ON GROUP GRADED RINGS SATISFYING POLYNOMIAL IDENTITIES

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(Received 2 November, 1993)

A number of classical theorems of ring theory deal with nilness and nilpotency of the Jacobson radical of various ring constructions (see [**10**], [**18**]). Several interesting results of this sort have appeared in the literature recently. In particular, it was proved in [**1**] that the Jacobson radical of every finitely generated PI-ring is nilpotent. For every commutative semigroup ring RS, it was shown in [**11**] that if $J(R)$ is nil then $J(RS)$ is nil. This result was generalized to all semigroup algebras satisfying polynomial identities in [**15**] (see [**16**, Chapter 21]). Further, it was proved in [**12**] that, for every normal band B, if $J(R)$ is nilpotent, then $J(RB)$ is nilpotent. A similar result for special band-graded rings was established in [**13**, Section 6]. Analogous theorems concerning nilpotency and local nilpotency were proved in [**2**] for rings graded by finite and locally finite semigroups.

This paper is devoted to the radicals of group graded rings, which have been actively investigated by many authors (see [**10**], [**14**]). Let G be a group. An associative ring $R = \bigoplus_{g \in G} R_g$ is said to be *G-graded* (*strongly G-graded*) if $R_g R_h \subseteq R_{gh}$ (respectively, $R_g R_h = R_{gh}$) for all $g, h \in G$.

First, we consider algebras over a field of characteristic zero. In this case our result will be also of interest in connection with the well-known problem of finding necessary and sufficient conditions for the Jacobson radical to be homogeneous. An ideal I of $R = \bigoplus_{g \in G} R_g$ is said to be *homogeneous* if $I = \bigoplus_{g \in G} I \cap R_g$. This problem has not been solved even for u.p.-groups (see [**7**], [**8**], [**10**]). Polynomial identities give what appears to be the first natural sufficient condition which is applicable to the case of arbitrary groups.

THEOREM 1. *Let G be a group with identity e, and let $R = \bigoplus_{g \in G} R_g$ be a G-graded PI-algebra over a field of characteristic zero. If the Jacobson radical $J(R_e)$ is nil, then $J(R)$ is a homogeneous nil ideal of R.*

The following corollary to the main theorem is worth mentioning.

COROLLARY 2. *Let G be a group with identity e, and let $R = \bigoplus_{g \in G} R_g$ be a strongly G-graded PI-algebra over a field of characteristic zero. If $J(R_e)$ is nilpotent, then $J(R)$ is nilpotent.*

It is impossible to replace strongly graded algebras by ordinary group graded algebras in Corollary 2. Indeed, let A be the free commutative algebra with free generators $a_1, a_2, \ldots$. Denote by I the ideal of A generated by $a_1, a_2^2, a_3^3, \ldots$. Then A/I is positively graded, and so $A/I = \bigoplus_{z \in Z} A_z$ where Z is the infinite cyclic group. Although $A_0 = 0$, it is clear that $J(A/I) = A/I$ is not nilpotent.

One cannot omit the restriction on the characteristic of the field neither in Theorem

https://doi.org/10.1017/S0017089500031104 Published online by Cambridge University Press

1 nor in Corollary 2, as the results on the Jacobson radical of group algebras show (see [**10**, Section 44]). However, the conclusion of Theorem 1 concerning nilness holds for arbitrary PI-rings. Namely, the following graded analogue of the celebrated Braun's Nullstellensatz for PI-rings (see [**1**], [**18**]) is valid.

THEOREM 3. *Let G be a group with identity e, and let $R=\bigoplus_{g\in G} R_g$ be a G-graded PI-ring. If $J(R_e)$ is nil, then $J(R)$ is nil, too.*

Throughout $R=\bigoplus_{g\in G} R_g$ will be a G-graded ring, G a group with identity e. Let T be a subset of G. Put $R_T=\bigoplus_{t\in T} R_t$. For any $r\in R$, say $r=\sum_{g\in G} r_g$ where $r_g\in R_g$, we put $r_T=\sum_{t\in T} r_t$. If $I\subseteq R$ and $g\in G$, then we put $I_g=I\cap R_g$ and $I_T=I\cap R_T$. Further, for $r\neq 0$, denote the set of all non-zero homogeneous summands r_g of r by $H(r)$, and put $\operatorname{supp}(r)=\{g\in G\mid r_g\neq 0\}$. Put $H(0)=0$, $\operatorname{supp}(0)=\varnothing$. Clearly, $H(r)$ and $\operatorname{supp}(r)$ are finite sets. Let $H(I)=\bigcup_{r\in I} H(r)$. Then $H(R)$ is the set of all *homogeneous* elements of R. Evidently, $H(R)$ is a multiplicative subsemigroup of R. By the *length* of r we mean $|\operatorname{supp}(r)|$. Recall that a semigroup S is said to be *permutational* if there exists $n>1$ such that, for any n elements $x_1,\dots,x_n$ of S, their product can be rearranged as $x_1\dots x_n=x_{\sigma 1}\dots x_{\sigma n}$ for a non-trivial permutation σ. Every PI-ring (or PI-algebra) satisfies a multilinear identity, i.e. an identity of the form

$$x_1\dots x_n+\sum_{1\neq\sigma\in S_n} k_\sigma x_{\sigma 1}\dots x_{\sigma n}=0, \tag{1}$$

where S_n is the symmetric group, k_σ are integers (elements of the field in the case of algebras, see [**17**]). Let us begin with a known lemma (see [**10**, Proposition 6.18]), which will be used repeatedly.

LEMMA 4. *Let R be a G-graded ring, H a subgroup of G. Then $J(R_H)\supseteq R_H\cap J(R)$.*

LEMMA 5. *Let R be a G-graded PI-ring, I a homogeneous ideal of R contained in $J(R)$. If I_e is nil, then I is nil.*

Proof. Take any element r in $H(I)$. Since I is homogeneous, $r\in I$. Let $r\in J(R)\cap R_g$, where $g\in G$. If g is a periodic element, then there exists a positive integer n such that $r^n\in I\cap R_e=I_e$, and so r is nilpotent. Further, assume that g is not periodic. Denote by T the infinite cyclic group generated in G by g. Lemma 4 shows that $r\in J(R_T)$, and therefore r is nilpotent again in view of [**10**, Theorem 32.5]. Thus $H(I)$ is a multiplicative nil subsemigroup of R. Since R satisfies a polynomial identity, it follows from [**17**, Theorem 1.6.36], that I is nil, as required.

LEMMA 6. *Let G be a permutational group, R a G-graded PI-ring. If $J(R_e)$ is nil, then $J(R)$ is nil.*

Proof. By [**16**, Theorem 19.8], G is finite-by-abelian-by-finite. Take any $r\in J(R_G)$. Denote by S the subgroup generated in G by the support of r. It is easily seen that S is also finite-by-abelian-by-finite. Lemma 4 implies $r\in J(R_S)$, and so without loss of

https://doi.org/10.1017/S0017089500031104 Published online by Cambridge University Press

generality we may assume that G is finitely generated itself. Then assertion (2.2) of [**4**] tells us that G is abelian-by-finite, i.e. G has an abelian normal subgroup A of finite index. Then A is finitely generated, too (see [**9**]). Therefore G contains a torsion-free abelian subgroup T of finite index. By [**10**, Corollary 22.8], R is graded by the finite group G/T with the identity component R_T. Therefore [**15**, Lemma 1.1(1)], shows that it suffices to prove that $J(R_T)$ is nil. However, $J(R_T)$ is homogeneous by [**10**, Theorem 30.28], because T is torsion-free abelian. Lemma 5 completes the proof.

LEMMA 7. *Let G be a permutational group, R a G-graded PI-algebra over a field of characteristic zero. If $J(R_e)$ is nil, then $J(R)$ is homogeneous.*

Proof. We shall verify that $H(J(R))$ consists of nilpotent elements. Then [**17**, Theorem 1.6.36], will show that $H(J(R))$ generates a homogeneous nil ideal I in R, and so $J(R)=I$ is homogeneous.

Pick any $r \in J(R)$ and $g \in \operatorname{supp}(r)$. We claim that r_g is nilpotent. As in the beginning of the proof of Lemma 6, we may assume that G has a torsion-free abelian subgroup T of finite index. If we look at the natural G/T-gradation of R and apply [**10**, Theorem 30.28 (b)], and the fact that our field has characteristic zero, then we conclude that $J(R)$ is G/T-homogeneous. We may assume that the whole $\operatorname{supp}(r)$ is contained in one T-coset of G (otherwise we would pass to the G/T-homogeneous summand of r corresponding to the coset containing g). Since G/T is finite, there exists a positive integer n such that $rr_g^n \in R_T$. Given that $J(R_e)$ is nil, [**10**, Theorem 32.5], implies that all the homogeneous summands of rr_g^n are nilpotent. Therefore r_g is nilpotent, as required.

Proof of Theorem 1. By Lemma 5 the largest homogeneous ideal I of R contained in $J(R)$ is nil. Obviously, R/I is a G-graded ring, and $J(R/I)=J(R)/I$. Therefore it suffices to prove Theorem 1 for R/I. To simplify the notation we may assume that from the very beginning $I=0$.

Suppose that $J(R)\neq 0$. Choose a non-zero element r with a minimal length in $J(R)$. Denote by T and S the subgroup and, respectively, subsemigroup generated in G by $\operatorname{supp}(r)$. Let $M=M(r)$ be the multiplicative subsemigroup generated in R by $H(r)$. We claim that $H(r)$ consists of nilpotent elements.

If S is permutational, then the group T is permutational too, by [**16**, Theorem 19.8], and so all elements in $H(r)$ are nilpotent in view of Lemmas 6 and 7.

Further, consider the case where S is not permutational. Let n be the degree of a multilinear identity (1) of R. There exist elements $s_1,\ldots,s_n$ in S such that $s_1\ldots s_n \neq s_{\sigma 1}\ldots s_{\sigma n}$ for all $\sigma \in S_n$ such that $\sigma \neq 1$. Clearly, there exist $x_1,\ldots,x_n \in M$ such that $x_i \in R_{s_i}$ for all $i=1,\ldots,n$. Applying (1) to the elements $x_1,\ldots,x_n$ we get

$$x_1\ldots x_n \in R_{s_1\ldots s_n} \cap \sum_{1\neq\sigma\in S_n} R_{s_{\sigma 1}\ldots s_{\sigma n}}, \tag{2}$$

whence $x_1\ldots x_n=0$. It follows that $y_1\ldots y_m=0$ for some $y_1,\ldots,y_m \in H(r)$. Then we can choose m and the $y_1,\ldots,y_m$ such that $y_1\ldots y_{m-1}\neq 0$. Now look at the product $y_1\ldots y_{m-1}r$. It also belongs to $J(R)$ but has fewer homogeneous summands than r. Hence $y_1\ldots y_{m-1}r=0$ by the choice of r. Since G is a group, we get $y_1\ldots y_{m-1}H(r)=0$. Further, we can look at $y_1\ldots y_{m-2}rH(r)=0$ and infer $y_1\ldots y_{m-2}(H(r)^2)=0$. Reasoning like this m times, we conclude $(H(r))^m=0$. In particular, all elements in $H(r)$ are nilpotent, again.

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Denote by L the ideal generated in $H(R)$ by $H(r)$. Each non-zero element of L is a homogeneous summand of a certain element of positive minimal length in $J(R)$. It follows from what we have proved that L is a nil ideal of $H(R)$. Hence L is locally nilpotent by [**17**, Theorem 1.6.36]. Therefore L generates a nil subalgebra K in R. Evidently, K is a homogeneous ideal of R, and so $K \subseteq I$, contradicting the fact that $I = 0$. Thus $J(R) = I$, and so $J(R)$ is a homogeneous nil ideal of R.

Proof of Corollary 2. By Theorem 1 we get $J(R) = \bigoplus_{g \in G} I_g$, where $I_g = I \cap R_g$. Denote by n the nilpotency index of $J(R_e)$. Lemma 4 implies $I_e \subseteq J(R_e)$, and so $(I_e)^n = 0$. We claim that $J(R)^{2n} = 0$.

To this end we need only to verify that $x_1 \ldots x_{2n} = 0$ for arbitrary homogeneous elements $x_1, \ldots, x_{2n} \in J(R)$. Let $x_i \in I_{g_i}$, where $i = 1, \ldots, 2n$. Given that $x_2 \in R_{g_2} = R_{g_1^{-1}} R_{g_1 g_2}$, we can find $y_{j_1}^{(1)} \in R_{g_1^{-1}}$ and $z_{j_1}^{(1)} \in R_{g_1 g_2}$ such that $x_2 = \sum_{j_1} y_{j_1}^{(1)} z_{j_1}^{(1)}$. Then $x_1 y_{j_1}^{(1)} \in I_e$ for all j_1. Further, suppose that for some $i = 2, \ldots, n$ elements $y_{j_{i-1}}^{(i-1)}$ and $z_{j_{i-1}}^{(i-1)}$ have been introduced such that $x_{2i-2} = \sum_{j_{i-1}} y_{j_{i-1}}^{(i-1)} z_{j_{i-1}}^{(i-1)}$ and all $z_{j_{i-1}}^{(i-1)} \in R_{g_1 \ldots g_{2i-2}}$. Then $x_{2i} \in R_{g_{2i}} = R_{(g_1 \ldots g_{2i-1})^{-1}} R_{g_1 \ldots g_{2i}}$, and so there exist homogeneous elements $y_{j_i}^{(i)} \in R_{(g_1 \ldots g_{2i-1})^{-1}}$ and $z_{j_i}^{(i)} \in R_{g_1 \ldots g_{2i}}$ such that $x_{2i} = \sum_{j_i} y_{j_i}^{(i)} z_{j_i}^{(i)}$ and all $z_{j_{i-1}}^{(i-1)} x_{2i-1} y_{j_i}^{(i)}$ belong to I_e. Therefore

$$x_1 \ldots x_{2n} = \sum_{j_1, \ldots, j_n} (x_1 y_{j_1}^{(1)})(z_{j_1}^{(1)} x_3 y_{j_2}^{(2)})(z_{j_2}^{(2)} x_5 y_{j_3}^{(3)}) \ldots (z_{j_{n-1}}^{(n-1)} x_{2n-1} y_{j_n}^{(n)}) z_{j_n}^{(n)}$$

$$\in I_e^n \sum_{j_n} z_{j_n}^{(n)} = 0,$$

which completes the proof.

Proof of Theorem 3. We shall prove that $J(R) = B(R)$, where $B(R)$ denotes the Baer radical of R. To this end we show that $R/B(R)$ is semisimple. Suppose to the contrary that $J(R/B(R)) \neq 0$. Since $R/B(R)$ is a subdirect product of prime PI-rings, there exists a prime PI-ring $\bar{R}$ and a homomorphism f of R onto $\bar{R}$ such that $f(J(R)) \neq 0$. By Posner's theorem (see [**17**]) $\bar{R}$ is contained in a matrix ring D_m, where D is a division ring. For any $r \in R$ and $T \subseteq R$, put $\bar{r} = f(r)$, $\bar{T} = f(T)$.

Consider the set L of all $x \in H(R)$ such that either $\bar{x} = 0$ or $\bar{x}$ has the smallest non-zero rank in D_m. By [**16**, Theorem 1.6], all non-zero elements of $\bar{L}$ lie in the same completely 0-simple factor F of the multiplicative semigroup D_m. Obviously, $\bar{L}$ is a multiplicative ideal of $\overline{H(R)}$, and so L is a multiplicative ideal of $H(R)$.

Put $M = LJ(R)L = \{xay \mid a \in J(R), x, y \in L\}$. Clearly, M is a multiplicative subsemigroup of R, because $J(R)$ is an ideal of R. Denote by I the subring generated in R by L. It follows that I is an ideal of R and $\bar{I}$ is an ideal of $\bar{R}$. Since $\bar{I} \neq 0$ and $\bar{R}$ is prime, we get $\overline{IJ(R)} \neq 0$, and so $\overline{IJ(R)I} \neq 0$. Since I consists of all finite sums of elements of L, we get $\bar{M} \neq 0$. We shall show that every element in $\bar{M}$ is nilpotent.

Fix any non-zero $\bar{w} \in \bar{M}$. There exists $w = axb$, such that $a, b \in L$, $x \in J(R)$, $x = \sum_{k=1}^{s} x_k$, and all x_k are homogeneous. Elements $\bar{a}$ and $\bar{b}$ belong to the same completely 0-simple factor F of D_m. By [**16**, Theorem 1.3], we can represent F as a Rees

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matrix semigroup $F = \mathcal{M}(G^0; I, \Lambda; P)$. (The definitions and standard properties of completely 0-simple semigroups and Rees matrix semigroups can be also found in each of the following monographs [**3**], [**5**], [**6**].) Let $\bar{a} = (i, g, \lambda)$ and $\bar{b} = (j, h, \mu)$, where $i, j \in I$, $\lambda, \mu \in \Lambda$, $g, h \in G$. It is routine to verify that $\overline{ax_k b} \in (i, G^0, \mu)$ for every $k = 1, \ldots, s$.

If $p_{\mu i} = 0$, then $(i, G^0, \mu)^2 = 0$ in F, and therefore $(\overline{axb})^2 = 0$ in F. Hence every element $(\overline{ax_k b})(\overline{ax_l b})$ has a smaller rank in D_m than $\bar{a}$. Therefore $(\overline{ax_k b})(\overline{ax_l b}) = 0$ where $1 \le k, l \le s$. Thus $\bar{w}^2 = 0$, as required.

Further, consider the case where $p_{\mu i} \neq 0$. By [**16**, Lemma 1.4], $P = (i, G, \mu)$ is a multiplicative subgroup of D_m. Put $T = \{g \in G \mid \bar{R}_g \cap P \neq 0\}$. Clearly, T is a subsemigroup of G. Let n be the degree of a multilinear identity (1) satisfied in R. For any $g_1, \ldots, g_n \in T$, we can choose $r_k \in R_{g_k}$ such that $\overline{r_k} \in P$, where $k = 1, \ldots, n$. Applying (1) we get (2), which implies that T is permutational. By [**16**, Theorem 19.8], T generates a permutational subgroup Q in G. Lemma 6 shows that $J(R_Q)$ is nil.

Since all non-zero summands $\overline{ax_k b}$ belong to P, where $k = 1, \ldots, s$, we get $ax_k b \in R_Q$, and so $axb \in J(R_Q)$ by Lemma 4. Therefore $w = axb$ is a nilpotent element.

Thus $\bar{M}$ is a multiplicative nil subsemigroup of D_m. Hence $\bar{M}^q = 0$ for some $q > 1$ by [**18**, Proposition 2.6.30]. Since M is, evidently, closed under multiplication by elements of $H(R)$, it follows that $\bar{M}$ generates a nilpotent ideal in $\bar{R}$. This contradicts the primeness of $\bar{R}$ and completes the proof.

REFERENCES

1. A. Braun, The nilpotency of the radical in a finitely generated PI-ring, *J. Algebra* **89** (1984), 375–396.

2. M. V. Clase and E. Jespers, On the Jacobson radical of semigroup graded rings, *J. Algebra* **169** (1994), 79–97.

3. A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, vol. 1 (American Mathematical Society, 1961).

4. M. Curzio, P. Longobardi, M. Maj and D. J. S. Robinson, A permutational property of groups, *Arch. Math. (Basel)* **44** (1985), 385–389.

5. P. M. Higgins, *Techniques of semigroup theory* (Oxford University Press, 1992).

6. J. M. Howie, *An introduction to semigroup theory*, London Mathematical Society Monographs No. 7 (Academic Press, 1976).

7. E. Jespers, Radicals of graded rings, *Theory of radicals, Szekszárd*, 1991, Colloq. Math. Soc. János Bolyai **61** (North Holland, 1993), 109–130.

8. E. Jespers, J. Krempa and E. R. Puczyłowski, On radicals of graded rings, *Comm. Algebra* **10** (1982), 1849–1854.

9. M. I. Kargapolov and Ju. I. Merzljakov, *Fundamentals of the Theory of Groups* (Springer, 1979).

10. G. Karpilovsky, *The Jacobson radical of classical rings*, Pitman Monographs (Longman/John Wiley, 1991).

11. W. D. Munn, The algebra of a commutative semigroup over a commutative ring with unity, *Proc. Roy. Soc. Edinburgh Sect. A* **99** (1985), 387–398.

12. W. D. Munn, The Jacobson radical of a band ring, *Math. Proc. Cambridge Philos. Soc.* **105** (1989), 277–283.

13. W. D. Munn, A class of band-graded rings, *J. London Math. Soc.* **45** (1992), 1–16.

14. C. Năstăsescu and F. Van Oystaeyen, *Graded ring theory* (North Holland, 1982).

15. J. Okniński, On the radical of semigroup algebras satisfying polynomial identities, *Math. Proc. Cambridge Philos. Soc.* **99** (1986), 45–50.

https://doi.org/10.1017/S0017089500031104 Published online by Cambridge University Press

16. J. Okniński, *Semigroup algebras* (Marcel Dekker, 1991).
17. L. H. Rowen, *Polynomial identities in ring theory* (Academic Press, 1980).
18. L. H. Rowen, *Ring theory* (Academic Press, 1988).

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