Bull. Aust. Math. Soc. **102** (2020), 506–516
doi:10.1017/S0004972720000167

SEQUENTIAL COLLISION-FREE OPTIMAL MOTION PLANNING ALGORITHMS IN PUNCTURED EUCLIDEAN SPACES

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(Received 17 January 2020; accepted 20 January 2020; first published online 13 March 2020)

Abstract

In robotics, a topological theory of motion planning was initiated by M. Farber. We present optimal motion planning algorithms which can be used in designing practical systems controlling objects moving in Euclidean space without collisions between them and avoiding obstacles. Furthermore, we present the multi-tasking version of the algorithms.

2010 *Mathematics subject classification*: primary 55R80; secondary 55P10, 68T40.

Keywords and phrases: configuration spaces, punctured Euclidean spaces, robotics, topological complexity, higher motion planning algorithms.

1. Introduction

The robot motion planning problem usually ignores dynamics and other differential constraints and focuses primarily on the translations and rotations required to move the robot. Here we have in mind an infinitesimal mass particle as an object, for example, an infinitesimally small ball (see Figure 1).

We consider a multi-robot system consisting of k distinguishable robots moving in Euclidean space $\mathbb{R}^d$ ($d \geq 2$) without collisions and avoiding r stationary obstacles (where $r \geq 0$). We focus primarily on the translations required to move the robot. The associated state space or configuration space for this mechanical system is the classical ordered configuration space $F(\mathbb{R}^d - Q_r, k)$ of k distinct points on the punctured Euclidean space $\mathbb{R}^d - Q_r$ (see [3] for the notion of ordered configuration spaces). Here $Q_r = \{q_1, \dots, q_r\}$ represents the set of r obstacles q_j. Explicitly,

$$F(\mathbb{R}^d - Q_r, k) = \{(x_1, \dots, x_k) \in (\mathbb{R}^d)^k \mid x_i \neq x_j \text{ for } i \neq j \text{ and } x_i \neq q_j \text{ for any } i, j\},$$

The first author would like to thank São Paulo Research Foundation (FAPESP), Grant No. 2018/23678-6, for financial support.

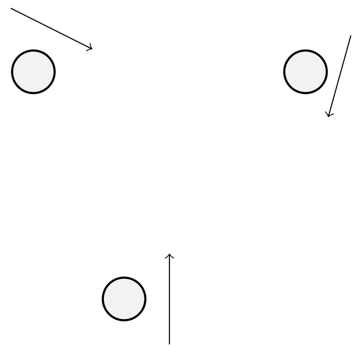

Figure 1. Multi-robot system.

equipped with the subspace topology of the Cartesian power $(\mathbb{R}^d)^k$. Note that the ith coordinate of a point $(x_1, \ldots, x_n) \in F(\mathbb{R}^d - Q_r, k)$ represents the state or position of the ith moving object, so that the condition $x_i \neq x_j$ reflects the collision-free requirement and the condition $x_i \neq q_j$ reflects the avoiding obstacle requirement.

The collision-free sequential robot motion planning problem (à la Rudyak) is to control simultaneously these k robots without collisions between them and avoiding obstacles, where one is interested, in addition to initial–final states, in $n - 2$ intermediate states. To solve this problem we need to find an *nth sequential collision-free optimal motion planning algorithm* on the state space $F(\mathbb{R}^d - Q_r, k)$ (see Section 2). A central challenge of modern robotics (see, for example, Latombe [7] and LaValle [8]) is to design explicit and suitably optimal motion planners. This involves challenges in modelling planning problems, designing efficient algorithms and developing robust implementations.

Sequential collision-free optimal motion planning algorithms in Euclidean space without obstacles were given by the authors in [10]. The purpose of the present work is to address the punctured case.

To give sequential collision-free optimal motion planning algorithms, we need to know the smallest possible number of regions of continuity for any nth sequential collision-free motion planning algorithm, that is, the value of $\mathrm{TC}_n(F(\mathbb{R}^d - Q_r, k))$, where TC_n is the nonreduced version of the nth sequential topological complexity (see Section 2 for the definition). This value was computed by González and Grant in [6].

Theorem 1.1 [6]. *For $d, k, n \geq 2$ and $r \geq 0$,*

$$TC_n(F(\mathbb{R}^d - Q_r, k)) = \begin{cases} n(k-1)+1 & \text{if } r = 0 \text{ and } d \text{ is odd;} \\ n(k-1) & \text{if } r = 0 \text{ and } d \text{ is even;} \\ nk & \text{if } r = 1 \text{ and } d \text{ is even;} \\ nk+1 & \text{otherwise.} \end{cases}$$

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We can say that a higher optimal motion planning algorithm in $F(\mathbb{R}^d, k+1)$ induces a higher optimal motion planning algorithm in $F(\mathbb{R}^d - Q_1, k)$ (see Remark 2.1) with nk regions of continuity for $d \geq 2$. This is because $F(\mathbb{R}^d, k+1)$ and $F(\mathbb{R}^d - Q_1, k)$ are homotopy equivalent. Indeed, $F(\mathbb{R}^d - Q_1, k)$ is a deformation retract of $F(\mathbb{R}^d, k+1)$. For this reason, we focus on the $r \geq 2$ case.

We present a higher optimal motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$ with $nk+1$ regions of continuity. We show that this algorithm works for any $d \geq 2$, $n \geq 2$, $r \geq 2$ and $k \geq 2$. Moreover, it gives an alternative constructive proof for the inequality $\mathrm{TC}_n(F(\mathbb{R}^d - Q_r, k)) \leq nk+1$, which was proved in [6] using tools from homotopy theory.

2. Preliminary results

The notion of the nth sequential or higher topological complexity was introduced by Rudyak in [9] and further developed in [1]. We follow [10] to recall the basic definitions and properties.

For a topological space X, let PX denote the space of paths $\gamma : [0,1] \to X$, equipped with the compact-open topology. For $n \geq 2$, one has the evaluation fibration

$$e_n : PX \to X^n, \quad e_n(\gamma) = \left(\gamma(0), \gamma\left(\frac{1}{n-1}\right), \ldots, \gamma\left(\frac{n-2}{n-1}\right), \gamma(1)\right). \tag{2.1}$$

An *nth sequential motion planning algorithm* is a section $s\colon X^n \to PX$ of the fibration e_n, that is, a (not necessarily continuous) map satisfying $e_n \circ s = id_{X^n}$. A continuous nth sequential motion planning algorithm in X exists if and only if the space X is contractible. This fact gives, in a natural way, the definition of the following numerical invariant. The *nth sequential topological complexity* $\mathrm{TC}_n(X)$ of a path-connected space X is the Schwarz genus of the evaluation fibration (2.1). In other words, the nth sequential topological complexity of X is the smallest positive integer $\mathrm{TC}_n(X) = k$ for which the product X^n is covered by k open subsets $X^n = U_1 \cup \cdots \cup U_k$ such that for any $i = 1, 2, \ldots, k$, there exists a continuous section $s_i : U_i \to PX$ of e_n over U_i (that is, $e_n \circ s_i = i_{U_i}$), where $i_U : U \hookrightarrow X^n$ denotes the inclusion map. An nth sequential motion planning algorithm $s := \{s_i : U_i \to PX\}_{i=1}^k$ is called *optimal* if $k = \mathrm{TC}_n(X)$.

One of the basic properties of TC_n is its homotopy invariance, that is, if X and Y are homotopy equivalent, then $\mathrm{TC}_n(X) = \mathrm{TC}_n(Y)$ for any $n \geq 2$. Furthermore, the nth sequential motion planning algorithms on X and Y are explicitly related.

Remark 2.1 (Homotopy invariance). Suppose that X *dominates* Y, by which we mean that there are maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq id_Y$. Choose a homotopy $H : Y \times [0,1] \to Y$ with $H_0 = id_Y$ and $H_1 = f \circ g$. Let $\mathrm{TC}_n(X) = k$ and let $s := \{s_i : U_i \to PX\}_{i=1}^k$ be an nth sequential motion planning algorithm to X with $X^n = U_1 \cup \cdots \cup U_k$ and $e_n \circ s_i = i_{U_i}$. Set $V_i := (g \times \cdots \times g)^{-1}(U_i) \subseteq Y^n$, for $i = 1, \ldots, k$,

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and define $\hat{s}_i : V_i \to PY$ by the formula

$$\hat{s}_i(y_1,\dots,y_n)(t) = \begin{cases} H_{3(n-1)t}(y_1), & 0 \le t \le \dfrac{1}{3(n-1)}; \\ f\Big(s(g(y_1),\dots,g(y_n))\Big(3t - \dfrac{1}{n-1}\Big)\Big), & \dfrac{1}{3(n-1)} \le t \le \dfrac{2}{3(n-1)}; \\ H_{3-3(n-1)t}(y_2), & \dfrac{2}{3(n-1)} \le t \le \dfrac{1}{n-1}; \\ H_{3(n-1)t-3}(y_2), & \dfrac{1}{n-1} \le t \le \dfrac{4}{3(n-1)}; \\ f\Big(s(g(y_1),\dots,g(y_n))\Big(3t - \dfrac{3}{n-1}\Big)\Big), & \dfrac{4}{3(n-1)} \le t \le \dfrac{5}{3(n-1)}; \\ H_{6-3(n-1)t}(y_3), & \dfrac{5}{3(n-1)} \le t \le \dfrac{2}{n-1}; \\ \quad\vdots & \\ H_{3(n-1)t-3(n-2)}(y_{n-1}), & \dfrac{n-2}{n-1} \le t \le \dfrac{3n-5}{3(n-1)}; \\ f\Big(s(g(y_1),\dots,g(y_n))\Big(3t - \dfrac{2n-3}{n-1}\Big)\Big), & \dfrac{3n-5}{3(n-1)} \le t \le \dfrac{3n-4}{3(n-1)}; \\ H_{3(n-1)-3(n-1)t}(y_n), & \dfrac{3n-4}{3(n-1)} \le t \le 1. \end{cases}$$

Then $Y^n = V_1 \cup \cdots \cup V_k$ and $e_n \circ \hat{s}_i = i_{V_i}$. Thus, $\hat{s} := \{\hat{s}_i : V_i \to PY\}_{i=1}^k$ is an nth sequential motion planning algorithm to Y and hence $\mathrm{TC}_n(Y) \le k = \mathrm{TC}_n(X)$.

In particular, if X and Y are homotopy equivalent, we have $\mathrm{TC}_n(X) = \mathrm{TC}_n(Y) = k$. Furthermore, if $s := \{s_i : U_i \to PX\}_{i=1}^k$ is an optimal nth sequential motion planning algorithm to X, then $\hat{s} := \{\hat{s}_i : V_i \to PY\}_{i=1}^k$, as above, is an optimal nth sequential motion planning algorithm to Y.

Remark 2.2 (Farber's TC and Rudyak's higher TC). Note that TC_2 coincides with Farber's topological complexity, which is defined in terms of motion planning algorithms for a robot moving between initial–final configurations [4]. The more general TC_n is Rudyak's higher topological complexity of the motion planning problem, whose input requires, in addition to initial–final states, $n-2$ intermediate states of the robot. As in [10], we will use the expression 'motion planning algorithm' as a substitute for 'nth sequential motion planning algorithm for $n = 2$'.

Since (2.1) is a fibration, the existence of a continuous motion planning algorithm on a subset A of X^n implies the existence of a corresponding continuous motion planning algorithm on any subset B of X^n deforming to A within X^n. Such a fact is argued in [10] in a constructive way, generalising [5, Example 6.4] (the latter given for $n = 2$). Of course, this suits our implementation-oriented objectives.

Remark 2.3 (Constructing motion planning algorithms via deformations—higher case [10]). Let $s_A : A \to PX$ be a continuous motion planning algorithm defined on a subset

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A of X^n. Suppose that a subset $B \subseteq X^n$ can be continuously deformed within X^n into A. Choose a homotopy $H : B \times [0, 1] \to X^n$ such that $H(b, 0) = b$ and $H(b, 1) \in A$ for any $b \in B$. Let $h_1, \ldots, h_n$ be the Cartesian components of H, that is, $H = (h_1, \ldots, h_n)$. The formula

$$s_B(b)(\tau) = \begin{cases} h_1(b, 3(n-1)\tau), & 0 \le \tau \le \dfrac{1}{3(n-1)}; \\ s_A(H(b,1))\Big(3\tau - \dfrac{1}{n-1}\Big), & \dfrac{1}{3(n-1)} \le \tau \le \dfrac{2}{3(n-1)}; \\ h_2(b, 3 - 3(n-1)\tau), & \dfrac{2}{3(n-1)} \le \tau \le \dfrac{1}{n-1}; \\ h_2(b, 3(n-1)\tau - 3), & \dfrac{1}{n-1} \le \tau \le \dfrac{4}{3(n-1)}; \\ s_A(H(b,1))\Big(3\tau - \dfrac{3}{n-1}\Big), & \dfrac{4}{3(n-1)} \le \tau \le \dfrac{5}{3(n-1)}; \\ h_3(b, 6 - 3(n-1)\tau), & \dfrac{5}{3(n-1)} \le \tau \le \dfrac{2}{n-1}; \\ \quad\vdots & \\ h_{n-1}(b, 3(n-1)\tau - 3(n-2)), & \dfrac{n-2}{n-1} \le \tau \le \dfrac{3n-5}{3(n-1)}; \\ s_A(H(b,1))\Big(3\tau - \dfrac{2n-3}{n-1}\Big), & \dfrac{3n-5}{3(n-1)} \le \tau \le \dfrac{3n-4}{3(n-1)}; \\ h_n(b, 3(n-1) - 3(n-1)\tau), & \dfrac{3n-4}{3(n-1)} \le \tau \le 1, \end{cases}$$

defines a continuous section $s_B : B \to PX$ of (2.1) over B. Hence, a deformation of B into A and a continuous motion planning algorithm defined on A determine an explicit continuous motion planning algorithm defined on B.

2.1. Tame motion planning algorithms. Although the definition of $\mathrm{TC}_n(X)$ deals with open subsets of X^n admitting continuous sections of the evaluation fibration (2.1), for practical purposes, the construction of explicit nth sequential motion planning algorithms is usually done by partitioning the whole space X^n into pieces, over each of which a continuous section (2.1) is set. Since any such partition necessarily contains subsets which are not open (recall that X has been assumed to be path-connected), we need to be able to operate with subsets of X^n of a more general nature.

Recall that a topological space X is a *Euclidean neighbourhood retract* (ENR) if it can be embedded into a Euclidean space $\mathbb{R}^d$ with an open neighbourhood U, $X \subset U \subset \mathbb{R}^d$, admitting a retraction $r : U \to X$, $r|_U = id_X$.

Example 2.4. A subspace $X \subset \mathbb{R}^d$ is an ENR if and only if it is locally compact and locally contractible; see [2, Ch. 4, Section 8]. This implies that all finite-dimensional polyhedra, smooth manifolds and semi-algebraic sets are ENRs.

Let X be an ENR. An nth sequential motion planning algorithm $s : X^n \to PX$ is said to be *tame* if X^n splits as a pairwise-disjoint union $X^n = F_1 \cup \cdots \cup F_k$, where each F_i is an ENR and each restriction $s|_{F_i}: F_i \to PX$ is continuous. The subsets F_i in such a decomposition are called *domains of continuity* for s.

Proposition 2.5 [9, Proposition 2.2]. *If X is an ENR, then $TC_n(X)$ is the minimal number of domains of continuity $F_1, \dots, F_k$ for tame nth sequential motion planning algorithms $s : X^n \to PX$.*

Remark 2.6. In the final paragraph of the introduction, we noted that we will construct optimal nth sequential motion planners in $F(\mathbb{R}^d - Q_r, k)$. We can now be more precise: we actually construct nth sequential tame motion planning algorithms with the advertised optimality property.

3. A tame motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$

In this section we present a tame motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$ for $r \geq 2$. The algorithm with $2k + 1$ regions of continuity works for any $d \geq 2, r \geq 2$ and $k \geq 2$. The algorithm is optimal in the sense that it has the smallest possible number of regions of continuity.

We think of $F(\mathbb{R}, k + r)$ as a subspace of $F(\mathbb{R}^d, k + r)$ via the standard embedding $\mathbb{R} \hookrightarrow \mathbb{R}^d$, $x \mapsto (x, 0, \dots, 0)$. By the m-homogeneous property of $\mathbb{R}^d$, we can suppose that $Q_r = \{q_1, \dots, q_r\} \subset \mathbb{R}$ with $q_1 < q_2 < \cdots < q_r$ and $|q_{i+1} - q_i| = 1$.

We assume that $d \geq 2$ and we write the first two standard basis elements as $e_1 = (1, 0, \dots, 0)$ and $e_2 = (0, 1, 0, \dots, 0)$ in $\mathbb{R}^d$. Let $p : \mathbb{R}^d \to \mathbb{R}, (x_1, \dots, x_q) \mapsto x_1$ be the projection onto the first coordinate. For a configuration $C \in F(\mathbb{R}^d, k + r)$, where $C = (x_1, \dots, x_{k+r})$ with $x_i \in \mathbb{R}^d$, $x_i \neq x_j$ for $i \neq j$, consider the set of projection points

$$P(C) = \{p(x_1), \dots, p(x_{k+r})\},$$

where $p(x_i) \in \mathbb{R}$, $i = 1, \dots, k + r$. The cardinality of this set will be denoted by $\mathrm{cp}(C)$. Note that $\mathrm{cp}(C)$ can be any number $1, 2, \dots, k + r$.

The configuration space $F(\mathbb{R}^d - Q_r, k)$ is the fibre of the Fadell–Neuwirth fibration $\pi_{k+r,r} : F(\mathbb{R}^d, k + r) \to F(\mathbb{R}^d, r)$, given by $(x_1, \dots, x_{k+r}) \mapsto (x_1, \dots, x_r)$. Indeed, the space $F(\mathbb{R}^d - Q_r, k)$ can be identified with the space

$$\pi^{-1}_{k+r,r}(q_1, \dots, q_r) = \{(q_1, \dots, q_r, x_{r+1}, \dots, x_{r+k}) : (x_{r+1}, \dots, x_{r+k}) \in F(\mathbb{R}^d - Q_r, k)\}.$$

We recall the tame motion planning algorithm in $F(\mathbb{R}^d, k + r)$ given in [10] for any $d \geq 2$. This algorithm has domains of continuity $W_2, W_3, \dots, W_{2k+2r}$, where

$$W_l = \bigcup_{i+j=l} A_i \times A_j,$$

and A_i is the set of all configurations $C \in F(\mathbb{R}^d, k + r)$ with $\mathrm{cp}(C) = i$. For each $i = 1, \dots, r - 1$, the set

$$A_i \cap F(\mathbb{R}^d - Q_r, k) = \varnothing,$$

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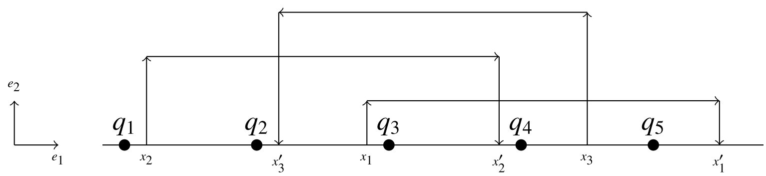

FIGURE 2. Section over $F(\mathbb{R} - Q_r, k) \times F(\mathbb{R} - Q_r, k)$. Vertical arrows pointing upwards (downwards) describe the first (last) third of the path $\Gamma^{C,C'}$, whereas horizontal arrows describe the middle third of $\Gamma^{C,C'}$.

because, for any configuration $C \in F(\mathbb{R}^d - Q_r, k) = \pi^{-1}_{k+r,r}(q_1, \dots, q_r) \subset F(\mathbb{R}^d, k+r)$, we have $\mathrm{cp}(C) \geq r$. Then, for each $l = 2, \dots, 2r-1$,

$$W_l \cap (F(\mathbb{R}^d - Q_r, k) \times F(\mathbb{R}^d - Q_r, k)) = \varnothing.$$

Thus, for the remainder of the paper, we will consider $i \in \{r, r+1, \dots, k+r\}$ and $l \in \{2r, 2r+1, \dots, 2k+2r\}$.

Set $A_i^\circ := A_i \cap F(\mathbb{R}^d - Q_r, k)$ and $W_l^\circ = W_l \cap (F(\mathbb{R}^d - Q_r, k) \times F(\mathbb{R}^d - Q_r, k))$. It is easy to see that $W_l^\circ = \bigcup_{i+j=l} A_i^\circ \times A_j^\circ$. From [10], the closure (relative to the topology on $F(\mathbb{R}^d, k+r)$) of each set A_i is contained in the union of the sets A_j with $j \leq i$, that is,

$$\overline{A_i} \subset \bigcup_{j \leq i} A_j.$$

Hence, the closure (relative to the topology on $F(\mathbb{R}^d - Q_r, k)$) of each set A_i° is contained in the union of the sets A_j° with $j \leq i$, that is,

$$\overline{A_i^\circ} \subset \bigcup_{j \leq i} A_j^\circ. \tag{3.1}$$

The sets $A_i^\circ = A_i \cap F(\mathbb{R}^d - Q_r, k)$, where $i = r, r+1, \dots, k+r$, are ENRs because they are semi-algebraic sets.

Next, we will construct a tame motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$ having $2k+1$ domains of continuity $W_{2r}^\circ, \dots, W_{2k+2r}^\circ$.

3.1. Section over $F(\mathbb{R} - Q_r, k) \times F(\mathbb{R} - Q_r, k)$. We start with two configurations $C = (q_1, \dots, q_r, x_{r+1}, \dots, x_{k+r})$ and $C' = (q_1, \dots, q_r, x'_{r+1}, \dots, x'_{k+r})$ in $F(\mathbb{R} - Q_r, k)$. Let $\Gamma^{C,C'}$ be the path in $F(\mathbb{R}^d - Q_r, k)$ from C to C' depicted in Figure 2.

Explicitly, $\Gamma^{C,C'}$ has components $(q_1, \dots, q_r, \Gamma^{C,C'}_{r+1}, \dots, \Gamma^{C,C'}_{k+r})$ defined by

$$\Gamma_i^{C,C'}(t) = \begin{cases} x_i + (3ti)e_2 & \text{for } 0 \leq t \leq \frac{1}{3}; \\ x_i + ie_2 + (3t-1)(x_i' - x_i) & \text{for } \frac{1}{3} \leq t \leq \frac{2}{3}; \\ x_i' + i(3-3t)e_2 & \text{for } \frac{2}{3} \leq t \leq 1. \end{cases} \tag{3.2}$$

This yields a continuous motion planning algorithm $\Gamma : F(\mathbb{R} - Q_r, k) \times F(\mathbb{R} - Q_r, k) \to PF(\mathbb{R}^d - Q_r, k)$.

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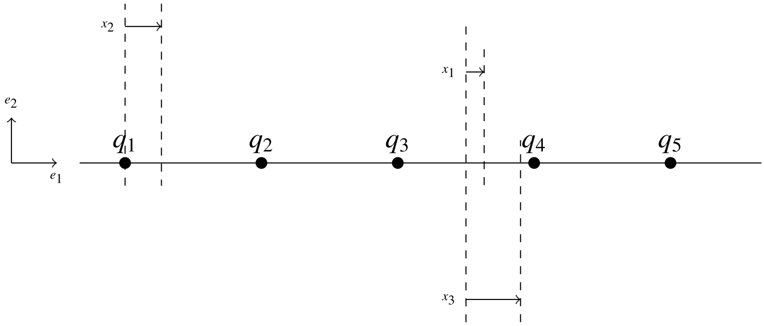

FIGURE 3. Desingularisation deformation.

3.2. The set A°_{k+r}. Choose a configuration $C = (x_1, \dots, x_r, x_{r+1}, \dots, x_{k+r}) \in A_{k+r}$. The map $\varphi : A_{k+r} \times [0, 1] \to F(\mathbb{R}^d, k + r)$ given by the formula

$$\varphi_i(C, t) = x_i + t(p(x_i) - x_i), \quad i = 1, \dots, k + r,$$

defines a continuous deformation of A_{k+r} onto $F(\mathbb{R}, k)$ inside $F(\mathbb{R}^d, k)$ (see [10]). If $C = (q_1, \dots, q_r, x_{r+1}, \dots, x_{k+r}) \in A^\circ_{k+r} = A_{k+r} \cap F(\mathbb{R}^d - Q_r, k)$, then

$$\varphi_i(C, t) = q_i \quad \text{for } i = 1, \dots, r,$$

because $p(q_i) = q_i$. Thus, the restriction of φ on $A_{k+r} \cap F(\mathbb{R}^d - Q_r, k)$ defines a continuous deformation of $A^\circ_{k+r} = A_{k+r} \cap F(\mathbb{R}^d - Q_r, k)$ onto $F(\mathbb{R} - Q_r, k)$ inside $F(\mathbb{R}^d - Q_r, k)$. As in Remark 2.3, this yields a continuous motion planning algorithm on $A^\circ_{k+r} \times A^\circ_{k+r}$.

3.3. The sets A°_i. For a configuration $C = (x_1, \dots, x_{k+r}) \in A_i$, where $i \geq 2$, set

$$\epsilon(C) := \frac{1}{k + r} \min\{|p(x_r) - p(x_s)| : \ p(x_r) \neq p(x_s)\}.$$

Consider a configuration $C = (q_1, \dots, q_r, x_{r+1}, \dots, x_{k+r})$ with $C \in A_i \cap F(\mathbb{R}^d - Q_r, k)$ (here we note that $i \geq r \geq 2$). For $t \in [0, 1]$, define

$$D^i(C, t) = \begin{cases} (q_1, \dots, q_r, z_{r+1}(C, t), \dots, z_{k+r}(C, t)) & \text{if } r - 1 < i < k + r, \\ C & \text{if } i = k + r, \end{cases}$$

where $z_j(t) = x_j + t(j - 1)\epsilon(C)e_1$ for $j = r + 1, \dots, k + r$. This defines a continuous 'desingularisation' deformation $D^i : A^\circ_i \times [0, 1] \to F(\mathbb{R}^d - Q_r, k)$ of A°_i into A°_{k+r} inside $F(\mathbb{R}^d - Q_r, k)$ depicted in Figure 3.

Again, Remark 2.3 yields a continuous motion planning algorithm on any subset $A^\circ_i \times A^\circ_j$ for $i, j \in \{r, r + 1, \dots, k + r\}$.

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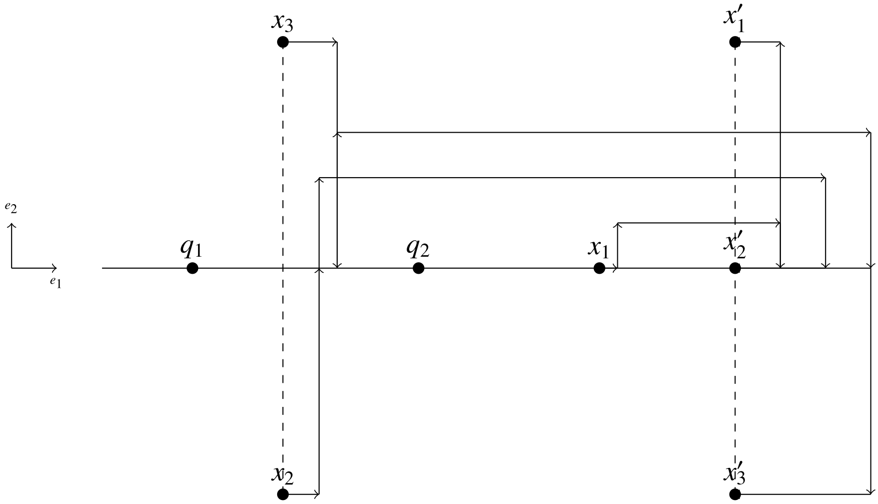

FIGURE 4. The motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$.

3.4. Combining regions of continuity. We have constructed continuous motion planning algorithms

$$\sigma_{i,j} \colon A_i^\circ \times A_j^\circ \to PF(\mathbb{R}^d - Q_r, k), \quad i, j = r, r+1, \ldots, k+r,$$

by applying iteratively the construction in Remark 2.3. For $i, j \in \{r, r+1, \ldots, k+r\}$, the sets $A_i^\circ \times A_j^\circ$ are pairwise-disjoint ENRs covering $F(\mathbb{R}^d - Q_r, k) \times F(\mathbb{R}^d - Q_r, k)$. The resulting estimate $\mathrm{TC}(F(\mathbb{R}^d - Q_r, k)) \leq (k+1)^2$ can be improved by noticing that the sets $A_i^\circ \times A_j^\circ$ can be repacked into $2k+1$ pairwise-disjoint ENRs each admitting its own continuous motion planning algorithm. Indeed, (3.1) implies that $A_i^\circ \times A_j^\circ$ and $A_{i'}^\circ \times A_{j'}^\circ$ are 'topologically disjoint' in the sense that $\overline{A_i^\circ \times A_j^\circ} \cap (A_{i'}^\circ \times A_{j'}^\circ) = \varnothing$, provided $i + j = i' + j'$ and $(i, j) \neq (i', j')$. Consequently, for $2r \leq \ell \leq 2k + 2r$, the motion planning algorithms $\sigma_{i,j}$ having $i + j = \ell$ determine a (well-defined) continuous motion planning algorithm on the ENR

$$W_\ell = \bigcup_{i+j=\ell} A_i^\circ \times A_j^\circ.$$

We have thus constructed a (global) tame motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$ having the $2k+1$ domains of continuity $W_{2r}, W_{2r+1}, \ldots, W_{2k+2r}$ (see Figure 4).

REMARK 3.1. We note that the algorithm given here is not a restriction from the algorithm given in [10].

4. A higher tame motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$

In this section we present an optimal tame nth sequential motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$, which generalises in a natural way the algorithm presented in the

https://doi.org/10.1017/S0004972720000167 Published online by Cambridge University Press

previous section. As indicated in the introduction, the algorithm has $nk+1$ regions of continuity, works for any $d, k, n \geq 2$ and is optimal. The algorithm we present in this section can be used in designing practical systems controlling sequential motion of many objects moving in Euclidean space without collisions and avoiding obstacles.

4.1. Section over $F(\mathbb{R} - Q_r, k)^n = F(\mathbb{R} - Q_r, k) \times \cdots \times F(\mathbb{R} - Q_r, k)$. Recall that we take the standard embedding $\mathbb{R} := \{(x, 0, \dots, 0) \in \mathbb{R}^d : x \in \mathbb{R}\}$, so that $F(\mathbb{R} - Q_r, k)$ is naturally a subspace of $F(\mathbb{R}^d - Q_r, k)$. Consequently, the motion planning algorithm $\Gamma : F(\mathbb{R} - Q_r, k) \times F(\mathbb{R} - Q_r, k) \to PF(\mathbb{R}^d - Q_r, k)$ given by (3.2) yields a continuous nth motion planning algorithm

$$\Gamma_n : F(\mathbb{R} - Q_r, k) \times \cdots \times F(\mathbb{R} - Q_r, k) \to PF(\mathbb{R}^d - Q_r, k)$$

given by concatenation of paths

$$\Gamma_n(C_1, \dots, C_n) = \Gamma(C_1, C_2) * \cdots * \Gamma(C_{n-1}, C_n). \tag{4.1}$$

4.2. Motion planning algorithms $\sigma_{j_1,\dots,j_n}$. We now go back to the notation introduced in the previous section, where, for $r \leq i \leq k+r$, we constructed ENRs A_i° covering $F(\mathbb{R}^d - Q_r, k)$, and concatenated homotopies $A_i^\circ \times [0,1] \to F(\mathbb{R}^d - Q_r, k)$ deforming A_i° into $F(\mathbb{R} - Q_r, k)$. Together with the motion planning algorithm Γ_n, by Remark 2.3, these deformations yield continuous nth motion planning algorithms

$$\sigma_{j_1,\dots,j_n} : A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ \to PF(\mathbb{R}^d - Q_r, k), \quad j_1, \dots, j_n = r, r+1, \dots, k+r.$$

Indeed, the desingularisation deformation $D^{j_1} \times \cdots \times D^{j_n}$ takes $A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ$ into $(A_{k+r}^\circ)^n$. Then we apply the deformation $\varphi \times \cdots \times \varphi$ (n times), which takes $(A_{k+r}^\circ)^n$ into $F(\mathbb{R} - Q_r, k)^n$, and finally we apply Remark 2.3. Let us emphasise that the above description of $\sigma_{j_1,\dots,j_n}$ is fully implementable.

4.3. Combining regions of continuity. For $j_1, \dots, j_n = r, r+1, \dots, k+r$, the ENRs $A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ$ are mutually disjoint and cover the whole product space $F(\mathbb{R}^d - Q_r, k)^n$. Proposition 2.5 gives the estimate $\mathrm{TC}_n(F(\mathbb{R}^d - Q_r, k)) \leq (k+1)^n$. The motion planning algorithms $\sigma_{j_1,\dots,j_n}$ can be improved by combining the domains of continuity to yield $nk+1$ covering ENRs W_ℓ, $\ell = nr, nr+1, \dots, nk+nr$, each admitting a continuous nth motion planning algorithm. Explicitly, let

$$W_\ell = \bigcup_{j_1+\cdots+j_n=\ell} A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ,$$

where $\ell = nr, nr+1, \dots, nk+nr$. By (3.1), any two distinct n-tuples $(j_1, \dots, j_n)$ and $(j'_1, \dots, j'_n)$ with $j_1 + \cdots + j_n = j'_1 + \cdots + j'_n$ determine topologically disjoint sets $A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ$ and $A_{j'_1}^\circ \times \cdots \times A_{j'_n}^\circ$ in $F(\mathbb{R}^d - Q_r, k)^n$, that is,

$$\overline{A_{j_1}^\circ \times \cdots \times A_{j_n}^\circ} \cap (A_{j'_1}^\circ \times \cdots \times A_{j'_n}^\circ) = \varnothing.$$

Therefore, the motion planning algorithms $\sigma_{j_1,\dots,j_n}$ with $j_1 + \cdots + j_n = \ell$ jointly define a continuous motion planning algorithm on W_ℓ. We have thus constructed a tame nth sequential motion planning algorithm in $F(\mathbb{R}^d - Q_r, k)$ having $nk+1$ domains of continuity $W_{nr}, W_{nr+1}, \dots, W_{nk+nr}$.

https://doi.org/10.1017/S0004972720000167 Published online by Cambridge University Press

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https://doi.org/10.1017/S0004972720000167 Published online by Cambridge University Press