Notes

107.14 Does a trapezium exist whose side lengths form a geometric progression?

It is known that there is no trapezium whose lengths of consecutive sides form an arithmetic progression [1]. Is this true also for a geometric progression?

Let in trapezium $ABCD$ with $BC \parallel AD$ the lengths of consecutive sides form a geometric progression with common ratio $q > 1$.

Obviously it is enough to consider two cases. In the first case the geometric progression starts at AB and in the second case it starts at BC.

Case 1. Without loss of generality we can assume that $AB = 1$, then $BC = q, CD = q^2, DA = q^3$.

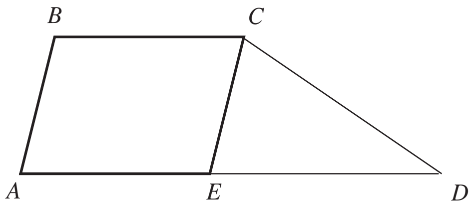

FIGURE 1

Let $CE \parallel AB$ (see Figure1). It is necessary and sufficient that the sides of triangle ECD satisfy the triangle inequality. Since $CE = AB < CD$, we have to check the inequalities: (a) $CE + ED > CD$ and (b) $CE + CD > ED$.

(a) $1 + q^3 - q > q^2 \Leftrightarrow q^3 - q^2 - q + 1 > 0 \Leftrightarrow (q + 1)(q - 1)^2 > 0$. Since $q > 1$, this inequality obviously holds.

(b) $1 + q^2 > q^3 - q \Leftrightarrow q^3 - q^2 - q - 1 < 0$. $f(q) = q^3 - q^2 - q - 1$ has $q_{\text{max}} = -\frac{1}{3}$ and $q_{\text{min}} = 1$ and $f\left(-\frac{1}{3}\right) < 0$. Then $f(q)$ has only one real root (see Figure 2), which can be approximately calculated: $q_0 \approx 1.839$. Since $q > 1$, we obtain $1 < q < q_0$.

Thus for $1 < q < q_0$ there exists a trapezium $ABCD$ with the lengths of consecutive sides AB, BC, CD, DA forming a geometric progression with common ratio q.

Case 2: Let $BC = 1$, $CD = q$, $DA = q^2$, $AB = q^3$. Let us check the triangle inequality for ΔECD: $EC = q^3$, $CD = q$, $ED = q^2 - 1$. Since $CE > CD$ it is enough to check the inequalities: (a) $CD + ED > CE$ and (b) $CD + EC > ED$.

(a) $q + q^2 - 1 > q^3 \Leftrightarrow q^3 - q^2 - q + 1 < 0$. Since $q > 1$ this inequality does not hold (see (a) of Case1) and so there is no need to check (b).

So, the trapezium exists only if the geometric progression ($q > 1$) of the lengths of the sides begins at one of the legs of the trapezium and $q < q_0 \approx 1.839$.

https://doi.org/10.1017/mag.2023.60 Published online by Cambridge University Press

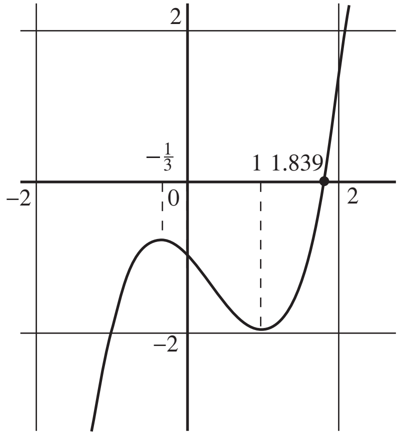

FIGURE 2

References

1. M. Stupel, V. Oxman, Trapezium whose side lengths form an arithmetic progression, *Math. Gaz.* **107** (March 2023) pp. 147

10.1017/mag.2023.60

VICTOR OXMAN
Western Galilee College, Acre, Israel
MOSHE STUPEL
Shaanan College, Gordon College, Haifa, Israel

107.15 Fruit diophantine equation

Introduction

A popular problem making the rounds in social media (stated as a problem about distribution of fruits) is the possibility of finding positive integer solutions in x, y, z of the equation that appears in the statement below. In this short note, we prove:

Theorem 1

The equation

$$y^2 - xyz + z^2 = x^3 - 5$$

has no integer solutions.

Note that the special case asserting that $y^2 = x^3 - 5$ has no integer solutions is already of interest.

The problem originated from the question "What is the smallest unsolved diophantine equation?" that was posed by user Zidane in MathOverflow [1]. In this question, the notion of size of a polynomial is the following one: