be a real-valued functional on $L^\infty(T)$ which satisfies:*

(i) *$F(x + y) = F(x) + F(y)$ when $xy = 0$ a.e.,*

(ii) *F is uniformly continuous on each bounded subset of $L^\infty(T)$,*

(iii) *$F(x_n) \to F(x)$ whenever $\{x_n\}_{n \geqq 1}$ converges boundedly almost everywhere to $x \in L^\infty(T)$.*

†Since the submission of this paper, two related papers [**4**; **7**] have appeared. Of the two, [**4**] is more closely related to this work.

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Then there exists a function $\varphi \in \mathrm{Car}^\infty(T)$ *such that*

$$(2.1) \qquad F(x) = \int_T (\varphi \circ x)\, d\mu = \int_T \varphi(x(t), t)\, d\mu(t).$$

Moreover, φ *can be taken to satisfy*

$$(2.2) \qquad \varphi(0, \cdot) = 0 \ a.e.,$$

and is then unique up to sets of the form $R \times N$ *with* N *a null set in* T.

Conversely, for every $\varphi \in \mathrm{Car}^\infty(T)$ *satisfying* (2.2), (2.1) *defines a functional satisfying* (i), (ii), *and* (iii).

Remarks. (1) The final statement of the theorem is valid for any $\varphi \in \mathrm{Car}^\infty(T)$ satisfying

$$(2.3) \qquad \int_T (\varphi \circ 0)\, d\mu = 0.$$

Moreover, condition (i) on F can be modified in such a way that this result applies to *all* $\varphi \in \mathrm{Car}^\infty(T)$. Namely, we could replace (i) by

(i′) $F(x+y) - F(x) - F(y) = \text{const} = C_F$ *whenever* $xy = 0$ *a.e.*

Note that then the functional $F_1(x) = F(x) + C_F$ satisfies (i), (ii), and (iii).

(2) Unlike the results in [**10**; **11**], the present characterization does not require a hypothesis concerning the non-atomic nature or almost non-atomic nature of T. The same holds true for Theorem 2 below.

Proof of Theorem 1. It follows from (i) and (iii) that for each real number h the real-valued set function a_h defined by $a_h(S) = F(h\chi_s)$ is countably additive and absolutely continuous relative to μ. Hence by the Radon-Nikodym theorem there corresponds to each h a function $\varphi_h \in L^1(T)$, unique up to a null set, such that

$$F(h\chi_S) = \int_S \varphi_h\, d\mu.$$

The functions φ_h with h rational will be utilized below in constructing the function φ occurring in (2.1). This construction applies the following lemma whose proof will be deferred until later.

LEMMA. *Given any* $\eta > 0$ *there is a measurable set* $S_\eta = \bigcup_{i=1}^\infty S_{\eta,i}$ *such that*

(1) $\mu(T - S_\eta) < \eta$, $\mu(S_{\eta,i}) < \infty$, $i = 1, 2, \ldots,$

(2) *on* $S_{\eta,i}$ *there exists for each pair of numbers* $M, \epsilon > 0$ *a* $\delta = \delta_i(\epsilon, M) > 0$ *such that for rational* h *and* h' *we have*

$$h, h' \in [-M, M] \quad \text{and} \quad |h - h'| < \delta \Rightarrow \sup_{t \in S_{\eta,i}} |\varphi_h(t) - \varphi_{h'}(t)| \leqq \epsilon.$$

Now select a sequence $\eta_m \to 0$ and define a function $\varphi: R \times T \to R$ as follows:

$$(2.4) \qquad \varphi(c,t) = \begin{cases} \lim\limits_{\substack{h \to c;\\ (h \text{ rational})}} \varphi_h(t) & \text{for } t \in S = \bigcup\limits_{m=1}^\infty S_{\eta_m}, \\ 0 & \text{for } t \in T - S. \end{cases}$$

It follows from the lemma that this defines φ unambiguously and that $\varphi(\cdot, t)$ is continuous for each $t \in T$. Moreover, since $T - S$ is a null set, for each $c \in R$ the function $\varphi(c, \cdot)$ is the almost everywhere pointwise limit of a sequence of measurable functions φ_h and is therefore measurable. Thus φ is of Carathéodory type for T. Further, since for c rational we have

$$\varphi(c, t) = \varphi_c(t) \quad \text{a.e.},$$

it is clear that $\varphi(c, \cdot) \in L^1(T)$ for c rational and that φ satisfies (2.2). It remains to be shown that (2.1) holds. For this we shall utilize Vitalli's convergence theorem.

Suppose that $x \in L^\infty(T)$ is a simple function with rational values, i.e.

$$x = \sum_{k=1}^{N} c_k \chi_{T_k}, \qquad c_k \text{ rational, } \{T_k\} \text{ disjoint.}$$

Then, using (i),

$$F(x) = \sum_{k=1}^{N} F(c_k \chi_{T_k}) = \sum_{k=1}^{N} \int_{T_k} \varphi_{c_k}\, d\mu$$

$$= \int_T (\varphi \circ (\sum c_k \chi_{T_k}))\, d\mu = \int_T (\varphi \circ x)\, d\mu.$$

Thus (2.1) holds in this special case.

Now each $x \in L^\infty(T)$ is the limit almost everywhere as well as in norm of a sequence x_n of simple functions with rational values,

$$x_n \to x \quad \text{a.e. and in } L^\infty(T).$$

Since $\varphi \in \operatorname{Car}(T)$, it follows that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{a.e.} \tag{2.5}$$

In addition, the sequence $\varphi \circ x_n \in L^1(T)$ is uniformly absolutely continuous, i.e.

$$\int_R |\varphi \circ x_n|\, d\mu \to 0 \quad \text{as } \mu(R) \to 0, \text{ uniformly in } n. \tag{2.6}$$

Otherwise there would exist for some $\epsilon > 0$ a sequence of sets $R_m \subset T$ with $\mu(R_m) < 3^{-m}$ and a corresponding sequence $\varphi \circ x_{n_m}$ such that

$$\int_{R_m} |\varphi \circ x_{n_m}|\, d\mu > \epsilon.$$

It follows that each R_m possesses a subset R_m' satisfying

$$\left| \int_{R_m'} (\varphi \circ x_{n_m})\, d\mu \right| > \epsilon/2.$$

Now the functions $y_m = x_{n_m} \chi_{R_m'}$ form a bounded set in $L^\infty(T)$ since the x_n

form such a set, and hence $y_m \to 0$ boundedly almost everywhere. Moreover, y_m being a rational-valued simple function implies that

$$F(y_m) = \int_T (\varphi \circ y_m)\, d\mu = \int_{R_m'} (\varphi \circ x_{n_m})\, d\mu.$$

However, by the construction of R_m' this implies that the $F(y_m)$ do not converge to zero, contradicting property (iii).

Furthermore, the sequence $\varphi \circ x_n$ has the property that for each $\epsilon > 0$ there exists a set R_ϵ such that $\mu(R_\epsilon) < \infty$ and

$$\int_{T-R_\epsilon} |\varphi \circ x_n|\, d\mu < \epsilon \quad \text{for all } n. \tag{2.7}$$

Otherwise, for some $\epsilon > 0$ there would exist an expanding sequence of sets R_m with $\mu(R_m) < \infty$ and $\bigcup_{m=1}^{\infty} R_m = T$ and a corresponding sequence $\varphi \circ x_{nm}$ such that $\int_{T-R_m} |\varphi \circ x_{nm}|\, d\mu > \epsilon$. Thus for some $R_m'' \subset T - R_m$,

$$\left| \int_{R_m''} (\varphi \circ x_{nm})\, d\mu \right| > \epsilon/2.$$

The functions $y_m = x_{nm}\chi_{R_m''}$ satisfy $y_m \to 0$ boundedly almost everywhere, while the formula

$$F(y_m) = \int_T (\varphi \circ y_m)\, d\mu = \int_{R_m''} (\varphi \circ x_{nm})\, d\mu$$

implies that the $F(y_m)$ do not converge to zero, contradicting (iii).

Since the sequence $\varphi \circ x_n$ in $L^1(T)$ satisfies (2.5)–(2.7), it follows by Vitalli's convergence theorem (see [**5**, p. 150]) that $\varphi \circ x$ belongs to $L^1(T)$ and that $\varphi \circ x_n \to \varphi \circ x$ in $L^1(T)$, whereby

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_T (\varphi \circ x_n)\, d\mu = \int_T (\varphi \circ x)\, d\mu.$$

Thus $\varphi \in \mathrm{Car}^\infty(T)$ and (2.1) holds. The uniqueness of φ follows from the fact that by (2.2),

$$F(c\chi_S) = \int_T \chi_S \varphi(c, t)\, d\mu = \int_S \varphi_c\, d\mu.$$

Considering only rational c, we see that this condition determines $\varphi(c,\cdot)$ up to a null set, and hence determines $\varphi \in \mathrm{Car}(T)$ up to sets of the form $R \times N$ as claimed. This completes the proof of the first half.

For the converse let φ be a function in $\mathrm{Car}^\infty(T)$ which satisfies condition (2.2). Then the functional F defined by (2.1) obviously satisfies (i). We proceed to show that (ii) holds. Otherwise there would exist numbers $A, a > 0$ such that corresponding to each positive integer n there is a pair of functions $x_n, y_n \in L^\infty(T)$ satisfying

$$\begin{gathered} \|x_n\|_\infty, \|y_n\|_\infty \leqq A, \qquad \|x_n - y_n\|_\infty < 1/n, \\ \|\varphi \circ x_n - \varphi \circ y_n\|_1 > a. \end{gathered} \tag{2.8}$$

Consider first the case in which $\mu(T)$ is finite and set $S_1 = T$. Select a subsequence of x_n, y_n as follows. By the absolute continuity of the indefinite integral of $\varphi \circ x_1 - \varphi \circ y_1$ there exists an $\epsilon_1 > 0$ such that

$$\int_S |\varphi \circ x_1 - \varphi \circ y_1| \, d\mu < a/3 \quad \text{whenever } \mu(S) < 2\epsilon_1.$$

Obviously, $\epsilon_1 < \frac{1}{2}\mu(T)$. Since $\varphi(\cdot, t)$ is continuous for almost all $t \in T$, it is uniformly continuous on the set $[-A, A] \subset R$ for such t. Thus for each ϵ,

$$T = \bigcup_{n=1}^{\infty} \left\{ t | c_1, c_2 \in [-A, A]|, |c_1 - c_2| \leqq \frac{1}{n} \Rightarrow |\varphi(c_1, t) - \varphi(c_2, t)| \leqq \epsilon \right\} \cup N,$$

where $\mu(N) = 0$. Hence by selecting n_2 sufficiently large one can find a measurable set T_2 satisfying

$$|(\varphi \circ x_{n_2})(t) - (\varphi \circ y_{n_2})(t)| \leqq \frac{a}{3\mu(T)} \quad \text{for } t \in T_2,$$

and $\mu(T - T_2) < \epsilon_1$. By (2.8), this implies that with $S_2 = T - T_2$,

$$\int_{S_2} |\varphi \circ x_{n_2} - \varphi \circ y_{n_2}| \, d\mu > 2a/3, \qquad \mu(S_2) < \epsilon_1.$$

Again, since the indefinite integral of $\varphi \circ x_{n_2} - \varphi \circ y_{n_2}$ is absolutely continuous, there exists an $\epsilon_2 > 0$ such that

$$\int_S |\varphi \circ x_{n_2} - \varphi \circ y_{n_2}| \, d\mu < a/3 \quad \text{wherever } \mu(S) < 2\epsilon_2.$$

Obviously, $2\epsilon_2 < \frac{1}{2}\mu(S_2)$. Again by the uniform continuity of $\varphi(\cdot, t)$ on $[-A, A]$ for almost all t, there exists an n_3 sufficiently large and a corresponding set T_3 such that

$$|(\varphi \circ x_{n_3})(t) - (\varphi \circ y_{n_3})(t)| < \frac{a}{3\mu(T)} \quad \text{for } t \in T_3,$$

and $\mu(T - T_3) < \epsilon_2$. By (2.8), this implies that with $S_3 = T - T_3$,

$$\int_{S_3} |\varphi \circ x_{n_3} - \varphi \circ y_{n_3}| \, d\mu > 2a/3, \qquad \mu(S_3) < \epsilon_2.$$

Proceeding with this construction we obtain a subsequence x_{nk}, y_{nk} and a corresponding sequence of sets S_k satisfying

$$\int_{S_k} |\varphi \circ x_{nk} - \varphi \circ y_{nk}| \, d\mu > 2a/3, \qquad \int_{S_{k+1}} |\varphi \circ x_{nk} - \varphi \circ y_{nk}| \, d\mu < a/3,$$

and $\mu(S_k) < \epsilon_{k-1} < \mu(S_{k-1})/2$. Now define $R_k = S_k - \bigcup_{l=k+1}^{\infty} S_l$. The sets R_k are disjoint. Moreover,

$$\mu\left(\bigcup_{l=k+1}^{\infty} S_l\right) < 2\mu(S_{k+1}) < 2\epsilon_k$$

so that, recalling how the ϵ_j are defined, one has

$$\int_{R_k} |\varphi \circ x_{nk} - \varphi \circ y_{nk}| \, d\mu > a/3.$$

Define

$$x = \sum_{l=1}^{\infty} x_{nk} \chi_{R_k}, \qquad y = \sum_{k=1}^{\infty} y_{nk} \chi_{R_k}.$$

By construction, $x, y \in L^\infty(T)$, so that $\varphi \circ x, \varphi \circ y \in L^1(T)$, and

$$\int_{R_k} |\varphi \circ x - \varphi \circ y| \, d\mu = \int_{R_k} |\varphi \circ x_{nk} - \varphi \circ y_{nk}| \, d\mu > a/3, \qquad k = 1, 2, \ldots .$$

Since the R_k are disjoint, this is a contradiction.

Consider now the case $\mu(T) = \infty$ and assume that (2.8) holds. One constructs sequences of functions $\{x_{nk}\}$, $\{y_{nk}\}$ and a sequence of disjoint sets $\{R_k\}$ such that

$$\mu(R_k) < \infty, \qquad \int_{R_k} |\varphi \circ x_{nk} - \varphi \circ y_{nk}| \, d\mu > a/2. \tag{2.9}$$

The procedure is again inductive. Let R_1 be a set of finite measure such that

$$\int_{R_1} |\varphi \circ x_1 - \varphi \circ y_1| \, d\mu > a/2.$$

This is possible by (2.8). Then, by the result in the preceding paragraph, for n_2 sufficiently large,

$$\int_{R_1} |\varphi \circ x_{n_2} - \varphi \circ y_{n_2}| \, d\mu < a/2.$$

Hence there exists a set $R_2 \subset T - R_1$ such that $\mu(R_2) < \infty$ and

$$\int_{R_2} |\varphi \circ x_{n_2} - \varphi \circ y_{n_2}| \, d\mu > a/2.$$

Again since $\mu(R_1 \cup R_2) < \infty$, we have by our earlier result that for n_3 sufficiently large,

$$\int_{R_1 \cup R_2} |\varphi \circ x_{n_3} - \varphi \circ y_{n_3}| \, d\mu < a/2.$$

Hence there exists a set $R_3 \subset T - (R_1 \cup R_2)$ such that $\mu(R_3) < \infty$ and

$$\int_{R_3} |\varphi \circ x_{n_3} - \varphi \circ y_{n_3}| \, d\mu > a/2.$$

Proceeding in this fashion one arrives at sequences of functions $\{x_{nk}\}$, $\{y_{nk}\}$ and of disjoint sets $\{R_k\}$ for which (2.9) holds. Now define

$$x = \sum_{k=1}^{\infty} x_{nk} \chi_{R_k}, \qquad y = \sum_{k=1}^{\infty} y_{nk} \chi_{R_k}.$$

By construction,

$$\int_{R_k} |\varphi \circ x - \varphi \circ y| \, d\mu > a/2, \qquad k = 1, 2, \ldots,$$

contradicting the fact that $\varphi \circ x, \varphi \circ y \in L^1(T)$.

There remains the proof of (iii). Let x_n be a sequence such that

$$x_n \to x \quad \text{a.e.}, \qquad ||x_n||_\infty, ||x||_\infty \leqq A. \tag{2.10}$$

Since $\varphi \in \mathrm{Car}(T)$, it follows that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{a.e.}, \tag{2.11}$$

while by (ii),

$$||\varphi \circ x_n||_1, ||\varphi \circ x||_1 \leqq M = M(A). \tag{2.12}$$

It will be shown that (iii) holds by proving that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{in } L^1 \text{ norm}.$$

The argument again utilizes Vitalli's convergence theorem. First, we show that for every sequence $\{x_n\}$ satisfying (2.10), the functions $\varphi \circ x_n$ have uniformly absolutely continuous indefinite integrals, i.e.

$$\int_U |\varphi \circ x_n| \, d\mu \to 0 \text{ as } \mu(U) \to 0, \quad \text{uniformly in } n. \tag{2.13}$$

For otherwise there would exist for certain numbers $A, \alpha > 0$ a sequence $\{x_n\}$ satisfying (2.10) and a corresponding sequence of measurable sets S_n such that

$$\mu(S_n) \to 0, \qquad \int_{S_n} |\varphi \circ x_n| \, d\mu > 2\alpha.$$

It then follows that there exists for each S_n a measurable subset S_n' such that

$$\mu(S_n') \to 0, \qquad \left| \int_{S_n'} (\varphi \circ x_n) \, d\mu \right| > \alpha.$$

By extracting a subsequence if necessary, we may assume without loss of generality that all the integrals in the above formula have the same sign, say positive. That is,

$$\mu(S_n') \to 0, \qquad \int_{S_n'} (\varphi \circ x_n) \, d\mu > \alpha. \tag{2.14}$$

Now by a construction analogous to that used in the proof of (ii) we can extract a subsequence $\{x_{nk}\}$ such that the corresponding sets S_{nk}' satisfy

$$\mu(S_{nk+1}') < \epsilon_k/2 < \mu(S_{nk}')/4, \tag{2.15}$$

where $\epsilon_k > 0$ is selected so that

$$\mu(U) < \epsilon_k \Rightarrow \int_U |\varphi \circ x_{nk}| \, d\mu < \alpha/2.$$

Namely, with $n_1 = 1$ and with $n_1 < \ldots < n_k$ already chosen, select $n_{k+1} > n_k$ to be the smallest integer such that

$$\mu(S_{n_{k+1}}') < \epsilon_k/2.$$

It then follows that

$$\text{(2.16)} \qquad \int_{R_k} (\varphi \circ x_{n_k})\, d\mu > \alpha/2, \quad \text{where } R_k = S_{n_k}' - \bigcup_{j=k+1}^{\infty} S_{n_j}',$$

since (2.15) implies that

$$\mu\left(\bigcup_{j=k+1}^{\infty} S_{n_j}'\right) < \epsilon_k.$$

Consider now the function

$$y = \sum_{k=1}^{\infty} x_{n_k} \chi_{R_k}.$$

By (2.16) one has $\varphi \circ y \notin L^1(T)$, a conclusion which contradicts the fact that $\varphi \in \mathrm{Car}^{\infty}(T)$.

Next, we show that the functions $\varphi \circ x_n$ are uniformly equicontinuous, i.e. given an $\epsilon > 0$ there is a measurable set S_ϵ satisfying

$$\text{(2.17)} \qquad \mu(S_\epsilon) < \infty, \qquad \int_{T-S_\epsilon} |\varphi \circ x_n|\, d\mu < \epsilon \quad \text{uniformly in } n.$$

For otherwise there would exist for certain numbers $A, \epsilon > 0$ a sequence satisfying (2.10) which fails to satisfy (2.17) for any set S of finite measure. We could then extract a subsequence $\{x_{n_k}\}$ and a disjoint sequence of sets R_k such that

$$\mu(R_k) < \infty, \left| \int_{R_k} (\varphi \circ x_{n_k})\, d\mu \right| > \epsilon/4, \qquad k = 1, 2, \ldots.$$

Namely, let n_1 be chosen so that $||x_{n_1}||_1 \geqq \epsilon$. There then exists a measurable set U_1 such that

$$\left| \int_{U_1} (\varphi \circ x_{n_1})\, d\mu \right| \geqq \epsilon/2,$$

and hence a set $R_1 \subset U_1$ such that

$$\mu(R_1) < \infty, \qquad \left| \int_{R_1} (\varphi \circ x_{n_1})\, d\mu \right| > \epsilon/4.$$

In general, with $n_1 < \ldots < n_k$ already chosen, select $n_{k+1} > n_k$ to be the smallest integer such that

$$\int_{T-S_k} |\varphi \circ x_{n_{k+1}}|\, d\mu \geqq \epsilon, \qquad \text{where } S_k = \bigcup_{j=1}^{k} R_j.$$

There then exists a measurable set $U_{k+1} \subset T - S_k$ such that

$$\left| \int_{U_{k+1}} (\varphi \circ x_{n_{k+1}})\, d\mu \right| \geqq \epsilon/2,$$

and hence a set $R_{k+1} \subset U_{k+1}$ such that

$$\mu(R_{k+1}) < \infty, \qquad \left| \int_{R_{k+1}} (\varphi \circ x_{n_{k+1}})\, d\mu \right| > \epsilon/4.$$

By extracting a further subsequence if necessary we may assume without loss of generality that all the integrals in the above formula have the same sign, say positive. Thus,

$$\text{(2.18)} \qquad \mu(R_k) < \infty, \qquad \int_{R_k} (\varphi \circ x_{n_k})\, d\mu > \epsilon/4, \qquad k = 1, 2, \dots.$$

Consider now the function

$$y = \sum_{k=1}^{\infty} x_{n_k} \chi_{R_k}.$$

By (2.18) and the disjointness of the sets R_k, we see that $\varphi \circ y \notin L^1(T)$, which contradicts the fact that $\varphi \in \mathrm{Car}^\infty(T)$.

However, (2.11), (2.13), and (2.17) imply the L^1-convergence of $\{\varphi \circ x_n\}$ to $\varphi \circ x$, which ensures (iii).

Proof of the Lemma. In the following we restrict the symbols h and r to denote *rational numbers*. Consider first the case of a finite measure space. To begin with we show that, for each $M > 0$ and each positive integer n, the contracting sequence of measurable sets $A_j^{M,n} = \{t|\ |\varphi_h(t) - \varphi_{h'}(t)| > 1/n$ for some $h, h' \in [-M, M]$ with $|h - h'| < 1/j\}$, $j = 1, 2, \dots$, converges to a null set. Otherwise for some fixed $c > 0$,

$$\mu(A_j^{M,n}) \geqq c, \qquad j = 1, 2, \dots.$$

Now

$$A_j^{M,n} \subset \bigcup_{h\in[-M,M]} \ \bigcup_{r\in[-1/j,1/j]} B_{h,r} = \bigcup_{h\in[-M,M]} B_h^{(j)},$$

where

$$B_{h,r} = \{t|\ [\varphi_h(t) - \varphi_{h+r}(t)] > 1/n\}, \qquad B_h^{(j)} = \bigcup_{r\in[-1/j,1/j]} B_{h,r}.$$

Enumerating the rationals in $[-M, M]$ and $[-1/j, 1/j]$ as $h_1, h_2, \dots$ and $r_1, r_2, \dots$, respectively, define the sets $C_{h_k}^{(j)}$ and C_{h_k,r_l} as follows:

$$C_{h_k}^{(j)} = B_{h_k}^{(j)} - \bigcup_{i=1}^{k-1} B_{h_i}^{(j)}, \qquad k = 1, 2, \dots,$$

$$C_{h_k,r_l} = B_{h_k,r_l} - \bigcup_{i=1}^{l-1} B_{h_k,r_i}, \qquad l = 1, 2, \dots.$$

For each j define the functions x_j and y_j by

$$\text{(2.19)} \qquad x_j = \sum_{k=1}^{\infty} h_k \chi_{C_{h_k}^{(j)}},$$

$$\text{(2.20)} \qquad y_j = \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} (h_k + r_l) \chi_{C_{h_k,r_l}}.$$

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By construction, x_j and y_j are in $L^\infty(T)$ and satisfy

(2.21) $$\|x_j\|_\infty,\ \|y_j\|_\infty \leqq M + 1,$$

(2.22) $$\|x_j - y_j\|_\infty \leqq 1/j.$$

Moreover, for $N, N' \to \infty$,

$$\sum_{k=1}^{N} h_k \chi_{C_{h_k}(j)} \to x_j \quad \text{boundedly a.e.,}$$

and

$$\sum_{k=1}^{N} \sum_{l=1}^{N'} (h_k + r_l) \chi_{C_{h_k, r_l}} \to y_j \quad \text{boundedly a.e. .}$$

Hence by (i) and (iii) and the definition of φ_h, we have:

$$F(x_j) - F(y_j) = \sum_{k=1}^{\infty} F(h_k \chi_{C_{h_k}(j)}) - \sum_{k=1}^{\infty} \sum_{l=1}^{\infty} F((h_k + r_l)\chi_{C_{h_k, r_l}})$$

$$= \int_T \sum_{k=1}^{\infty} \left[\varphi_{h_k} \chi_{C_{h_k}(j)} - \sum_{l=1}^{\infty} \varphi_{h_k + r_l} \chi_{C_{h_k, r_l}} \right] d\mu > \frac{1}{n} \mu\Bigl(\bigcup_{h \in [-M, M]} B_h^{(j)} \Bigr) \geqq \frac{1}{n} c,$$

$$j = 1, 2, \ldots,$$

contradicting (ii).

It follows from the above that with M given there exists for each $\eta > 0$ a set S_η^M satisfying

(2.23) for each $\epsilon > 0$ there exists a $\delta = \delta(\epsilon, M) > 0$ such that $h, h' \in [-M, M]$ and $|h - h'| < \delta \Rightarrow |\varphi_h(t) - \varphi_{h'}(t)| \leqq \epsilon$ for $t \in S_\eta^M$,

(2.24) $$\mu(T - S_\eta^M) < \eta.$$

For by the preceding paragraph one can select for each integer n an index j_n such that

$$\mu(A_{j_n}^{M,n}) < \eta/2^n, \qquad n = 1, 2, \ldots .$$

Then the set S_η^M, defined by

(2.25) $$S_\eta^M = T - \bigcup_{n=1}^{\infty} A_{j_n}^{M,n},$$

satisfies (2.23) and (2.24).

In addition, the set S_η defined by

(2.26) $$S_\eta = \bigcap_{M=1}^{\infty} S_{\eta/2}{}^{M} M$$

is readily seen to satisfy (2.23) and (2.24) for all M. Thus the lemma is proved in case T is a finite measure space (with $S_{\eta, i} = S_\eta$ for $i = 1, 2, \ldots$).

Now suppose that $\mu(T) = \infty$. By hypothesis,

$$T = \bigcup_{i=1}^{\infty} T_i \quad \text{with } \mu(T_i) < \infty.$$

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Using the result established in the preceding paragraphs, we construct sets $S_{\eta,i} \subset T_i$, $i = 1, 2, \ldots$, by defining

$$S_{\eta,i} = S_{\eta/2^i} \qquad \text{(relative to the measure space } T_i\text{)}.$$

It is then clear that the set $S_\eta \subset T$ which is defined by

$$S_\eta = \bigcup_{i=1}^{\infty} S_{\eta,i}$$

satisfies all the requirements stated in the lemma.

COROLLARY 1. *With T non-atomic let F be a real-valued functional on a subspace V of $M(T)$ such that $V \supset L^\infty(T)$. Suppose that F satisfies the following conditions:*

(i) *$F(x + y) = F(x) + F(y)$ when $xy = 0$ a.e.,*

(ii) *F is uniformly continuous on each bounded subset of $L^\infty(T)$,*

(iii)′ *$F(x_n) \to F(x)$ whenever $\{x_n\}_{n\geqq 1} \in V$ converges a.e. to $x \in L^\infty(T)$.*

Then there exists a function φ in Car(T) *such that*

$$F(x) = \int_T (\varphi \circ x)\, d\mu \quad \text{for } x \in V \tag{2.27}$$

and $F: V \to R$ is bounded. In fact,

$$\varphi(V) = R_\varphi \subset L^1(T) \text{ is bounded.} \tag{2.28}$$

Moreover, φ can be taken to satisfy (2.2) *and is then unique in the same sense as in Theorem* 1.

Conversely, for every $\varphi \in$ Car$^\infty(T)$ *which satisfies conditions* (2.2) *and* (2.28) *[the latter for $V = L^\infty(T)$], the functional defined by* (2.27) *satisfies* (i), (ii), *and* (iii) *with $V = M(T)$.*

Proof. Observe that the functional $F_1 = F|L^\infty(T)$ satisfies (i), (ii), and (iii) of Theorem 1, and hence is given by

$$F_1(x) = \int_T (\varphi \circ x)\, d\mu \quad \text{for } x \in L^\infty(T), \tag{2.29}$$

for some $\varphi \in \text{Car}^\infty(T)$.

We show first that $\varphi(L^\infty(T)) \subset L^1(T)$ is bounded. Since every $x \in M(T)$ is the limit almost everywhere of a sequence $x_n \in L^\infty(T)$, it will then follow by Fatou's lemma that $|\varphi \circ x|$ being the almost everywhere limit of $\{|\varphi \circ x_n|\}$ is in $L^1(T)$ and is norm bounded by the same constant as $\{|\varphi \circ x_n|\}$. Thus $\varphi(M(T)) \subset L^1(T)$ is also a bounded set. Suppose that $\varphi(L^\infty(T))$ were unbounded. Then there exists a sequence $x_n \in L^\infty(T)$ such that

$$\|\varphi \circ x_n\|_1 = c_n \to \infty. \tag{2.30}$$

It follows that there exists a subset $A_n \subset T$ such that

$$|F(x_n \chi_{A_n})| = \left| \int_{A_n} (\varphi \circ x_n)\, d\mu \right| \geqq c_n/2 \to \infty. \tag{2.31}$$

Consider first the case $\mu(T) < \infty$. Then since T is non-atomic, there exists for each sufficiently large n a subset A_n' of A_n such that

$$|F(x_n\chi_{A_n'})| \geqq 1, \qquad \mu(A_n') \leqq 2/c_n. \tag{2.32}$$

However, since $x_n\chi_{A_n'} \to 0$ a.e., (2.32) contradicts (iii)$'$.

Now suppose that $T = \bigcup_{i=1}^{\infty} T_i$, $\mu(T_i) < \infty$. The preceding argument shows that for each m there is a constant N_m such that

$$||\varphi \circ x||_1 \leqq N_m \text{ for } x \text{ such that supp } x \subset \bigcup_{i=1}^{m} T_i.$$

By extracting a subsequence we can assume that in (2.20), $c_m > 3N_m$. Consequently, there exist sets $A_m \subset T - \bigcup_{i=1}^{m} T_i$ such that

$$||\varphi \circ x_m\chi_{A_m}||_1 > 2N_m, \qquad m = 1, 2, \ldots .$$

It then follows that for some subset $A_m' \subset A_m$,

$$|F(x_m\chi_{A_m'})| = \left|\int_{A_m'} (\varphi \circ x_m)\, d\mu\right| > N_m. \tag{2.33}$$

Now

$$x_m\chi_{A_m'} \to 0 \quad \text{a.e.,}$$

and hence (2.33) contradicts (iii)$'$.

For the converse, suppose that $\varphi \in \mathrm{Car}^\infty(T)$ satisfies (2.2) and (2.28). By the argument given earlier, it follows that $\varphi(M(T)) \subset L^1(T)$ is bounded, so that F is defined on $M(T)$. Property (i) is obvious and (ii) is a consequence of the theorem. It only needs to be shown that (iii)$'$ holds. Suppose that $x_n \in V$, $x \in L^\infty(T)$ and $x_n \to x$ a.e. Then it can be shown just as in the proof of the theorem that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{in } L^1 \text{ norm}$$

and therefore

$$F(x_n) = \int (\varphi \circ x_n)\, d\mu \to \int (\varphi \circ x)\, d\mu = F(x).$$

Remark. It is easy to show by examples that on atomic measure spaces, (i), (ii), and (iii)$'$ do not imply (2.28). On the other hand, the above proof shows that for all T, if $\varphi \in \mathrm{Car}^\infty(T)$ and φ satisfies (2.2) and (2.28), then F satisfies (i), (ii), and (iii)$'$.

THEOREM 2. *With T as in Theorem 1, let F be a real-valued functional on $L^p(T)$, $1 \leqq p < \infty$, which satisfies the following conditions:*

(i) *$F(x + y) = F(x) + F(y)$ when $xy = 0$ a.e.,*

(ii$_p$) *F is continuous on $L^p(T)$,*

(iii$_p$) *F is uniformly continuous relative to the L^∞ norm on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.*

Then there exists a function $\varphi \in \mathrm{Car}^p(T)$ such that

$$F(x) = \int_T (\varphi \circ x)\, d\mu \quad \text{for } x \in L^p(T). \tag{2.34}$$

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Moreover, φ can be taken to satisfy

$$\varphi(0, \cdot) = 0 \quad \text{a.e.} \tag{2.2}$$

and is then unique up to sets of the form $R \times N$ with N a null set in T.

Conversely, for every $\varphi \in \mathrm{Car}^p(T)$ satisfying (2.2), *the formula* (2.34) *defines a functional satisfying* (i), (ii$_p$), *and* (iii$_p$).

Remarks. (1) Observe that when F is a *linear* functional, (ii$_p$) signifies uniform continuity on bounded subsets of $L^p(T)$ and hence implies (iii$_p$). In addition, for such cases the function φ necessarily has the form $\varphi(x, t) = xu(t)$ for some locally summable function u. Thus the present result includes the Riesz representation theorem, modulo a proof that $u \in L^q(T)$ is necessary and sufficient in order that the above φ be in $\mathrm{Car}^p(T)$.

(2) Combining Theorem 2 with results in [**9**], it follows even for the case of non-linear F that F is locally bounded on $L^p(T)$. However, F is generally *not* uniformly continuous on bounded subsets of $L^p(T)$. (See the remark following Corollary 2.)

(3) This result provides a significant strengthening of a result stated in [**6**] (see Corollary 2).

Proof of Theorem 2. By hypothesis, $T = \bigcup_{i=1}^{\infty} T_i$ where the T_i are disjoint subsets of finite measure. Thus $L^\infty(T_i)$ can be identified in the obvious way with a subspace of $L^\infty(T)$, $i = 1, 2, \ldots$. Define

$$F_i = F|L^\infty(T_i), \qquad i = 1, 2, \ldots .$$

Then (i), (ii$_p$), and (iii$_p$) imply that each of the functionals F_i satisfies the hypotheses of Theorem 1, the validity of (iii) being a consequence of (ii$_p$) and the dominated convergence theorem. Hence there exist functions $\varphi_i \in \mathrm{Car}(T_i)$, unique up to null sets, which satisfy (2.2) on T_i and

$$F_i(x) = \int_{T_i} (\varphi_i \circ x)\, d\mu \quad \text{for } x \in L^\infty(T_i), \qquad i = 1, 2, \ldots . \tag{2.35}$$

Now define $\varphi: R \times T \to R$ by means of

$$\varphi(h, \cdot)|T_i = \varphi_i(h, \cdot), \qquad h \in R, i = 1, 2, \ldots . \tag{2.36}$$

It is clear that $\varphi \in \mathrm{Car}(T)$ and that φ satisfies (2.2). It remains to show that (2.34) holds for $x \in L^p(T)$. Now for each simple function x the set $A = \mathrm{supp}(x)$ has finite measure. Hence by the reasoning above,

$$F(x) = \int_A (\psi \circ x)\, d\mu, \quad \text{where } \psi \in \mathrm{Car}^\infty(A).$$

Now by uniqueness of the Radon-Nikodym derivative of the set function $a(S) = F(a\chi_S)$ we have, on the sets $(\bigcup_{i=1}^{n} T_i) \cap A$, and hence on A, that $\varphi \circ x|_A = \psi \circ x$ a.e. Thus

$$F(x) = \int_A (\varphi \circ x)\, d\mu = \int_T (\varphi \circ x)\, d\mu.$$

Therefore (2.34) has been established for simple functions.

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To show that (2.34) holds for all $x \in L^p(T)$, we again utilize the Vitalli convergence theorem. Notice that each $x \in L^p(T)$ is the limit almost everywhere as well as in norm of a sequence x_n of simple functions,

$$x_n \to x \quad \text{a.e.} \quad \text{and in } L^p(T). \tag{2.37}$$

Since $\varphi \in \mathrm{Car}(T)$, it follows that

$$\varphi \circ x_n \to \varphi \circ x \quad \text{a.e.} \tag{2.38}$$

In addition, the indefinite integrals of the sequence $\varphi \circ x_n \in L^1(T)$ are uniformly absolutely continuous, i.e.

$$\int_U |\varphi \circ x_n| \, d\mu \to 0 \quad \text{as } \mu(U) \to 0, \text{ uniformly in } n. \tag{2.39}$$

Otherwise there would exist for some $a > 0$, a sequence of sets $U_m \subset T$ with $\mu(U_m) < 3^{-m}$, and a corresponding sequence $\varphi \circ x_{nm}$ such that

$$\int_{U_m} |\varphi \circ x_{nm}| \, d\mu > a.$$

It follows that each U_m (even if U_m is an atom) would possess a subset U_m' satisfying

$$\left| \int_{U_m'} (\varphi \circ x_{nm}) \, d\mu \right| > a/2.$$

Now by (2.37) and the Vitalli convergence theorem [**5**, p. 150] applied to the x_n, the functions x_n form a bounded set in $L^p(T)$ and

$$\lim_{\mu(U) \to 0} \int_U |x_n|^p \, d\mu = 0$$

uniformly in n. Hence the functions $y_m = x_{nm}\chi_{U_m'}$ lie in a bounded subset of $L^p(T)$ and satisfy $y_m \to 0$ in $L^p(T)$. Moreover, since y_m is a simple function,

$$F(y_m) = \int_T (\varphi \circ y_m) \, d\mu = \int_{U_m'} (\varphi \circ x_{nm}) \, d\mu.$$

However, by the construction of U_m', this formula implies that the $F(y_m)$ do not converge to zero, contradicting (ii$_p$).

Finally, the sequence $\varphi \circ x_n$ has the property that for each $\epsilon > 0$ there exists a set U_ϵ such that $\mu(U_\epsilon) < \infty$ and

$$\int_{T - U_\epsilon} |\varphi \circ x_n| \, d\mu < \epsilon \quad \text{for all } n. \tag{2.40}$$

Otherwise for some $\epsilon > 0$ there exists an expanding sequence of sets U_m with $\mu(U_m) < \infty$, $\bigcup_{m=1}^{\infty} U_m = T$, and a corresponding sequence $\varphi \circ x_{nm}$ such that

$$\int_{T - U_m} |\varphi \circ x_{nm}| \, d\mu > \epsilon.$$

Thus (even if $T - U_m$ is an atom) for some $U_m'' \subset T - U_m$,

$$\left| \int_{U_m''} (\varphi \circ x_{nm})\, d\mu \right| > \epsilon/2.$$

By (2.37) and the Vitalli convergence theorem, the indefinite integrals of the functions x_n are equicontinuous with respect to μ, so that the functions $y_m = x_{nm}\chi_{U_m''}$ satisfy $y_m \to 0$ in $L^p(T)$. However, the formula

$$F(y_m) = \int_T (\varphi \circ y_m)\, d\mu = \int_{U_m''} (\varphi \circ x_{nm})\, d\mu$$

implies that the $F(y_m)$ do not converge to zero, contradicting (ii$_p$).

Since the sequence $\varphi \circ x_n$ satisfies (2.36)–(2.40), it follows by the Vitalli convergence theorem that $\varphi \circ x$ is in $L^1(T)$ and that $\varphi \circ x_n \to \varphi \circ x$ in $L^1(T)$, whereby

$$F(x) = \lim_{n\to\infty} F(x_n) = \lim_{n\to\infty} \int_T (\varphi \circ x_n)\, d\mu = \int_T (\varphi \circ x)\, d\mu.$$

Thus (2.34) holds for all $x \in L^p(T)$. The uniqueness of φ assuming that (2.2) holds, is immediate, since Theorem 1 then asserts the uniqueness of $\varphi|T_i$, $i = 1, 2, \ldots$.

For the converse we proceed as follows. Suppose that φ is a function in $\mathrm{Car}^p(T)$ which satisfies (2.2). Then (i) obviously holds. Moreover, for any S such that $\mu(S) < \infty$, the restriction $\varphi|S$ is in $\mathrm{Car}^p(S)$. This implies in particular that $\varphi|S$ is in $\mathrm{Car}^\infty(S)$ and satisfies (2.2). Thus the validity of (iii$_p$) follows from Theorem 1. On the other hand, (ii$_p$) is a consequence of a theorem of Nemitskiĭ [**9**, p. 32] which asserts that every $\varphi \in \mathrm{Car}^p(T)$ yields a *continuous* transformation from $L^p(T)$ to $L^1(T)$ by $x \to \varphi \circ x$. Indeed, the continuity of the function $x \to \int_T (\varphi \circ x)\, d\mu$ is a direct consequence of the continuity of the above transformation.

COROLLARY 2. *With T as above, there exists for every real-valued functional F on $L^p(T)$, $1 \leqq p < \infty$, which satisfies the following conditions:*

(i) *$F(x + y) = F(x) + F(y)$ when $xy = 0$ a.e.,*

(ii$_p'$) *F is uniformly continuous on each bounded subset of $L^p(T)$,*

a function $\varphi \in \mathrm{Car}^p(T)$ such that

$$F(x) = \int_T (\varphi \circ x)\, d\mu \qquad \text{for } x \in L^p(T).$$

Moreover, φ can be taken to satisfy (2.2), *and is then unique up to sets of the form $R \times N$ with $N \subset T$ a null set.*

Remark. The converse to this corollary is false except for a purely atomic space T consisting of a finite number of atoms. That is, φ being in $\mathrm{Car}^p(T)$ and satisfying (2.2) does not in other cases ensure that (ii$_p'$) holds. To see

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this, let $T = \bigcup_{i=1}^{\infty} T_i$, where $0 < \mu(T_i) < \infty$ and T_i are disjoint. Then the function

$$\varphi(h, t) = \sum_{i=1}^{\infty} f_i(h) h^p \chi_{T_i}(t)$$

is in $\mathrm{Car}^p(T)$ provided that each $f_i: R \to R$ is continuous and satisfies $|f_i| \leqq 1$. However, it is easy to prevent *uniform* continuity on certain bounded sets in $L^p(T)$ by selecting the f_i to have appropriate zeros.

3. In this section we analyze transformations from $L^p(T)$ to $C(S)$.

THEOREM 3. *With T as in Theorem 1 let A be a transformation on $L^\infty(T)$ with values in $C(S)$, where S is a compact Hausdorff space. Suppose that A satisfies the conditions*

(i_A) *$A(x + y) = A(x) + A(y)$ when $xy = 0$ a.e.,*

(ii_A) *A is uniformly continuous on each bounded subset of $L^\infty(T)$,*

(iii_A) *$A(x_n) \to A(x)$ whenever $\{x_n\}_{n \geqq 1}$ converges boundedly a.e. to $x \in L^\infty(T)$.*

Then there exists a transformation $\Phi: S \to \mathrm{Car}^\infty(T)$ such that

$$A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu = \int_T \Phi(s; x(t), t)\, d\mu(t). \tag{3.1}$$

The transformation Φ can be taken to satisfy

(a) *$\Phi(s) \circ 0 = 0$ a.e. for all $s \in S$,*

in which case $\Phi(s)$ is unique, for each s, up to sets of the form $R \times N$ with N a null set T. Moreover, Φ has the following additional properties:

(b) *the mapping $s \mapsto \Phi(s) \circ x$ is weakly continuous for each $x \in L^\infty(T)$,*

(c) *the mapping $x \mapsto \Phi(s) \circ x \in L^1(T)$ is uniformly continuous on each bounded subset of $L^\infty(T)$, uniformly in s,*

(d) *if $x_n \to x$ boundedly a.e., then*

(1) *$\lim_{\mu(E) \to 0} \int_E (\Phi(s) \circ x_n)\, d\mu = 0$ uniformly in s and n,*

(2) *for any expanding sequence E_j such that $\bigcup E_j = T$,*

$$\lim_{E_j \uparrow T} \int_{T - E_j} (\Phi(s) \circ x_n)\, d\mu = 0 \quad \textit{uniformly in } s \textit{ and } n.$$

Conversely, every transformation $\Phi: S \to \mathrm{Car}^\infty(T)$ satisfying (a), (b), (c), *and* (d) *determines by means of* (3.1) *a transformation $A: L^\infty(T) \to C(S)$ satisfying* (i_A), (ii_A), (iii_A).

Proof. If A satisfies (i_A), (ii_A), and (iii_A), then for each fixed $s \in S$ the functional defined by $F_s(x) = A(x)(s)$ satisfies (i), (ii), and (iii) of Theorem 1. Hence by Theorem 1 there exists an element $\Phi(s) \in \mathrm{Car}^\infty(T)$ satisfying (a) for which the representation

$$F_s(x) = A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu$$

holds, and $\Phi(s)$ is unique up to sets of the form $R \times N$ with N a null set in T.

To show that (b), (c), and (d) are satisfied, we proceed as follows. According to (i_A) and (iii_A), F_s determines for each $x \in L^\infty(T)$ a μ-continuous measure ν_x by means of

$$(3.2) \qquad \nu_x(G) = F_s(x\chi_G) = \int_T (\Phi(s) \circ x\chi_G)\, d\mu.$$

Using (a) we can rewrite this as follows:

$$(3.3) \qquad \nu_x(G) = \int_G (\Phi(s) \circ x)\, d\mu = A(x\chi_G)(s).$$

Thus for any $x, y \in L^\infty(T)$ we have

$$(3.4) \qquad \int_G [\Phi(s) \circ x - \Phi(s) \circ y]\, d\mu = \nu_x(G) - \nu_y(G) = A(x\chi_G)(s) - A(y\chi_G)(s).$$

Now the total variation of the signed measure $\nu_x - \nu_y$ is given by

$$(3.5) \qquad \operatorname{Var}(\nu_x - \nu_y) = \int_T |\Phi(s) \circ x - \Phi(s) \circ y|\, d\mu = \sup_{G \in \Sigma} [A(x\chi_G)(s) - A(y\chi_G)(s)] - \inf_{G' \in \Sigma} [A(x\chi_{G'})(s) - A(y\chi_{G'})(s)].$$

However, by (ii_A) we see that on each bounded subset B of $L^\infty(T)$ there exists for each $\epsilon > 0$ a δ, independent of s, such that for $x, y \in B$, $||x - y||_\infty < \delta$ the right side of equation (3.5) is less than ϵ. This yields (c).

To show that (b) holds, we observe first that, as a consequence of (c), for each $x \in L^\infty(T)$ the family

$$\mathscr{R}_x = \{\Phi(s) \circ x | \, s \in S\}$$

is a bounded subset of $L^1(T)$ (here the bounded subset of $L^\infty(T)$ is taken as $B_x = \{y \in L^\infty(T) | \, ||y||_\infty \leqq ||x||_\infty\}$). Moreover, since A maps $L^\infty(T)$ into $C(S)$, we have for each $E \in \Sigma$:

$$(3.6) \qquad \int_T (\Phi(s) \circ x\chi_E)\, d\mu = \int_T \chi_E(\Phi(s) \circ x)\, d\mu = A(x\chi_E)(s)$$

is continuous with respect to s. It then follows by (i_A) and (a) that

$$(3.7) \qquad \int_T z(\Phi(s) \circ x)\, d\mu \in C(S)$$

for every measurable function z whose range is a finite set. Since these functions are dense in $L^\infty(T) = L^1(T)'$ and $\mathscr{R}_x$ is a bounded subset of $L^1(T)$, it follows that (3.7) holds for all $z \in L^\infty(T)$, which yields (b).

To prove (d) we argue by contradiction. If (d) (1) were false, then there would exist a sequence x_n converging to x boundedly almost everywhere and a sequence of triples (E_m, s_m, x_{nm}) with $\mu(E_m) < 1/m$ such that for some fixed $\alpha > 0$,

$$(3.8)\qquad \left|\int_{E_m} (\Phi(s_m) \circ x_{nm})\, d\mu\right| > \alpha, \qquad m = 1, 2, \ldots .$$

By compactness of S we may assume without loss of generality that $s_m \to s_0$. Moreover, by (iii$_A$) we have for each $G \in \Sigma$:

$$(3.9)\quad \nu_{x_n, s_m}(G) = \int_G (\Phi(s_m) \circ x_n)\, d\mu = A(x_n\chi_G)(s_m) \to A(x\chi_G)(s_m)$$
$$= \int_G (\Phi(s_m) \circ x)\, d\mu \text{ as } n \to \infty,$$

uniformly in m. The continuity of $A(x\chi_G)$ now implies that

$$(3.10)\qquad \lim_{m,n\to\infty} \nu_{x_n, s_m}(G) = A(x\chi_G)(s_0) = \nu_{x, s_0}(G).$$

Therefore it follows by the Vitalli-Hahn-Saks theorem [**5**, p. 158] that

$$(3.11)\quad \lim_{\mu(E)\to 0} \nu_{x_n, s_m}(E) = \lim_{\mu(E)\to 0} \int_E (\Phi(s_m) \circ x_n)\, d\mu = 0 \quad \text{uniformly in } m \text{ and } n,$$

which contradicts (3.8). If (d) (2) were false, then there would exist a sequence x_n converging boundedly to x almost everywhere and a sequence of triples (E_m', s_m, x_{nm}), with $\{E_m'\}$ an expanding family in Σ whose union is T, such that for some fixed $\alpha > 0$, we have:

$$(3.12)\qquad \left|\int_{T-E_m'} (\Phi(s_m) \circ x_{nm})\, d\mu\right| > \alpha, \qquad m = 1, 2, \ldots .$$

Again we may assume that $s_m \to s_0$, so that (3.10) holds. Therefore it follows by Nikodym's corollary to the Vitalli-Hahn-Saks theorem [**5**, p. 160] that

$$(3.13)\qquad \lim_{m\to\infty} \int_{T-E_m'} (\Phi(s_m) \circ x_{nm})\, d\mu = 0 \quad \text{uniformly in } m \text{ and } n,$$

which contradicts (3.12).

For the converse we observe by Theorem 1 that (i$_A$), (ii$_A$), and $\mathscr{R}_A \subset C(s)$ all follow directly from (a), (b), and (c). To prove (iii$_A$) we observe that x_n converging to x boundedly almost everywhere implies by (d) (2) that for each $\epsilon > 0$ there exists a set E_ϵ, with $\mu(E_\epsilon) < \infty$, such that

$$\int_{T-E_\epsilon} (\Phi(s) \circ x_n)\, d\mu < \epsilon \quad \text{uniformly in } s \text{ and } n.$$

Now bounded almost everywhere convergence of x_n to x implies that on the set E_ϵ this convergence is almost uniform. Hence by (d) (1) there exists a subset $F_\epsilon \subset E_\epsilon$ such that

$$\left|\int_{F_\epsilon} (\Phi(s) \circ x_n)\, d\mu\right| < \epsilon \quad \text{uniformly in } n \text{ and } s$$

while the convergence of x_n to x on $E_\epsilon - F_\epsilon$ is uniform. Thus by (i_A), we have

$$(3.14)\quad |A(x_n)(s) - A(x)(s)| = \left| \int_{T-E_\epsilon} (\Phi(s) \circ x_n)\, d\mu - \int_{T-E_\epsilon} (\Phi(s) \circ x)\, d\mu + \int_{F_\epsilon} (\Phi(s) \circ x_n)\, d\mu - \int_{F_\epsilon} (\Phi(s) \circ x)\, d\mu + \int_{E_\epsilon - F_\epsilon} (\Phi(s) \circ x_n - \Phi(s) \circ x)\, d\mu \right| \leqq 4\epsilon + \int_{E_\epsilon - F_\epsilon} |\Phi(s) \circ x_n - \Phi(s) \circ x|\, d\mu.$$

Then by (ii_A) we have, for sufficiently large n:

$$(3.15)\qquad |A(x_n)(s) - A(x)(s)| \leqq 5\epsilon \quad \text{uniformly in } s.$$

Since $\epsilon > 0$ was arbitrary, this yields (iii_A).

We now give an analogue for $L^p(T)$, $1 \leqq p < \infty$.

THEOREM 4. *With T as in Theorem* 1, *let A be a transformation on $L^p(T)$ with values in $C(S)$, where S is a compact Hausdorff space. Suppose that A satisfies the conditions*

(i_A) *$A(x + y) = A(x) + A(y)$ when $xy = 0$ a.e.,*

(ii_{A_p}) *A is continuous on $L^p(T)$,*

(iii_{A_p}) *A is uniformly continuous relative to the L^∞ norm on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.*

Then there exists a transformation $\Phi: S \to \mathrm{Car}^p(T)$ such that

$$(3.16)\qquad A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu.$$

The transformation Φ can be taken to satisfy:

(a) *$\Phi(s) \circ 0 = 0$ a.e. for all $s \in S$,*

in which case $\Phi(s)$ is unique, for each s, up to sets of the form $R \times N$ with N a null set in T. Moreover, Φ has the following additional properties:

(b_p) *the mapping $s \mapsto \Phi(s) \circ x \in L^1(T)$ is weakly continuous for each $x \in L^p(T)$,*

(c_p) *the mapping $x \mapsto \Phi(s) \circ x \in L^1(T)$ is weakly continuous (using the norm topology on $L^p(T)$), uniformly in s,*

(d_p) *the mapping $x \mapsto \Phi(s) \circ x$ is uniformly continuous (relative to the L^∞ norm), uniformly in s, on each bounded subset of $L^\infty(T)$ which is supported by a set of finite measure.*

Conversely, every transformation $\Phi: S \to \mathrm{Car}^p(T)$ satisfying (a), (b_p), (c_p), *and* (d_p) *determines by means of* (3.16) *a transformation $A: L^p(T) \to C(S)$ satisfying* (i_A), (ii_{A_p}), *and* (iii_{A_p}).

Proof. If A satisfies (i_A), (ii_{A_p}), and (iii_{A_p}), then for each fixed $s \in S$ the functional defined by $F_s(x) = A(x)(s)$ satisfies (i), (ii_p), and (iii_p). Hence by Theorem 2 there exists an element $\Phi(s) \in \text{Car}^p(T)$ satisfying (a) for which the representation

$$F_s(x) = A(x)(s) = \int_T (\Phi(s) \circ x)\, d\mu$$

holds, and $\Phi(s)$ is unique up to sets of the form $R \times N$. To show that (b_p), (c_p), and (d_p) hold, we proceed as follows. According to (i_A) and (ii_{A_p}), F_s determines for each $x \in L^p(T)$ a μ-continuous measure ν_x by means of

$$\nu_x(G) = F_s(x\chi_G) = \int_T (\Phi(s) \circ x\chi_G)\, d\mu. \tag{3.17}$$

Using (a) we can rewrite this as follows

$$\nu_x(G) = \int_G (\Phi(s) \circ x)\, d\mu = A(x\chi_G)(s). \tag{3.18}$$

Thus the variation of the signed measure ν_x is given by

$$\begin{aligned} \text{Var}(\nu_x) &= \int_T |\Phi(s) \circ x|\, d\mu \\ &= \sup_{G \in \Sigma} A(x\chi_G)(s) - \inf_{G' \in \Sigma} A(x\chi_{G'})(s). \end{aligned} \tag{3.19}$$

We now show that for each x, the right side of equation (3.19) is bounded. Since x is in $L^p(T)$, we deduce by equicontinuity of the indefinite integral of $|x|^p$ that corresponding to each ϵ there is a set $E_\epsilon \in \Sigma$, $\mu(E_\epsilon) < \infty$, such that $\|x\chi_{T-E_\epsilon}\|_p < \epsilon$. We can require without loss of generality that E_ϵ contain at most finitely many atoms, $E_1, \ldots, E_{n_\epsilon}$. Moreover, by absolute continuity of the indefinite integral of x there exists a δ such that

$$\|x\chi_E\|_p < \epsilon \quad \text{whenever } \mu(E) < \delta. \tag{3.20}$$

Now by (ii_{A_p}), A is continuous at $0 \in L^p(T)$. Hence on taking ϵ sufficiently small we deduce that

$$\begin{aligned} &(*)\ |A(x\chi_F)(s)| \leqq 1 \quad \text{uniformly in } s, \text{ whenever } \mu(F) \leqq \delta, \\ &(**)\ |A(x\chi_F)(s)| \leqq 1 \quad \text{uniformly in } s, \text{ whenever } F \subset T - E_\epsilon. \end{aligned} \tag{3.21}$$

Now for any $G \in \Sigma$ we have, by (i_A),

$$\begin{aligned} |A(x\chi_G)(s)| &= \Big| A(x\chi_{G\cap(T-E_\epsilon)})(s) + \sum_{i=1}^{n_\epsilon} A(x\chi_{G\cap E_i})(s) \\ &\qquad + A(x\chi_{G\cap(E-\cup_{i=1}^{n_\epsilon} E_i)})(s) \Big| \\ &\leqq 1 + \sum_{i=1}^{n_\epsilon} |A(x\chi_{E_i})(s)| + |A(x\chi_{G\cap(E_\epsilon - \cup_{i=1}^{n_\epsilon} E_i)})(s)|. \end{aligned} \tag{3.22}$$

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Moreover, by splitting the non-atomic subset $E_\epsilon - \bigcup_{i=1}^{n_\epsilon} E_i$ into parts of measure less than δ and applying (3.21) (*), we obtain the estimate

$$(3.23)\quad |A(x\chi_{G\cap(E_\epsilon - \bigcup_{i=1}^{n_\epsilon} E_i)})(s)| \leqq \frac{\mu(E_\epsilon - \bigcup_{i=1}^{n_\epsilon} E_i)}{\delta} + 1 \leqq \frac{\mu(E_\epsilon)}{\delta} + 1.$$

Combining (3.22) and (3.23) we deduce that

$$(3.24)\quad |A(x\chi_G)(s)| \leqq 2 + \frac{\mu(E_\epsilon)}{\delta} + \sum_{i=1}^{n_\epsilon} ||A(x\chi_{E_i})||_\infty \equiv M_x$$

uniformly in s and G.

Therefore by (3.19) it follows that the set

$$(3.25)\qquad B_x = \{\Phi(s) \circ x\chi_E |\, E \in \Sigma,\, s \in S\}$$

is a bounded subset of $L^1(T)$.

Now since A takes $L^p(T)$ into $C(S)$ we have for each $E \in \Sigma$:

$$(3.26)\qquad \int_T (\Phi(s) \circ x\chi_E)\, d\mu = \int_T \chi_E(\Phi(s) \circ x)\, d\mu = A(x\chi_E)(s)$$

is continuous with respect to s. It then follows by (i_A) that

$$(3.27)\qquad \int_T z(\Phi(s) \circ x)\, d\mu \quad \text{is in } C(S)$$

for every simple function z. Since the simple functions are dense in $L^\infty(T) = L^1(T)'$ and B_x is a bounded subset of $L^1(T)$, it follows that (3.27) holds for all $z \in L^\infty(T)$, which yields (b_p).

To show that (c_p) holds, let $\{x_n\}_{n\geqq 1}$ denote a sequence converging to $x = x_0$ in $L^p(T)$. Then the indefinite pth power integrals of the $\{x_n\}_{n\geqq 0}$ are uniformly absolutely continuous and equicontinuous with respect to μ. Hence it follows by the technique used in deriving (3.24) that

$$B_{\{x_n\}} = \{\Phi(s) \circ x_n\chi_E |\, E \in \Sigma,\, s \in S,\, n \geqq 0\}$$

is a bounded subset of $L^1(T)$.

Now for each $E \in \Sigma$, $x_n\chi_E$ converges to $x \in \chi_E$ in $L^p(T)$ and hence by (ii_{A_p}) we have

$$(3.28)\quad \int_T (\Phi(s) \circ x_n\chi_E)\, d\mu = \int_T \chi_E(\Phi(s) \circ x_n)\, d\mu \to \int_T \chi_E(\Phi(s) \circ x)\, d\mu$$

uniformly in s.

It then follows by (i_A) that

$$(3.29)\qquad \int_T z(\Phi(s) \circ x_n)\, d\mu \to \int_T z(\Phi(s) \circ x)\, d\mu \quad \text{uniformly in } s,$$

for every simple function z. Since the simple functions are dense in $L^\infty(T) = L^1(T)'$ and $B_{\{x_n\}}$ is a bounded subset of $L^1(T)$, it follows that (3.29) holds for all $z \in L^\infty(T)$, which yields (c_p). Finally, the transformation

$A_1 = A|L^\infty(E)$, for any E such that $\mu(E) < \infty$, satisfies (i_A), (ii_A), and (iii_A) of Theorem 3, the last following from (ii_{A_p}) by virtue of the Lebesgue dominated convergence theorem. Therefore (d_p) is a consequence of Theorem 3.

The converse is immediate.

Remark. Theorems 3 and 4 are well known in the linear case [**5**, p. 490].

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