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ON REAL PARTS OF POWERS OF COMPLEX PISOT NUMBERS

TOUFIK ZAÏMI

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Abstract

We prove that a nonreal algebraic number θ with modulus greater than 1 is a complex Pisot number if and only if there is a nonzero complex number λ such that the sequence of fractional parts $(\{\Re(\lambda\theta^n)\})_{n\in\mathbb{N}}$ has a finite number of limit points. Also, we characterise those complex Pisot numbers θ for which there is a convergent sequence of the form $(\{\Re(\lambda\theta^n)\})_{n\in\mathbb{N}}$ for some $\lambda \in \mathbb{C}^*$. These results are generalisations of the corresponding real ones, due to Pisot, Vijayaraghavan and Dubickas.

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1. Introduction

A Pisot number is a real algebraic integer greater than 1 whose other conjugates are of modulus less than 1, and a complex Pisot number is a nonreal algebraic integer with modulus greater than 1 whose other conjugates, except its complex conjugate, are of modulus less than 1 [2].

As usual we denote, respectively, by $\{\cdot\}$, $\|\cdot\|$, $\mathbb{S}$ and $\mathbb{S}_c$ the fractional part function, the usual distance of a real number to $\mathbb{Z}$, the set of Pisot numbers and the set of complex Pisot numbers. Throughout, when we speak about conjugates, the minimal polynomial, say M_θ, the trace and the degree of an algebraic number θ, without mentioning the basic field, this is meant over $\mathbb{Q}$.

It seems that the first two papers that discuss complex Pisot numbers explicitly are due to Kelly [15], generalising Salem's result [17] that $\mathbb{S}$ is closed, to show that $\mathbb{S} \cup \mathbb{S}_c$ is also closed, and to Chamfy [8], proving that 'the smallest' element of $\mathbb{S}_c$ has modulus $\sqrt{\theta_0} = 1.15\ldots$, where θ_0 is the smallest Pisot number [2]. In fact, the generalisation of Dufresnoy and Pisot's algorithm [12], proved in [8], has been used by Garth [14] to classify small elements of $\mathbb{S}_c$ and to deduce that there is no limit point in the set of complex Pisot numbers with modulus less than 1.17. Recall also

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that the related work of Cantor [7], concerning certain k-tuples of algebraic integers, contains a generalisation of Dufresnoy and Pisot's characterisation of the limit points of $\mathbb{S}$ [11], and the paper of Samet [18] is essentially concerned with complex Pisot numbers as limit points of complex Salem numbers. A complex Salem number is a nonreal algebraic integer with modulus greater than 1 whose other conjugates, except its complex conjugate, are of modulus at most 1, and having a conjugate with modulus 1. Recently, Bertin and the author [3] have shown that the family of complex Pisot numbers generating a nonreal number field is a complex Meyer set.

Another of Pisot's results about the distribution modulo one of powers of real numbers asserts that if the set, say $L = L(\theta, \lambda)$, of limit points of the sequence $(\{\lambda\theta^n\})_{n\in\mathbb{N}}$, where λ is a nonzero real number and θ is a real algebraic number greater than 1, has a finite number of limit points, then $\theta \in \mathbb{S}$ and $\lambda \in \mathbb{Q}(\theta)$ [16]. A weaker theorem, due to Vijayaraghavan [20], yields also that a real algebraic number $\theta > 1$ is a Pisot number when $\lim_{n\to\infty} \|\lambda\theta^n\| = 0$ for some $\lambda \in \mathbb{R}^*$. More details about these last two results may be found in [23]. Considering the inverse problem, Dubickas [10] obtained many results about the sets $L(\theta, \lambda)$ when $\theta \in \mathbb{S}$ and $\lambda \in \mathbb{Q}(\theta)$. The first one implies immediately that $L(\theta, \lambda)$ is a finite subset of $\mathbb{Q}$. Also, [10, Theorem 4] shows that, for any $\theta \in \mathbb{S}$, there is $\lambda \in \mathbb{R}^*$ such that $\mathrm{Card}(L(\theta, \lambda)) = 1$ if and only if $|M_\theta(1)| \geq 2$ or θ is a strong Pisot number. A Pisot number θ, with degree d, is said to be strong if $d = 1$, or if $d \geq 2$ and θ has a real positive conjugate which is greater than the absolute values of the $d - 2$ remaining conjugates of θ [4].

In the present paper, we are concerned with a generalisation of the above-mentioned results of Pisot, Vijayaraghavan and Dubickas to the complex case. By analogy with the real case, we denote by $L(\theta, \lambda)$ the set of limit points of the sequence $(\{\Re(\lambda\theta^n)\})_{n\in\mathbb{N}}$, where θ is a nonreal (complex) algebraic number and λ is a nonzero complex number. The first assertion in Theorem 1, below, is a characterisation of the elements of the set $\{(\theta, \lambda) \mid \theta \in \mathbb{S}_c, \lambda \in \mathbb{Q}(\theta) \text{ and } \lambda \neq 0\}$ among all pairs having a first coordinate which is a nonreal algebraic number with modulus greater than 1 and a nonzero complex second coordinate.

Theorem 1.1. *Let θ be a nonreal algebraic number with modulus greater than 1 and let λ be a nonzero complex number. Then the following equivalences hold.*

(i) *The set $L(\theta, \lambda)$ is finite $\Leftrightarrow$ $\theta \in \mathbb{S}_c$ and $\lambda \in \mathbb{Q}(\theta)$.*

(ii) $\lim_{n\to\infty} \|\Re(\lambda\theta^n)\| = 0 \Leftrightarrow \theta \in \mathbb{S}_c$ *and* $\lambda = 2\beta/\theta^N M'_\theta(\theta)$ *for some* $\beta \in \mathbb{Z}[\theta]$ *and* $N \in \mathbb{N} \cup \{0\}$.

Here, the notation $M'_\theta(\theta)$ means the usual derivative of the polynomial M_θ, evaluated at θ. The following result may be viewed as a nonreal version of the above-mentioned result of Dubickas.

Theorem 1.2. *Let θ be a nonreal algebraic number of modulus greater than 1. Then there is a nonzero complex number λ such that $L(\theta, \lambda)$ is a singleton if and only if θ is a complex Pisot number satisfying one of the following conditions:*

(i) *θ is quadratic;*

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(ii) *the degree of* θ *is at least* 3 *and* θ *has a real positive conjugate which is greater than the absolute values of the* $d-3$ *remaining conjugates of* θ *belonging to the unit disc;*
(iii) $M_\theta(1) \geq 2$*;*
(iv) *the degree of* θ *is at least* 4 *and* θ *has two real conjugates with the same modulus which is greater than the absolute values of the* $d-4$ *remaining conjugates of* θ*.*

It is worth noting that a related well-known characterisation of Pisot and Salem numbers (see for instance [6, Theorem 3.9]) asserts that a real algebraic number $\theta > 1$ is a Pisot number or a Salem number if and only for any $\varepsilon > 0$ there is $\lambda \in \mathbb{R}^*$ such that $\|\lambda\theta^n\| < \varepsilon$ for all $n \in \mathbb{N}$. A Salem number is a real algebraic integer greater than 1 whose other conjugates are of modulus at most 1, and having a conjugate with modulus 1. By the same arguments as in the proof of Theorem 1.1, we easily obtain the following analogous result.

THEOREM 1.3. *Let* θ *be a nonreal algebraic number of modulus greater than* 1*. Then* θ *is a complex Pisot number or a complex Salem number if and only if for any* $\varepsilon > 0$ *there is a nonzero complex number* λ *such that* $\|\Re(\lambda\theta^n)\| < \varepsilon$ *for all* $n \in \mathbb{N}$*.*

It is easy to see that the degree of a complex Salem number θ is at least 6, the conjugates of θ, other than θ, $\overline{\theta}$, $1/\theta$ and $1/\overline{\theta}$, have modulus 1 and the sequence $(\{\Re(\theta^n)\})_{n\in\mathbb{N}}$ is dense modulo one. To make clear the proof of the theorems, we prefer to describe, in the next section, the set $L(\theta, \lambda)$, where $\theta \in \mathbb{S}_c$, $\lambda \in \mathbb{Q}(\theta)$ and $\lambda \neq 0$. This allows us, in particular, to show the inverse implications in Theorem 1.1(i) and in Theorem 1.2. Theorem 1.3, and the remaining parts of the Theorems 1.1 and 1.2, are proved in Section 3. We shall use algebraic tools from [21, 23], instead of Fatou's lemma and the residue theorem (usually used in the real case; see for instance the proof of [2, Theorem 5.4.1] or the proof of [6, Lemma 2.2]), to show the direct implications in Theorems 1.1 and 1.3. All computations are done using PARI [1].

From now on, suppose that θ is a nonreal algebraic number with modulus greater than 1, $\sigma_1, \sigma_2, \ldots, \sigma_d$ are the distinct embeddings of $\mathbb{Q}(\theta)$ into $\mathbb{C}$, $\theta_1 := \sigma_1(\theta) = \theta$, $\theta_2 := \sigma_2(\theta) = \overline{\theta}, \ldots, \theta_d := \sigma_d(\theta)$ and $|\theta_1| = |\theta_2| > 1 \geq |\theta_3| \geq |\theta_4| \geq \cdots \geq |\theta_d|$ when θ is a complex Pisot number or a complex Salem number. We denote the minimal polynomial of θ by $M_\theta(x) := a_d x^d + a_{d-1}x^{d-1} + \cdots + a_0 \in \mathbb{Z}[x]$.

2. On certain sets $L(\theta, \lambda)$

Here we assume that $\theta \in \mathbb{S}_c$, $\lambda \in \mathbb{Q}(\theta)$ and $\lambda \neq 0$. To describe the corresponding set $L(\theta, \lambda)$, we shall follow the same scheme as in [10], with minor modifications.

Let $B = B(\theta, \lambda)$, $\in \mathbb{N}$ and $N \in \mathbb{N}$ be such that

$$t_n := \mathrm{Trace}(B\lambda\theta^n) \in \mathbb{Z}$$

for all $n \in \mathbb{N} \cap [N, \infty)$. Since $a_d\theta^d + a_{d-1}\theta^{d-1} + \cdots + a_0 = 0$, we also have $a_dB\lambda\theta^{n+d} + a_{d-1}B\lambda\theta^{n+d-1} + \cdots + a_0B\lambda\theta^n = 0$ and so

$$a_d t_{n+d} + a_{d-1}t_{n+d-1} + \cdots + a_0 t_n = 0 \quad \text{for all } n \geq N.$$

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Thus, $T := (t_n)_{n\geq N}$ is a linear recurrence sequence with companion polynomial M_θ. Moreover, as $\mathrm{Trace}(B\lambda\theta^n) = \sum_{j=1}^{d} \sigma_j(B\lambda\theta^n)$, we see that

$$\{\Re(2\lambda\theta^n)\} = \frac{t_n}{B} - [\Re(2\lambda\theta^n)] - \sum_{j=3}^{d} \theta_j^n \sigma_j(\lambda), \tag{2.1}$$

where $[\cdot]$ is the integer part function. Since the sequence $(t_n \pmod B)_{n\geq N}$ is eventually periodic (see for instance [9] or [22]), there are two rational integers $p \geq 1$ and $q \geq 0$ such that $t_{n+p} \equiv t_n \pmod B$ for all $n \geq \max(N, q)$. Setting $\overline{b_1 \ldots b_p}$ for the period of $(t_n \pmod B)_{n\geq N}$, we deduce that the set $R := \{r_1, \ldots, r_s\}$ of distinct terms of the sequence $(t_n \pmod B)_{n\geq N}$, occurring infinitely often, satisfies $R = \{b_1, \ldots, b_p\} \subseteq \{0, 1, \ldots, B-1\}$ and, using (2.1), we easily obtain the following assertion.

Proposition 2.1. *With the notation above,*

$$L(\theta, 2\lambda) \subseteq \{r_1/B, \ldots, r_s/B, 1\} \subseteq \{0, 1/B, \ldots, (B-1)/B, 1\}.$$

Moreover, if $0 \notin R$, *then* $L(\theta, 2\lambda) = \{r_1/B, \ldots, r_s/B\} \subseteq \{1/B, \ldots, (B-1)/B\}$.

Consider, for example, the polynomial $M_\theta(x) = x^3 + x^2 - 1$. Then M_θ is the reciprocal polynomial of the minimal polynomial of the smallest Pisot number, and θ satisfies condition (ii) of Theorem 1.2. Thus, by Proposition 2.2 below, $L(\theta, 2) = \{1\}$. If we set, for instance, $\lambda := 1/2$ and $B := 2$, then the sequence $(t_n \pmod 2)_{n\geq 0}$ is purely periodic with period $\overline{1110100}$, $R = \{0, 1\}$ and $L(\theta, 1) \subseteq \{0, 1/2, 1\}$ and so $L(\theta, 1) = \{1/2, 1\}$, since the conjugate of θ with modulus less than 1, namely $1/\theta_0$, is positive. A similar computation gives $L(\theta, 2/3) = \{1/3, 2/3, 1\}$, $L(\theta, 1/2) = \{1/4, 1/2, 3/4, 1\}$ and $L(\theta, 2/5) = \{1/5, 2/5, 3/5, 4/5, 1\}$. To obtain more information about the cardinality of the sets $L(\theta, \lambda)$, we may prove, similarly as in [25], a correspondence between the elements λ of the field $\mathbb{Q}(\theta)$ and the linear recurrence sequences with rational integer terms and companion polynomial M_θ, but this is far from our main objective. The result below describes some sequences of the form $(\{\Re(\lambda\theta^n)\})_{n\in\mathbb{N}}$, having only one limit point.

Proposition 2.2. *Suppose that the complex Pisot number* θ *satisfies condition (i) (respectively (ii), (iii), (iv)) in Theorem 1.2. Then* $L(\theta, 2) = \{0\}$ *(respectively* $L(\theta, 2) = \{1\}$, $L(\theta, 2/(1-\theta)M'_\theta(\theta)) = \{1/M_\theta(1)\}$, $L(\theta, 2+2\theta) = \{1\}$*).*

Proof. When $d = 2$, it is clear that $\{\Re(2\theta^n)\} = 0$ for $n \geq 1$, so $L(\theta, 2) = \{0\}$. If $d \geq 3$, $\theta_3 > |\theta_4|$, $\lambda := 1$ and $B(\theta, 1) := 1$, then the relation (2.1) yields

$$\{\Re(2\theta^n)\} + \theta_3^n\Big(1 + \sum_{j=4}^{d} (\theta_j/\theta_3)^n\Big) = s_n \in \mathbb{Z}$$

for all $n \geq 1$. Hence, $s_n = 1$ for n sufficiently large and so $\lim_{n\to\infty}\{\Re(2\theta^n)\} = 1$.

Similarly, if $d \geq 3$, $\theta_3 \in \mathbb{R}$, $\theta_4 \in \mathbb{R}$, $|\theta_3| = |\theta_4| > |\theta_5|$, $\lambda := 1 + \theta$ and $B(\theta, 1+\theta) := 1$, then $\theta_4 = -\theta_3$ and $\sum_{j=3}^{d} \theta_j^n \sigma_j(\lambda)$ has the same sign as $2\theta_3^n = (1 + \theta_3 + 1 + \theta_4)\theta_3^n$

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(respectively as $2\theta_3^{n+1} = (1 + \theta_3 - 1 - \theta_4)\theta_3^n$) when n is a sufficiently large even (respectively large odd) rational integer. So, by (2.1), $\{\Re(2\lambda\theta^n)\} + \sum_{j=3}^{d} \theta_j^n \sigma_j(\lambda) = 1$. Thus, $\lim_{n\to\infty}\{\Re(2\lambda\theta^n)\} = 1$ and $L(\theta, 2 + 2\theta) = \{1\}$.

Finally, suppose that $M_\theta(1) \geq 2$, and set $\lambda := 1/(1-\theta)M'_\theta(\theta)$ and $B := cM_\theta(1)$ for some $c \in \mathbb{N}$ in a way that makes $t_n \in \mathbb{Z}$ for all $n \in \mathbb{N}$ (for instance, choose c so that $c/(1-\theta)M'(\theta) \in \mathbb{Z}[\theta]$). Then an easy calculation shows that $t_n = c$ for all $n \leq d$ (see also [21, Lemma 4]) and so $t_n \equiv c \pmod{B}$ for all $n \in \mathbb{N}$, since $(t_n)_{n\in\mathbb{N}}$ is a linear recurrence sequence with companion polynomial M_θ. It follows that $p = s = 1$, $R = \{c\}$ and the second assertion in Proposition 2.1 gives immediately $L(\theta, 2\lambda) = \{c/B\} = \{1/M_\theta(1)\}$. □

3. Proofs of the theorems

Proof of Theorem 1.1. (i) Suppose that the sequence $(\{\Re(\lambda\theta^n)\})_{n\in\mathbb{N}}$ has a finite number of limit points, say $l_1, \ldots, l_r$, and let ε be a positive real number satisfying the inequality $\varepsilon < 1/\sum_{0\leq j\leq d} |a_j|$ and another relation, stated later in (3.6).

Similarly to the proof of [6, Theorem 2.5], Kronecker's approximation theorem shows there is a $b \in \mathbb{N}$ satisfying $\|bl_j\| < \varepsilon/2$ for all $j \in \{1, \ldots, r\}$, and so there exists $N_1 \in \mathbb{N}$ such that $\|\Re(b\lambda\theta^n)\| < \varepsilon$ for all $n \geq N_1$. From now on, assume that $n \geq N_1$. Writing

$$\Re(b\lambda\theta^n) = k_n + \varepsilon_n, \tag{3.1}$$

where $k_n \in \mathbb{Z}$ and $|\varepsilon_n| < \varepsilon$, we have, from the equations $b\lambda\theta^n \sum_{0\leq j\leq d} a_j\theta^j = 0$ and $\overline{b\lambda\theta^n} \sum_{0\leq j\leq d} a_j\overline{\theta^j} = 0$, that $\sum_{0\leq j\leq d} a_j \Re(b\lambda\theta^{n+j}) = 0$ and $\sum_{0\leq j\leq d} a_j k_{n+j} = -\sum_{0\leq j\leq d} a_j\varepsilon_{n+j}$, and so $|\sum_{0\leq j\leq d} a_j k_{n+j}| \leq \varepsilon \sum_{0\leq j\leq d} |a_j| < 1$. Hence,

$$a_0k_{n+j} + a_1k_{n+1} + \cdots + a_dk_{n+d} = 0,$$
$$a_0\varepsilon_{n+j} + a_1\varepsilon_{n+1} + \cdots + a_d\varepsilon_{n+d} = 0$$

and consequently there are complex numbers $\zeta_1, \zeta_2, \ldots, \zeta_d$ such that

$$\varepsilon_n = \zeta_1\theta_1^n + \zeta_2\theta_2^n + \cdots + \zeta_d\theta_d^n. \tag{3.2}$$

By considering the dual base $\{\gamma_k := (\sum_{j=1+k}^{d} a_j\theta^{j-1-k})/a_dM'_\theta(\theta) \mid k \in \{0, \ldots, d-1\}\}$ of the base $\{1, \theta, \ldots, \theta^{d-1}\}$ of $\mathbb{Q}(\theta)$, which satisfies the matrix equation

$$[\sigma_j(\gamma_{l-1})]_{\substack{1\leq j\leq d\\ 1\leq l\leq d}} [\theta_l^{j-1}]_{\substack{1\leq j\leq d\\ 1\leq l\leq d}} = I_d, \tag{3.3}$$

where I_d is the identity matrix, and j and l denote, respectively, the row and column numbers (for more details, see [23, Lemma 1]), we have the following relation, deduced from (3.2),

$$[\theta_l^{j-1}]_{\substack{1\leq j\leq d\\ 1\leq l\leq d}} \begin{bmatrix} \zeta_1\theta_1^n \\ \vdots \\ \zeta_d\theta_d^n \end{bmatrix} = \begin{bmatrix} \varepsilon_n \\ \vdots \\ \varepsilon_{n+d-1} \end{bmatrix}$$

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and then $\zeta_j\theta_j^n = \sigma_j(\gamma_0)\varepsilon_n + \sigma_j(\gamma_1)\varepsilon_{n+1} + \cdots + \sigma_j(\gamma_{d-1})\varepsilon_{n+d-1}$ for all $j \in \{1, \ldots, d\}$, and $|\zeta_j| \leq \varepsilon d(\max_{(j,l)\in\{1,\ldots,d\}^2} |\sigma_j(\gamma_{l-1})|)/|\theta_j|^n$. Also, we may also suppose, without loss of generality, that the first t conjugates of θ have modulus greater than 1 and the remaining ones belong to the closed unit disc. Then the last inequality gives immediately that $\zeta_1 = \zeta_2 = \cdots = \zeta_t = 0$. Furthermore, we have, by (3.1),

$$(b/2)\lambda\theta^n + (b/2)\overline{\lambda\theta^n} - \zeta_3\theta_3^n - \cdots - \zeta_d\theta_d^n = k_n,$$

$$[\theta_l^{j-1}]_{\substack{1\leq j\leq d\\ 1\leq l\leq d}} \begin{bmatrix} b\lambda\theta^n/2 \\ b\overline{\lambda\theta^n}/2 \\ -\zeta_3\theta_3^n \\ \vdots \\ -\zeta_d\theta_d^n \end{bmatrix} = \begin{bmatrix} k_n \\ k_{n+1} \\ k_{n+2} \\ \vdots \\ k_{n+d-1} \end{bmatrix}$$

and, using (3.3),

$$\lambda = \frac{2(\gamma_0 k_n + \gamma_1 k_{n+1} + \cdots + \gamma_{d-1}k_{n+d-1})}{b\theta^n} \in \mathbb{Q}(\theta) \tag{3.4}$$

and $\zeta_j = -(\sigma_j(\gamma_0)k_n + \sigma_j(\gamma_1)k_{n+1} + \cdots + \sigma_j(\gamma_{d-1})k_{n+d-1})/\theta_j^n = -\sigma_j(b\lambda/2)$ for all $j \in \{3, \ldots, d\}$. Thus, the other conjugates of the algebraic number λ are $\lambda_2 := \overline{\lambda}$ and $\lambda_3 := -2\zeta_3/b, \ldots, \lambda_d := -2\zeta_d/b$ and so $t = 2$, since otherwise $\zeta_3 = 0$, implying that $\lambda = 0$.

If a designates a nonzero rational integer that makes the numbers $2a\gamma_0/b\lambda$ and $2a\gamma_1/b\lambda, \ldots, 2a\gamma_{d-1}/b\lambda$ algebraic integers, we see from (3.4) that $a\theta^n$ is an algebraic integer for all $n \geq N_1$. It follows, from [23, Lemma 4], that θ is an algebraic integer and consequently θ is a complex Pisot number or a complex Salem number.

To conclude the proof, assume, on the contrary, that θ has a conjugate, say θ_d, with modulus 1, and consider, similarly to the argument in [6, Lemma 2.2], the polynomial $Q(x) := (x - \theta_1)(x - \theta_2)\cdots(x - \theta_{d-1}) = \sum_{j=0}^{d-1} \delta_j x^j$ and the quantity q defined by $q := \delta_{d-1}\varepsilon_{n+d-1} + \delta_{d-2}\varepsilon_{n+d-2} + \cdots + \delta_1\varepsilon_{n-1} + \delta_0\varepsilon_n$ which satisfies

$$|q| < \varepsilon \sum_{j=0}^{d-1} |\delta_j| < \varepsilon 2^{d-1}|\theta|^2. \tag{3.5}$$

From (3.2), $q = \sum_{j=0}^{d-1} \delta_j(\zeta_1\theta_1^{n+j} + \cdots + \zeta_{d-1}\theta_{d-1}^{n+j} + \zeta_d\theta_d^{n+j}) = \sum_{k=0}^{d} \zeta_k\theta_k^n Q(\theta_k)$ and so $q = \zeta_d\theta_d^n Q(\theta_d) = \zeta_d\theta_d^n M'_\theta(\theta_d)$, since $Q(\theta_1) = \cdots = Q(\theta_{d-1}) = 0$. Hence, $|q| = |\zeta_d M'_d(\theta_d)| = b|\lambda_d M'_d(\theta_d)|/2$ and it follows, by (3.5), that $b|\lambda_d M'_\theta(\theta_d)|/2^d|\theta|^2 < \varepsilon$. Letting

$$\varepsilon \leq \frac{\min_{1\leq j\leq d} |\sigma_j(\lambda M'_\theta(\theta))|}{2^d|\theta|^2}, \tag{3.6}$$

we immediately obtain a contradiction. Thus, θ is a complex Pisot number, and this completes the proof of Theorem 1.1(i), since Proposition 2.1 asserts that $L(\theta, \lambda) = L(\theta, 2(\lambda/2))$ is finite when $\theta \in \mathbb{S}_c$ and $\lambda \in \mathbb{Q}(\theta)$.

(ii) The condition $\lim_{n\to\infty} \|\Re(\lambda\theta^n)\| = 0$ implies that $L(\theta,\lambda) \subset \{0,1\}$. So, from Theorem 1.1(i), $\theta \in \mathbb{S}_c$ and, if we set $b = 1$ in the proof of Theorem 1.1(i), then (3.4) yields $\lambda \in 2\mathbb{Z}[\theta]/\theta^N M'_\theta(\theta)$. Conversely, suppose that $\theta \in \mathbb{S}_c$ and $\lambda \in 2\mathbb{Z}[\theta]/(\theta^{N_2} M'_\theta(\theta))$ for some $N_2 \in \mathbb{N}$. Then $(\lambda/2)\theta^{N_2} \in \mathbb{Z}[\theta]/M'_\theta(\theta)$ and so we have, from [23, Lemma 2], that $\mathrm{Trace}((\lambda/2)\theta^{N_2}\theta^n) \in \mathbb{Z}$ for all $n \in \mathbb{N} \cup \{0\}$. Thus, $\mathrm{Trace}((\lambda/2)\theta^n) \in \mathbb{Z}$ for all $n \geq N_2$, and the result follows immediately, by Proposition 2.1, with $B(\theta,\lambda/2) := 1$, since $L(\theta,\lambda) = L(\theta, 2(\lambda/2)) \subset \{0,1\}$, implying that $\lim_{n\to\infty} \|\Re(\lambda\theta^n)\| = 0$. □

Proof of Theorem 1.2. As signalled in the introduction, the inverse implication in Theorem 1.2 follows from Proposition 2.2. To prove the direct implication, consider a nonreal algebraic number θ with modulus greater than 1, and a nonzero complex number λ' such that the corresponding set $L(\theta,\lambda')$ contains a unique element, say l. Clearly, Theorem 1.1(i) gives $\theta \in \mathbb{S}_c$ and $\lambda := \lambda'/2 \in \mathbb{Q}(\theta)$ and so we have, with the notation of Section 2, that $s = 1$ and $l = r_1/B$ (respectively $l \in \{0,1\}$) when $r_1 \neq 0$ (respectively $r_1 = 0$).

Now assume, on the contrary, that θ does not satisfy any of the conditions (i)–(iv) in Theorem 1.2. Then $d \geq 3$, and the case $r_1 \neq 0$ cannot hold, since otherwise the equations $t_n \pmod B = r_1$, where $n \geq N$, yield $r_1 M_\theta(1) \equiv 0 \pmod B$ and so $M_\theta(1) \geq 2$ (recall that $(t_n)_{n\in\mathbb{N}}$ is a linear recurrence sequence with companion polynomial M_θ and $M_\theta(1) \geq 1$ for all $\theta \in \mathbb{S}_c$). Thus, $r_1 = 0$ and (2.1) gives $\{\Re(2\lambda\theta^n)\} + R_n \in \mathbb{Z}$, where $R_n = \sum_{j=3}^d \theta_j^n \sigma_j(\lambda)$. Hence, there is an $N_1 \in \mathbb{N}$ such that $\{Re(\lambda'\theta^n)\} + R_n = 0$ (respectively $\{\Re(\lambda'\theta^n)\} + R_n = 1$) for all $n \geq \max\{N,\ N_1\}$ when $l = 0$ (respectively $l = 1$).

To obtain a contradiction, we shall prove that the sequence $(R_n)_{n\in N}$ contains infinitely many positive terms and infinitely many negative terms. This happens, in particular, when $|\theta_3| > |\theta_4|$, as in this case θ_3 is a negative real number. Recall also by [24, Theorem 2], generalising a result of Smyth [19], that θ has at most four conjugates with the same modulus, and so to conclude we shall consider the following three cases.

Case 1: $|\theta_3| = |\theta_4| > |\theta_5|$. Then $\overline{\theta_3} = \theta_4$. Setting $\theta_3 := \rho e^{i\pi t}$ and $\sigma_3(\lambda) := \eta e^{i\pi s}$, where $(t,s,\rho) \in [0,1]^3$, $\eta > 0$ and $i^2 = -1$, we see that the sign of R_n is that of $\cos(s+nt)\pi$ for n sufficiently large. Hence, R_n will be both positive and negative for infinitely many n when $t \notin \mathbb{Q}$ (see also the proof of [10, Theorem 2]). Now suppose that t is rational. Then there is $v \in \mathbb{N}$ such that $\zeta := e^{i\pi t}$ is a primitive vth root of unity. Because the vth roots of unity, namely $\zeta, \zeta^2, \ldots, \zeta^v = 1$, are uniformly distributed on the unit circle, then so are the numbers $e^{i\pi(tk+s)}$ when k runs through the set $\{1,\ldots,v\}$. Thus, there is $(a,b) \in \{1,\ldots,v\}^2$ such that $\Re(e^{i\pi(ta+s)})\Re(e^{i\pi(tb+s)}) < 0$, and a simple calculation shows that $R_{nv+a}R_{nv+b} < 0$ when n is sufficiently large, that is, R_n is both positive and negative for infinitely many n.

Case 2: $|\theta_3| = |\theta_4| = |\theta_5| > |\theta_6|$. Then one of the three conjugates θ_3, θ_4 and θ_5 of θ is real and so, from a theorem of Ferguson [13], generalising a result of Boyd [5], there is a $P \in \mathbb{Z}[x]$ such that $M_\theta(x) = P(x^3)$. Thus, θ has three distinct conjugates, namely θ, $j\theta$ and $\bar{j}\theta$, where $j^2 + j + 1 = 0$, with modulus greater than 1.

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Case 3: $|\theta_3| = |\theta_4| = |\theta_5| = |\theta_6|$. By the above-mentioned result of Ferguson, we see that none of the numbers θ_3, θ_4, θ_5 and θ_6 is real, since otherwise $M_\theta(x) = P(x^4)$ for some $P \in \mathbb{Z}[x]$, and so θ has more than two conjugates with modulus greater than 1. Setting, for instance, $\overline{\theta_3} = \theta_4$, $\overline{\theta_5} = \theta_6$, $\theta_3 := \rho e^{i\pi t}$, $\sigma_3(\lambda) := \eta e^{i\pi s}$, $\theta_5 := \rho' e^{i\pi t'}$ and $\sigma_5(\lambda) := \eta' e^{i\pi s'}$, where $(t, t', s, s', \rho, \rho') \in [0, 1]^6$ and $(\eta, \eta') \in (0, \infty)^2$, we see that the sign of R_n is that of $\eta \cos(s + nt)\pi + \eta' \cos(s' + nt')\pi$ for n sufficiently large. By the same argument as in the first case, we see that R_n is both positive and negative for infinitely many n if the numbers t and t' are $\mathbb{Q}$-linearly dependent and, using again Kronecker's approximation theorem, we easily see that the quantities $\cos(s + nt)\pi$ and $\cos(s' + nt')$ may be both positive and both negative for infinitely many n when t and t' are $\mathbb{Q}$-linearly independent. □

Proof of Theorem 1.3. Let θ be a complex Pisot number or a complex Salem number. Then, by the result of [3] mentioned above, there is a complex Pisot number, say α, generating the field $\mathbb{Q}(\theta)$. Let $\varepsilon > 0$ and let $N_1 \in \mathbb{N}$ be such that $\sum_{3\le j\le d} |\alpha_j|^{N_1} < \varepsilon$. Then, for each $n \in \mathbb{N}$, we have $\sum_{3\le j\le d} |\alpha_j^{N_1}\theta_j^n| < \varepsilon$ and $|\mathrm{Trace}(\alpha\theta^n) - \Re(2\alpha^{N_1}\theta^n)| < \varepsilon$ and so $\|\Re(\lambda\theta^n)\| < \varepsilon$ for all $n \in \mathbb{N}$, where $\lambda = 2\alpha^{N_1}$.

To prove the converse, fix a positive real number ε less than $\sum_{0\le j\le d} |a_j|$ and set $\Re(\lambda\theta^n) = k_n + \varepsilon_n$, where $k_n \in \mathbb{Z}$ and $|\varepsilon_n| < \varepsilon$. Then, in the same way as at the beginning of the proof of Theorem 1.1(i), we find an $N \in \mathbb{N}$ such that the sequences $(k_n)_{n\ge N}$ and $(\varepsilon_n)_{n\ge N}$ are linear recurrence sequences with companion polynomial M_θ and conclude that θ is a complex Pisot number or a complex Salem number. □

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TOUFIK ZAÏMI, Department of Mathematics and Statistics, College of Science,
Al Imam Mohammad Ibn Saud Islamic University,
PO Box 90950,
Riyadh 11623, Saudi Arabia
e-mail: tmzaemi@imamu.edu.sa

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