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A Note on Class Field Theory for Two-Dimensional Local Rings

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Abstract. The class field theory for the fraction field of a two-dimensional complete normal local ring with finite residue field is established by S. Saito. In this paper, we investigate the index of the norm group in the K_2-idele class group for a finite Abelian extension of such fields and deduce that the existence theorem does not hold for almost fields in this case.

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Key words. class field theory, two-dimensional local ring, index of norm group, existence theorem, two-dimensional Abelian covering singularity.

1. Introduction

For a field K, the class field theory for K is the theory which describes the Galois group $G_K := \mathrm{Gal}(K^{\mathrm{ab}}/K)$ of the maximal Abelian extension K^{ab} over K in terms of a certain group C_K endowed with a topology which is defined by using K-groups of fields related to K. One of the main purposes of class field theory is to define the group C_K as above and a continuous homomorphism $\rho_K : C_K \longrightarrow G_K$, which is 'almost' isomorphic.

The class field theory has been proven for certain fields which have an 'arithmetic' origin. In fact, the following theorem is known by works of many people including Bloch, Parshin, Fesenko, K. Kato and S. Saito. (For precise statements, see [R] and the papers listed in the references of [R], cf. Section 2.)

THEOREM 1.1 (Kato, Parshin, Fesenko, Kato and Saito). *Let K be an n-dimensional local field ($n \geqslant 0$) or a field which is finitely generated over its prime field. Then, there exists a certain group C_K endowed with a topology which is defined by using K-groups of fields related to K and a continuous homomorphism $\rho_K : C_K \longrightarrow G_K$ such that the homomorphism $\rho_K^* : G_K^* \longrightarrow C_K^*$ induced by ρ_K is an isomorphism. Here, for a group endowed with a topology T, we denoted the group of continuous homomorphisms $T \longrightarrow \mathbb{Q}/\mathbb{Z}$ of finite order by T^*.*

Remark 1.2. (1) When K is a local field, one can take the multiplicative group $K^\times$ as the group C_K, and when K is a number field, one can take the idele class group of K as the group C_K.

(2) In the case of a two-dimensional local field, one can take the Milnor K-group $K_2^M(K)$ (endowed with the topology of Kato, Parshin and Fesenko) as the group C_K, as we review in Section 2. In the case of an n-dimensional local field ($n \geqslant 3$), one can take the topological Milnor K-group $K_n^{\mathrm{top}}(K)$ (endowed with the topology of Parshin and Fesenko) as the group C_K. In the case $n \geqslant 3$, this group is not a topological group in general, that is, the multiplication is not necessarily continuous (although it is sequentially continuous). This is why we called C_K as 'a group endowed with a topology'.

Let K be one of the fields in the above theorem and for a finite Abelian extension L over K, define the subgroup D_L of C_K by

$$D_L := \mathrm{Ker}(C_K \xrightarrow{\rho_K} G_K \longrightarrow \mathrm{Gal}(L/K)).$$

Then, by the above theorem, the correspondence

$$\Phi_K : \left\{ \begin{array}{c} \text{finite Abelian} \\ \text{extensions of } K \end{array} \right\} \longrightarrow \left\{ \begin{array}{c} \text{finite index open} \\ \text{subgroups of } C_K \end{array} \right\},$$

which is defined by $\Phi_K(L) := D_L$, gives a bijection.

For a finite Abelian extension L over K, one can define C_L and $\rho_L : C_L \longrightarrow G_L$ in the same way as C_K and ρ_K. Moreover, we have the following theorem:

THEOREM 1.3 (Kato, Parshin, Fesenko, Kato and Saito). *With the above notation, there exists a homomorphism $N : C_L \longrightarrow C_K$ which is induced by the norm homomorphisms of K-groups such that $NC_L \subset C_K$ is open and the following diagram is commutative*:

$$\begin{array}{ccc} C_L & \xrightarrow{\rho_L} & G_L \\ {\scriptstyle N}\downarrow & & \downarrow \\ C_K & \xrightarrow{\rho_K} & G_K, \end{array}$$

where the right vertical arrow is induced by the inclusion $K \subset L$.

Assume that one already knows Theorems 1.1 and 1.3. Then one can immediately obtain the following theorem, which is known as the existence theorem:

THEOREM 1.4. (Kato, Parshin, Fesenko, Kato and Saito). *Let K be as in Theorem 1.1 and L be a finite Abelian extension over K. Then we have $NC_L = D_L$. In particular,* *the correspondence*

$$\Psi_K : \begin{Bmatrix} \text{finite Abelian} \\ \text{extensions of } K \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{finite index open} \\ \text{subgroups of } C_K \end{Bmatrix},$$

which is defined by $\Psi_K(L) := NC_L$, *is well-defined and it gives a bijection.*

Now let us recall the class field theory for the fraction field K of a two-dimensional complete normal local ring A with finite residue field, which is proven by S. Saito. (For precise statements, see Section 2.) Let P be the closed point of $\operatorname{Spec} A$ and let $f: X \longrightarrow \operatorname{Spec} A$ be a proper birational morphism such that X is regular and $Y := f^{-1}(P)_{\mathrm{red}}$ is a simple normal crossing divisor. (In this paper, we will call a morphism $X \longrightarrow \operatorname{Spec} A$ satisfying this condition as *a resolution of* A.) Let Γ be the dual graph of Y and define the rank $r(K)$ of K by $r(K) := \operatorname{rk} H_1(\Gamma, \mathbb{Z})$.

Then we can describe the class field theory for K as follows:

THEOREM 1.5 (Saito). *Let the notations be as above. Then there exists a certain topological group* C_K *defined by using* K*-groups of fields related to* K *and a continuous homomorphism* $\rho_K: C_K \longrightarrow G_K$ *such that the homomorphism* $\rho_K^* : G_K^* \longrightarrow C_K^*$ *induced by* ρ_K *is surjective and* $\operatorname{Ker}(\rho_K^*) = \operatorname{Gal}(K^{\mathrm{cs}}/K)^* \cong (\mathbb{Q}/\mathbb{Z})^{r(K)}$ *holds. Here* K^{cs} *is the maximal cs extension of* K *(for definition, see Definition 2.3).*

Let the situation be as above and for a finite Abelian extension L of K, define the subgroup D_L by

$$D_L := \operatorname{Ker}(C_K \xrightarrow{\rho_K} G_K \longrightarrow \operatorname{Gal}(L/K)),$$

as before. Then, by the above theorem, the correspondence

$$\Phi_K : \begin{Bmatrix} \text{finite Abelian} \\ \text{extensions of } K \end{Bmatrix} \longrightarrow \begin{Bmatrix} \text{finite index open} \\ \text{subgroups of } C_K \end{Bmatrix}$$

defined by $\Phi_K(L) := D_L$ is surjective, and it is bijective if and only if $r(K) = 0$ holds.

On the other hand, for a finite Abelian extension L over K, one can define the norm homomorphism $N : C_L \longrightarrow C_K$ such that $NC_L \subset C_K$ is open and the diagram

$$\begin{array}{ccc} C_L & \xrightarrow{\rho_L} & G_L \\ {\scriptstyle N}\downarrow & & \downarrow \\ C_K & \xrightarrow{\rho_K} & G_K \end{array}$$

is commutative (see Section 2). If the ranks of K and L are equal to zero, one can see easily, as in the previous case, that $NC_L = D_L$ holds. (This fact is remarked by Koya [Ko].) But in general case, the situation is not so easy. Hence one may propose the following problem:

PROBLEM 1.6. *Let K be the fraction field of a two-dimensional complete normal local ring A with finite residue field. Then,*

(1) *For a finite Abelian extension $L \supset K$, is the index of the group NC_L in C_K finite? If it is finite, how can we bound this index?*
(2) *Assume the former part of the problem 1. is true. Then, is the correspondence*

$$\Psi_K : \left\{ \begin{array}{l} \text{finite Abelian} \\ \text{extensions of } K \end{array} \right\} \longrightarrow \left\{ \begin{array}{l} \text{finite index open} \\ \text{subgroups of } C_K \end{array} \right\},$$

which is defined by $\Psi_K(L) := NC_L$ surjective?
(3) *For a finite Abelian extension $L \supset K$, does the equality $NC_L = D_L$ hold?*

Remark 1.7. One can easily see the following: (3) implies (1), and if (3) is true for any L, then (2) is also true, since we have $\Psi_K = \Phi_K$. Moreover, as we will see later (Proposition 2.7), if (2) is true, then (3) is true for any L.

The purpose of this paper is to give an answer to the above problems.

Let K be as above, let L be a finite Abelian extension of K and put $d(L/K) := [L : K]$. Then, from the viewpoint of the class field theory for K, the following three quantities seem to be important:

$$\begin{aligned} d'(L/K) &:= [C_K : D_L] = [L : L \cap K^{\mathrm{cs}}], \\ r(L/K) &:= r(L) - r(K), \\ c(L/K) &:= [C_K : NC_L]. \end{aligned}$$

As for the relation between the above quantities, we obtained the following theorem, which is the first main result in this paper:

THEOREM 1.8. *Let l be a prime number. Let K be the fraction field of a two-dimensional complete normal local ring with finite residue field, and let L be a cyclic extension of K of degree l^n ($n \in \mathbb{N}$). For $0 \leqslant i \leqslant n$, let K_i be the intermediate field of K and L such that $[K_i : K] = l^i$ holds. Let m be the integer such that $L \cap K^{\mathrm{cs}} = K_m$ holds, and define r_i ($1 \leqslant i \leqslant n$) by $r_i := r(K_i) - r(K_{i-1})$. (So, in the above notation, we have*

$$d(L/K) = l^n, \qquad d'(L/K) = l^{n-m}, \qquad r(L/K) = \sum_{i=1}^{n} r_i.)$$

Then we have the following:

(1) *For any i, $r_i \geqslant 0$* holds.
(2) *r_i is divisible by $l^{i-1}(l-1)$. (Hence $r(L/K)$ is nonnegative and divisible by $l - 1$.)*

(3) *$NC_L \subset C_K$ is an open subgroup and the following inequality holds*:

$$c(L/K) \leqslant l^{n-m+\sum_{i=m+1}^{n} \frac{r_i}{l^{i-1}(l-1)}}.$$

(*In particular, the inequality*

$$c(L/K) \leqslant d'(L/K) l^{\frac{r(L/K)}{l-1}}$$

holds.) *Moreover, if $r(K) = 0$ holds, there exists another inequality*

$$l^{n+\lceil \frac{r(L/K)}{l^n} \rceil} \leqslant c(L/K),$$

where, for a real number a, $\lceil a \rceil$ denotes the least integer which is not less than a.

We have the following corollary:

COROLLARY 1.9. *Let K be as above and let L be a finite Abelian extension. Let $d(L/K)$, $d'(L/K)$, $r(L/K)$, $c(L/K)$ be as above and let $\mathcal{P}$ (resp. $\mathcal{P}'$) be the set of prime numbers which divides $d(L/K)$ (resp. $d'(L/K)$). Then,*

(1) *There exist non-negative integers a_l ($l \in \mathcal{P}$) such that $r(L/K) = \sum_{l \in \mathcal{P}} a_l(l-1)$ holds.*
(2) *$NC_L \subset C_K$ is open and the following inequality holds*:

$$c(L/K) \leqslant d'(L/K) \left(\prod_{l \in \mathcal{P}'} l^{\frac{1}{l-1}} \right)^{r(L/K)}.$$

In particular, the former statement of (1) in Problem 1.6 is valid. (But the validity itself is a rather easy consequence of class field theory (Lemma 3.1).)

In some cases, we can actually calculate the quantities $d'(L/K)$, $r(L/K)$ and $c(L/K)$ (see Section 4 and 5). For example, we can show the following:

PROPOSITION 1.10. *Let l be a prime and let $a \geqslant 0$ be an integer. Let k be a finite field which contains primitive* lth *root of unity and the order $|k|$ is greater than a. Let K be either* $\mathrm{Frac}(k[[x, y]])$ *or* $\mathrm{Frac}(W(k)[[x]])$. *Then, there exists a finite Abelian extension $L \supset K$ such that the following equalities hold*:

$$d'(L/K) = d(L/K) = l, \qquad r(L) = a(l-1), \qquad r(K) = 0, \qquad c(L/K) = l^{a+1}.$$

In particular, $NC_L = D_L$ does not hold in general, that is, 3. in Problem 1.6 is not always true.

Remark 1.11. The examples in the above proposition attain the equality in the first inequality in 2. of Theorem 1.8. So this inequality is best possible in a sense.

Moreover, using Proposition 1.10, we can show the following theorem, which is the second main result in this paper:

THEOREM 1.12. *Let A be a two-dimensional complete normal local ring with finite residue field k, and let K be the fraction field of A. Then*:

(1) *There exists an unramified extension $K' \supset K$ such that the correspondence*

$$\Psi_{K''} : \left\{ \begin{matrix} \text{finite Abelian} \\ \text{extensions of } K'' \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finite index open} \\ \text{subgroups of } C_{K''} \end{matrix} \right\}$$

given by $\Psi_{K''}(L) := NC_L$ is not surjective for any unramified extension $K' \subset K''$.

(2) If $k \neq \mathbb{F}_2$ *holds, then the correspondence*

$$\Psi_K : \left\{ \begin{matrix} \text{finite Abelian} \\ \text{extensions of } K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \text{finite index open} \\ \text{subgroups of } C_K \end{matrix} \right\}$$

given by $\Psi_K(L) := NC_L$ is not bijective.

Hence, (2) in Problem 1.6 is not true for many fields K. Moreover, the bijectivity of Ψ_K, which is known as the existence theorem (Theorem 1.4) in the case of the class field theory for the fields in Theorem 1.1, does not hold for almost fields in our case.

Remark 1.13. The behavior of quantities $d'(L/K)$, $r(L/K)$ and $c(L/K)$ ($K \subset L$ is as in Corollary 1.9) is not so simple. We will give two remarks concerning this.

(1) Let M be an intermediate field of K and L. Then it is clear that $r(L/K) = r(L/M) + r(M/K)$ holds. But it is not always true that $d'(L/K) = d'(L/M)d'(M/K)$ and $c(L/K) = c(L/M)c(M/K)$ hold. (See Examples 5.1 and 5.4.)

(2) By (2) of Theorem 1.8, the following statement holds: let $K \subset L$ be a cyclic extension such that $[L : K]$ is a power of a prime, and assume $r(K) = 0$ and $r(L) > 0$ hold. Then $d'(L/K) < c(L/K)$ holds and so NC_L is not equal to D_L. But this conclusion is not always true if we drop the assumption '$[L : K]$ is a power of a prime'. In fact, there exists an example of cyclic extension $K \subset L$ such that $[L : K] = 6$, $r(K) = 0$, $r(L) = 2$ and $d'(L/K) = c(L/K) = 6$ hold. (See Example 5.4.)

The content of each section is as follows: In Section 2, we will give a review of class field theory for two-dimensional local fields due to K. Kato and class field theory for the fraction fields of two-dimensional complete normal local rings with finite residue fields due to S. Saito. In Section 3, we will give a proof of Theorem 1.8 and Corollary 1.9. In Section 4, we will explain how to calculate the quantities $d'(L/K)$, $r(L/K)$ and $c(L/K)$ under certain assumptions. The method is based on the argument by Tsuchihashi ([T]) which gives a nice way to calculate the dual graph of two-dimensional Abelian covering singularities and the argument by Saito ([S1]) which allows us to connect the Galois group of the maximal cs extension (defined in Section 2) with the first homology group of certain dual graph. In Section 5, we will give two examples and show Proposition 1.10 and Theorem 1.12.

NOTATION. For an Abelian group M, denote the maximal torsion subgroup of M by M_{tor}. For a group G and a G-module M, denote the G-coinvariant of M by M_G. For a graph (by which we always mean a connected finite one-dimensional CW complex) Δ, denote the set of vertices (resp. edges) of Δ by vertex(Δ) (resp. edge(Δ)).

2. Review of Class Field Theory

In this section, we will review the class field theory for a two-dimensional local field developped by K. Kato, and the class field theory for the fraction field of a two-dimensional complete normal local ring with finite residue field developped by S. Saito.

Let F be a two-dimensional local field, that is, a complete discrete valuation field whose residue field is a local field. Then, on the second K-group $K_2(F)$ of F, Kato defined a certain topology which makes this group a topological group (see [Ka1]). Then, the class field theory for F is described as follows:

THEOREM 2.1 (Kato). *Let F be as above. Then there exists a canonical continuous homomorphism $\rho_F : K_2(F) \longrightarrow G_F$ (called the reciprocity map) which satisfies the following*:

(1) *The homomorphism $\rho_F^* : G_F^* \longrightarrow K_2(F)^*$ induced by ρ_F is an isomorphism.*
(2) *For a finite Abelian extension $F \subset F'$, the following diagram is commutative*:

$$\begin{array}{ccc} K_2(F') & \xrightarrow{\rho_{F'}} & G_{F'} \\ {\scriptstyle N}\downarrow & & \downarrow \\ K_2(F) & \xrightarrow{\rho_F} & G_F, \end{array}$$

where N is the norm homomorphism of K-groups and the right vertical arrow is induced by the inclusion $F \subset F'$. Moreover, the image $NK_2(F')$ of $K_2(F')$ in $K_2(F)$ is open.

As for the norm homomorphism of K-groups, we will use the following standard fact later:

PROPOSITION 2.2. *Let $F \subset F'$ be a finite separable extension of two-dimensional local fields and let O (resp. O') be the ring of integers in F (resp. F'). Then, if the extension $F \subset F'$ is unramified, the norm map induces the surjection $K_2(O') \longrightarrow K_2(O)$.*

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Next, let A be a two-dimensional complete normal local ring with finite residue field and let K be the fraction field of A. Let P_K be the set of prime ideals of height one in A and for $x \in P_K$, let A_x be the x-adic completion of the localization of A by the ideal x, and let K_x be the fraction field of A_x. (This is a two-dimensional local field.) Then define the topological group C_K (which is called the K_2-idele class group of K) as follows:

$$C_K := \left(\prod_{x \in P_K}{}' K_2(K_x) \right) \Big/ K_2(K),$$

where $\prod'$ denotes the restricted product with respect to $K_2(A_x) \subset K_2(K_x)$. The topology of C_K is defined as follows: For a finite subset S in P_K, define $C_{K,S}$ by

$$C_{K,S} := \mathrm{Im}\left(\prod_{x \in S} K_2(K_x) \times \prod_{x \notin S} K_2(A_x) \longrightarrow C_K \right).$$

Then, C_K is the inductive limit of $C_{K,S}$'s. First endow the group $\prod_{x \in S} K_2(K_x) \times \prod_{x \notin S} K_2(A_x)$ with the topology induced by $\prod_{x \in P_K} K_2(K_x)$, and then endow $C_{K,S}$ with the topology induced by the above group. Finally, endow C_K with the inductive limit topology induced by $C_{K,S}$'s. Denote the natural morphism $K_2(K_x) \longrightarrow C_K$ by ι_x.

Let $L \supset K$ be a finite extension of K. Then the integral closure B of A in L is a two-dimensional complete normal local ring with finite residue field which is finite over A. So one can define P_L, C_L etc. as in the case of K, by using B instead of A.

Next let us recall the notion of cs (= complete splitting) extension.

DEFINITION 2.3. Let K be as above. Then a (possibly infinite) Abelian extension $K \subset L$ is said to be a cs extension if any $x \in P_K$ splits completely by any finite Abelian extension $K \subset M$ satisfying $M \subset L$. Denote the maximal cs extenstion of K by K^{cs}.

Remark 2.4. We would like to note that, in this paper, a cs extension is assumed to be Abelian.

Then, the class field theory for K is described as follows:

THEOREM 2.5 (S. Saito). *Let K be as above. Then there exists a canonical continuous homomorphism $\rho_K : C_K \longrightarrow G_K$ (also called the reciprocity map) satisfying the following*:

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(1) *For any* $x \in P_K$, *the following diagram is commutative*:

$$\begin{array}{ccc} K_2(K_x) & \xrightarrow{\rho_{K_x}} & G_{K_x} \\ {\scriptstyle \iota_x}\downarrow & & \downarrow \\ C_K & \xrightarrow{\rho_K} & G_K, \end{array}$$

where ρ_{K_x} *is the reciprocity map of two-dimensional local field* K_x *and the right vertical arrow is the homomophism induced by the inclusion* $K \subset K_x$.

(2) *The homomorphism* $\rho_K^* : G_K^* \longrightarrow C_K^*$ *is surjective and* $\mathrm{Ker}(\rho_K^*) = \mathrm{Gal}\,(K^{\mathrm{cs}}/K)^* \cong (\mathbb{Q}/\mathbb{Z})^{r(K)}$ *holds.*

Remark 2.6. By statement (2) in the above theorem, the rank $r(K)$ of K depends only on A and independent of the choice of a resolution $f : X \longrightarrow \mathrm{Spec}\, A$.

One can see, as a corollary of the above theorem, that the following diagram is commutative for a finite Abelian extension $K \subset L$:

$$\begin{array}{ccc} C_L & \xrightarrow{\rho_L} & G_L \\ {\scriptstyle N}\downarrow & & \downarrow \\ C_K & \xrightarrow{\rho_K} & G_K, \end{array}$$

where the right vertical arrow is induced by the inclusion $K \subset L$ and $N : C_L \longrightarrow C_K$ is the homomorphism induced by the norm homomorphism of K-groups.

Finally in this section, we will show what we remarked in Remark 1.7.

PROPOSITION 2.7. *Let K be the fraction field of a two-dimensional complete normal local ring with finite residue field and assume the following*:

(1) *For any finite Abelian extension* $K \subset L$, NC_L *is a finite index open subgroup of* C_K. (*This is always true as we will see in Lemma 3.1.*)
(2) *The correspondence*

$$\Psi_K : \left\{ \begin{matrix} \textit{finite Abelian} \\ \textit{extensions of } K \end{matrix} \right\} \longrightarrow \left\{ \begin{matrix} \textit{finite index open} \\ \textit{subgroups of } C_K \end{matrix} \right\}$$

defined by $\Psi_K(L) := NC_L$ *is surjective.*

Then, for any finite Abelian extension $K \subset L$, *we have* $NC_L = D_L$.

Proof. Assume the conclusion is false and let $L \supset K$ be a counter-example such that $[C_K : D_L]$ is minimal. Then, since Ψ_K is surjective, there exists a finite Abelian extension $L' \supset K$ such that $D_L = NC_{L'}$ holds.

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Now let us assume that $NC_{L'} = D_{L'}$ holds. Then we have $D_L = D_{L'}$. Let L'' be the composite field LL'. Then we have

$$\begin{aligned} D_{L''} &= \mathrm{Ker}(C_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(L''/K)) \\ &= \mathrm{Ker}(C_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(L/K)) \cap \mathrm{Ker}(C_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(L'/K)) \\ &= D_L \cap D_{L'} = D_L. \end{aligned}$$

On the other hand, we have

$$\begin{aligned} &\mathrm{Im}(C_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(L''/K)) = \mathrm{Gal}(L''/L'' \cap K^{\mathrm{cs}}), \\ &\mathrm{Im}(C_K \longrightarrow G_K \longrightarrow \mathrm{Gal}(L/K)) = \mathrm{Gal}(L/L \cap K^{\mathrm{cs}}). \end{aligned}$$

Hence the natural map

$$\mathrm{Gal}(L''/L'' \cap K^{\mathrm{cs}}) \longrightarrow \mathrm{Gal}(L/L \cap K^{\mathrm{cs}})$$

is injective. So we have $L'' = (L'' \cap K^{\mathrm{cs}})L$. In particular, we have $L'K^{\mathrm{cs}} \subset LK^{\mathrm{cs}}$. By the same argument, we also have $LK^{\mathrm{cs}} \subset L'K^{\mathrm{cs}}$, so $LK^{\mathrm{cs}} = L'K^{\mathrm{cs}}$ holds. Let K_1, K_2 be finite cs extensions of K such that $L \subset L'K_1 \subset LK_2$ holds. Then, we have

$$\begin{aligned} &NC_L = NC_{LK_2} \subset NC_{L'K_1} = NC_{L'}, \\ &NC_{L'} = NC_{L'K_1} \subset NC_L. \end{aligned}$$

Hence, $NC_L = NC_{L'} = D_L$ holds and this contradicts to the definition of L. So we have $NC_{L'} \subsetneq D_{L'}$.

Then we have $[C_K : D_{L'}] < [C_K : D_L]$ and then it contradicts to the definition of L again. So the assertion is proved. $\square$

3. Proofs (I)

In this section, we prove Theorem 1.8 and Corollary 1.9. First we prove Theorem 1.8 (1).

Proof of Theorem 1.8 (1). For any element $\sigma \in \mathrm{Gal}(K_{i-1}^{\mathrm{sep}}/K_{i-1})$, $\sigma(K_i^{\mathrm{cs}})$ is a cs extension of $\sigma(K_i) = K_i$. Hence K_i^{cs} is a Galois extension of K_{i-1} and we have the following exact sequences:

$$\begin{aligned} &1 \longrightarrow \mathrm{Gal}(K_i^{\mathrm{cs}}/K_i) \longrightarrow \mathrm{Gal}(K_i^{\mathrm{cs}}/K_{i-1}) \longrightarrow \mathrm{Gal}(K_i/K_{i-1}) \longrightarrow 1, \\ &1 \longrightarrow \mathrm{Gal}(K_i^{\mathrm{cs}}/K_{i-1}^{\mathrm{cs}}) \longrightarrow \mathrm{Gal}(K_i^{\mathrm{cs}}/K_{i-1}) \longrightarrow \mathrm{Gal}(K_{i-1}^{\mathrm{cs}}/K_{i-1}) \longrightarrow 1, \end{aligned}$$

Then one can see that the cokernel of the composite

$$f : \mathrm{Gal}(K_i^{\mathrm{cs}}/K_i) \longrightarrow \mathrm{Gal}(K_i^{\mathrm{cs}}/K_{i-1}) \longrightarrow \mathrm{Gal}(K_{i-1}^{\mathrm{cs}}/K_{i-1})$$

is a $\mathbb{Z}/l\mathbb{Z}$-module. Let p be a prime distinct from l. Then the homomorphism

$$\mathrm{Gal}(K_i^{\mathrm{cs}}/K_i) \otimes \mathbb{Z}/p\mathbb{Z} \cong (\mathbb{Z}/p\mathbb{Z})^{r(K_i)} \longrightarrow \mathrm{Gal}(K_{i-1}^{\mathrm{cs}}/K_{i-1}) \otimes \mathbb{Z}/p\mathbb{Z} \cong (\mathbb{Z}/p\mathbb{Z})^{r(K_{i-1})}$$

induced by f is surjective. So $r_i := r(K_i) - r(K_{i-1})$ is nonnegative, as desired. □

Before giving the proof of Theorem 1.8 (2) and (3), we first prepare two lemmas.

LEMMA 3.1. *Let K be the fraction field of a two-dimensional complete normal local ring with finite residue field and let L be a finite Abelian extension of K. Then NC_L is a finite index open subgroup of C_K and we have the equality*

$$[C_K : NC_L] = [L : L \cap K^{\mathrm{cs}}][L^{\mathrm{cs}} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs}}].$$

Moreover, if $[L : K]$ is a power of a prime number l, then both sides are also powers of l.

Proof. First, by Proposition 2.2 and the class field theory for two-dimensional local field, we have the following properties:

(1) For any $x \in P_K$ and any $y \in P_L$ lying above x, $NK_2(L_y) \subset K_2(K_x)$ is open.
(2) Let $x \in P_x$ and let y be an element in P_L lying above x. Assume L_y is unramified over K_x and denote the ring of integers of L_y, K_x by O_y, O_x respectively. Then $NK_2(O_y) = K_2(O_x)$ holds.

One can easily check the fact that NC_L is open in C_K, by using the above properties and the definition of the topology of C_K.

Next let us consider the following diagram:

$$\begin{tikzcd} 0 \arrow[r] & \mathrm{Gal}(K^{\mathrm{cs}}/K)^* \arrow[r] \arrow[d] & G_K^* \arrow[r, "\rho_K^*"] \arrow[d] & C_K^* \arrow[r] \arrow[d, "N^*"'] & 0 \\ 0 \arrow[r] & \mathrm{Gal}(L^{\mathrm{cs}}/L)^* \arrow[r] & G_L^* \arrow[r, "\rho_L^*"] & C_L^* \arrow[r] & 0, \end{tikzcd}$$

where horizontal lines are exact. By using the snake lemma, we obtain the following exact sequence:

$$0 \longrightarrow \mathrm{Gal}(L/L \cap K^{\mathrm{cs}})^* \longrightarrow (C_K/NC_L)^* \longrightarrow \mathrm{Gal}(L^{\mathrm{cs}} \cap K^{\mathrm{ab}}/LK^{\mathrm{cs}})^* \longrightarrow 0.$$

Put $d := [L : K]$. Then, since NC_L is open in C_K and the composite $C_K \longrightarrow C_L \xrightarrow{N} C_K$ (where the first map is induced by the inclusion $K \subset L$) is the d-th power map, C_K/NC_L is a discrete $\mathbb{Z}/d\mathbb{Z}$-module. Then one can check easily that $|C_K/NC_L|$ is finite if and only if $|(C_K/NC_L)^*|$ is finite, and if they are finite, they are equal. Hence, it suffices to show the following:

(1) $|\mathrm{Gal}(L^{\mathrm{cs}} \cap K^{\mathrm{ab}}/LK^{\mathrm{cs}})|$ is finite.

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(2) If p is a prime number such that $(p, d) = 1$ holds, then $|\mathrm{Gal}(L^{\mathrm{cs}} \cap K^{\mathrm{ab}}/LK^{\mathrm{cs}})|$ is prime to p.

Put $G := \mathrm{Gal}(L^{\mathrm{cs}} \cap K^{\mathrm{ab}}/LK^{\mathrm{cs}})$. Since it is a subquotient of $\mathrm{Gal}(L^{\mathrm{cs}}/L) \cong \hat{\mathbb{Z}}^{r(L)}$, it is Abelian. For a prime p, let G_p be the pro-p completion of G. Then G_p is a finitely generated $\mathbb{Z}_p$-module, since it is a subquotient of $\mathbb{Z}_p^{r(L)}$. Hence, G_p^* can be expressed in the form

$$G_p^* = (\mathbb{Q}_p/\mathbb{Z}_p)^{a_p} \oplus T_p,$$

where $a_p \in \mathbb{N}$ and T_p is a finite Abelian p-group. On the other hand, since $(C_K/NC_L)^*$ is a $\mathbb{Z}/d\mathbb{Z}$-module, G^* is a $\mathbb{Z}/d\mathbb{Z}$-module. Hence G_p^* is also a $\mathbb{Z}/d\mathbb{Z}$-module. Hence $a_p = 0$ holds for any p (so $G_p = (G_p^*)^*$ is finite for all p) and if $(d, p) = 1$ holds, G_p is trivial. Hence $G = \prod_p G_p$ is finite and $|G|$ is prime to p if $(d, p) = 1$ holds, as desired. □

LEMMA 3.2. *Let K be the fraction field of a two-dimensional complete normal local ring A with finite residue field and let l be a prime number. Let L be a finite Abelian extension over K and assume that the image of the homomorphism*

$$f : C_K \xrightarrow{\rho_K} G_K \longrightarrow \mathrm{Gal}(L/K)$$

is isomorphic to $\mathbb{Z}/l^n\mathbb{Z}$ for some $n \in \mathbb{N}$. Then there exists an element x in P_K and $y \in P_L$ lying above x such that $[L_y : K_x] = l^n$ holds.

Proof. Let S be the subset of P_K which ramifies in the extension $K \subset L$, and define J by

$$J := \mathrm{Im}\left(\prod_{x \notin S} K_2(A_x) \longrightarrow C_K\right).$$

By Proposition 2.2, J is contained in NC_L, hence it is contained in $\mathrm{Ker}(f)$. So f factors through

$$C_K/J = \mathrm{Coker}\left(K_2(K) \longrightarrow \bigoplus_{x \notin S}(K_2(K_x)/K_2(A_x)) \oplus \bigoplus_{x \in S} K_2(K_x)\right).$$

Let g be the composite

$$\bigoplus_{x \in P_K} K_2(K_x) \xrightarrow{\pi} C_K/J \xrightarrow{\bar{f}} \mathrm{Gal}(L/K),$$

where π is the natural surjection and $\bar{f}$ is the map induced by f. Then, by definition, $\mathrm{Im}(g) \cong \mathbb{Z}/l^n\mathbb{Z}$ holds. For each $x \in P_K$, choose an element $y(x)$ in P_L lying above

x. Then the following diagram is commutative:

$$\begin{array}{ccc} \bigoplus_{x \in P_K} K_2(K_x) & \xrightarrow{\oplus \sigma_{K_x}} & \bigoplus_{x \in P_K} \mathrm{Gal}(L_{y(x)}/K_x) \\ \Big\| & & \Big\downarrow i \\ \bigoplus_{x \in P_K} K_2(K_x) & \xrightarrow{g} & \mathrm{Gal}(L/K), \end{array}$$

where σ_{K_x} is the composite

$$K_2(K_x) \xrightarrow{\rho_K} G_{K_x} \longrightarrow \mathrm{Gal}(L_{y(x)}/K_x)$$

and i is the sum of natural inclusions. Then, we have

$$\mathrm{Im}(g) = \sum_{x \in P_K} i \circ \sigma_x(K_2(K_x)) = \sum_{x \in P_K} i(\mathrm{Gal}(L_{y(x)}/K_x)).$$

Then, since $\mathrm{Im}(g) \cong \mathbb{Z}/l^n\mathbb{Z}$ holds, there exists an element $x \in P_K$ such that $i(\mathrm{Gal}(L_{y(x)}/K_x)) = \mathrm{Im}(g)$. Then we have $[L_{y(x)} : K_x] = l^n$, so we are done. $\square$

Now we will prove the key proposition for the proof of Theorems 1.8 (2) and (3).

PROPOSITION 3.3. *Let l, K and L be as in Theorem 1.8. Define the notations as follows*:

$L^{\mathrm{cs},l}$*(resp.*$K^{\mathrm{cs},l}$*) : the maximal pro-l cs extension of L (resp. K),*
$G := \mathrm{Gal}(L/K)(\cong \mathbb{Z}/l^n\mathbb{Z})$,
$M := \mathrm{Gal}(L^{\mathrm{cs},l}/L), \quad M_1 := M_G, \quad M_2 := M_1/M_{1,\mathrm{tor}}$,
F_i : *the cs extension of L corresponding to* $\mathrm{Ker}(M \to M_i)\,(i = 1, 2)$.

Then we have $F_1 = L^{\mathrm{cs},l} \cap K^{\mathrm{ab}}$ and $F_2 = LK^{\mathrm{cs},l}$.

Proof. First we show the equality $F_1 = L^{\mathrm{cs},l} \cap K^{\mathrm{ab}}$. Since $\mathrm{Ker}(M \to M_1)$ is G-invariant, the extension $K \subset F_1$ is Galois and there exists the following exact sequence:

$$1 \longrightarrow M_1 \longrightarrow \mathrm{Gal}(F_1/K) \longrightarrow G \longrightarrow 1.$$

Since the action of G on M_1 is trivial, M_1 is contained in the center of $\mathrm{Gal}(F_1/K)$. Then, since G is cyclic, the group $\mathrm{Gal}(F_1/K)$ is Abelian. Hence, we have $F_1 \subset L^{\mathrm{cs},l} \cap K^{\mathrm{ab}}$. On the other hand, since the action of G to the group $\mathrm{Ker}(\mathrm{Gal}(L^{\mathrm{cs},l} \cap K^{\mathrm{ab}}/K) \longrightarrow G))$ is trivial, we have $L^{\mathrm{cs},l} \cap K^{\mathrm{ab}} \subset F_1$. Hence we have the equality $F_1 = L^{\mathrm{cs},l} \cap K^{\mathrm{ab}}$.

Next we prove the inclusion $LK^{\mathrm{cs},l} \subset F_2$. Let us consider the composite of natural homomorphisms

$$\mathrm{Gal}(LK^{\mathrm{cs},l}/L) \longrightarrow \mathrm{Gal}(LK^{\mathrm{cs},l}/K) \longrightarrow \mathrm{Gal}(K^{\mathrm{cs},l}/K).$$

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One can easily see that it is injective. So $\mathrm{Gal}(LK^{\mathrm{cs},l}/L)$ is a free $\mathbb{Z}_l$-module. Moreover, since $\mathrm{Gal}(LK^{\mathrm{cs},l}/K)$ is Abelian, the action of G on $\mathrm{Gal}(LK^{\mathrm{cs},l}/L)$ is trivial. Hence there exists a natural surjection $M_2 \longrightarrow \mathrm{Gal}(LK^{\mathrm{cs},l}/L)$. So we have $LK^{\mathrm{cs},l} \subset F_2$.

Finally we prove the inclusion $F_2 \subset LK^{\mathrm{cs},l}$. Let us consider the following exact sequence:

$$1 \longrightarrow M_2 \longrightarrow \mathrm{Gal}(F_2/K) \xrightarrow{\pi} G \longrightarrow 1.$$

Since G acts on M_2 trivially and G is cyclic, $\mathrm{Gal}(F_2/K)$ is Abelian. Hence it is a finitely generated $\mathbb{Z}_l$-module. Let N be the intermediate field between K and F_2 which corresponds to $\mathrm{Gal}(F_2/K)_{\mathrm{tor}}$. Then, $\mathrm{Gal}(F_2/LN)$ is a torsion Abelian group, since it is a subgroup of $\mathrm{Gal}(F_2/N) = \mathrm{Gal}(F_2/K)_{\mathrm{tor}}$. On the other hand, it is a free $\mathbb{Z}_l$-module, since it is a subgroup of M_2. Hence we have $F_2 = LN$. Now we have the following claim:

CLAIM. *N is a cs extension of K.*

Proof. Let us consider the following homomorphism:

$$h : C_K \xrightarrow{\rho_K} G_K \longrightarrow \mathrm{Gal}(N/K).$$

It suffices to show that h is a zero map. Let $K \subset N'$ be a subextension of $K \subset N$ such that $\mathrm{Gal}(N'/K) \cong \mathbb{Z}/l^b\mathbb{Z}$ holds for some $b \in \mathbb{N}$ and consider the homomorphism

$$h' : C_K \xrightarrow{h} \mathrm{Gal}(N/K) \longrightarrow \mathrm{Gal}(N'/K) \cong \mathbb{Z}/l^b\mathbb{Z}.$$

Assume $|\mathrm{Im}(h')|$ is greater than l^n. Then, by Lemma 3.2, there exists an element $x \in P_K$ and an element $y \in P_{N'}$ lying above x such that $[N'_y : K_x] > l^n$ holds. On the other hand, choose an element $z \in P_{LN'}$ lying above y and let $w \in P_L$ be the element lying under z. Then, since $LN' \supset L$ is a cs extension, we have the inequality

$$[N'_y : K_x] \leqslant [(LN')_z : K_x] = [L_w : K_x] \leqslant l^n.$$

This is a contradiction, so we have $|\mathrm{Im}(h')| \leqslant l^n$.

Now assume that $\mathrm{Im}(h)$ is nontrivial and let α be a nontrivial element of $\mathrm{Im}(h)$. Then there exists a surjection $v : \mathrm{Gal}(N/K) \longrightarrow \mathbb{Z}/l^b\mathbb{Z}$ (for some $b \in \mathbb{N}$) such that the order of the image of α is greater than l^n, since $\mathrm{Gal}(N/K)$ is a free $\mathbb{Z}_l$-module. If we define N' to be the subfield of N corresponding to $\mathrm{Ker}(v)$, we have $|\mathrm{Im}(h')| > l^n$ and this contradicts to what we have shown in the above parapraph. So $\mathrm{Im}(h)$ is trivial, that is, N is a cs extension of K. $\square$

By the above claim, we have $F_2 = LN \subset LK^{\mathrm{cs},l}$, as is desired. $\square$

COROLLARY 3.4. *Let the notations be as in Theorem 1.8 and put* $G := \mathrm{Gal}(L/K)$, $M := \mathrm{Gal}(L^{\mathrm{cs}}/L)$. *Then we have* $c(L/K) = l^{n-m}|M_{G,\mathrm{tor}}|$.

Proof. By Lemma 3.1, we have $c(L/K) = l^{n-m}[L^{\mathrm{cs}} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs}}]$. Since the index $[L^{\mathrm{cs}} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs}}]$ is a power of l again by Lemma 3.1, we have

$$[L^{\mathrm{cs}} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs}}] = [L^{\mathrm{cs},l} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs},l}]$$

and so we have

$$c(L/K) = l^{n-m}[L^{\mathrm{cs},l} \cap K^{\mathrm{ab}} : LK^{\mathrm{cs},l}] = l^{n-m}|M_{G,\mathrm{tor}}|,$$

by Proposition 3.3. $\square$

Now we will give a proof of Theorems 1.8 (2) and (3).

Proof of Theorem 1.8 (2). By replacing L by K_i, we have only to show the assertion in the case $i = n$. Define the notations as follows:

$$\begin{aligned}
&G := \mathrm{Gal}(L/K), \qquad G' := \mathrm{Gal}(L/K_{n-1}),\\
&M := \mathrm{Gal}(L^{\mathrm{cs},l}/L), \qquad M' := \mathrm{Gal}(LK_{n-1}^{\mathrm{cs},l}/L),\\
&V := M \otimes_{\mathbb{Z}} \mathbb{Q}, \qquad V' := M' \otimes_{\mathbb{Z}} \mathbb{Q}.
\end{aligned}$$

Consider the following exact sequence:

$$1 \longrightarrow M' \longrightarrow \mathrm{Gal}(K_{n-1}^{\mathrm{cs},l}/K_{n-1}) \longrightarrow \mathrm{Gal}(L \cap K_{n-1}^{\mathrm{cs},l}/K_{n-1}) \longrightarrow 1.$$

From this sequence, one can see that M' is isomorphic to $\mathbb{Z}_l^{r(K_{n-1})}$. Hence r_n is equal to the dimension of $\mathrm{Ker}(V \to V')$ as $\mathbb{Q}_l$-vector space.

By Proposition 3.3, $M' = M_{G'}/M_{G',\mathrm{tor}}$ holds. So $V' = V_{G'}$ holds. Fix a generator σ of $G \cong \mathbb{Z}/l^n\mathbb{Z}$. Since V is a $\mathbb{Q}_l$-vector space with G-action, it can be regarded as a module over $A := \mathbb{Q}_l[\sigma]/(\sigma^{l^n} - 1)$. For $0 \leqslant j \leqslant n$, let ζ_{l^j} be a primitive l^j-th root of unity and put $A_j := \mathbb{Q}_l(\zeta_{l^j})$. Then one has the isomorphism of $\mathbb{Q}_l$-algebras

$$A \longrightarrow \prod_{j=0}^{n} A_j, \quad \sigma \mapsto (\zeta_{l^j})_{j=0}^{n}.$$

Let V_j be A_jV. Then we have the natural decomposition $V \cong \bigoplus_{j=0}^{n} V_j$. Then, V' is calculated as follows:

$$V' = V_{G'} = \bigoplus_{j=0}^{n} V_j/((\zeta_{l^j})^{l^{n-1}} - 1)V_j \cong \bigoplus_{j=0}^{n-1} V_j.$$

Hence, we have the isomorphism $\mathrm{Ker}(V \to V') \cong V_n$. Since V_n is a finite-dimensional A_n-vector spece, $r_n = \dim_{\mathbb{Q}_p} V_n$ is divisible by $[A_n : \mathbb{Q}_l] = l^{n-1}(l-1)$. $\square$

Proof of Theorem 1.8 (3). Let M be $\mathrm{Gal}(L^{\mathrm{cs},l}/L)$ and let G be $\mathrm{Gal}(L/K)$. Then, by Corollary 3.4, one has $c(L/K) = l^{n-m}|M_{G,\mathrm{tor}}|$.

Let $A, V, A_j, V_j\,(0 \leqslant j \leqslant n)$ be as in the proof of Theorem 1.8 (2), and let $a_j\,(1 \leqslant j \leqslant n)$ be the integer satisfying $V_j \cong A_j^{\oplus a_j}$ (as A_j-vector spaces). Then we have the following:

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CLAIM.

$$a_0 = r(K), \qquad a_j = \frac{r_j}{l^{j-1}(l-1)} \quad (1 \leqslant j \leqslant n).$$

Proof. By the similar argument to the proof of Theorem 1.8 (2), we obtain the equality

$$r(K_i) = \dim_{\mathbb{Q}_l}\left(\bigoplus_{j=0}^{i} V_j\right) = a_0 + \sum_{j=1}^{i} l^{j-1}(l-1)\,a_j$$

for $0 \leqslant i \leqslant n$. The claim follows from it. $\square$

Let π be the projection $V \longrightarrow \bigoplus_{j=0}^{m} V_j$ and define N and M_0 by $N := \pi(M)$, $M_0 := \mathrm{Ker}(\pi)$ respectively. Then we have the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & M_0 & \longrightarrow & M & \longrightarrow & N & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & \bigoplus_{j=m+1}^{n} V_j & \longrightarrow & V & \xrightarrow{\pi} & \bigoplus_{j=0}^{m} V_j & \longrightarrow & 0. \end{array}$$

Put $G' := \mathrm{Gal}(L/K_m)$. Then, by the similar argument to the proof of Theorem 1.8 (2), one can see that the morphism $\pi\colon V \longrightarrow \bigoplus_{j=0}^{m} V_j$ is nothing but the natural morphism $V \longrightarrow V_{G'}$. Hence we have $N \cong M_{G'}/M_{G',\mathrm{tor}}$. So, by Proposition 3.3, we have

$$N \cong \mathrm{Gal}(LK_m^{\mathrm{cs},l}/L) \cong \mathrm{Gal}(K_m^{\mathrm{cs},l}/K_m).$$

By taking G-coinvariant of the upper horizontal line in the above diagram, we obtain the exact sequence

$$M_{0,G} \longrightarrow M_G \longrightarrow N_G \longrightarrow 0.$$

Now note the following claim:

CLAIM. *N_G is a free $\mathbb{Z}_l$-module.*

Proof. Let $K_m^{\mathrm{cs},l}$ be the maximal pro-l cs extension of K_m. Then, since K_m is a cs extension of K, we have $K_m^{\mathrm{cs},l} \cap K^{\mathrm{ab}} \subset K^{\mathrm{cs},l}$. The inclusion in the other direction is trivial, so we have $K_m^{\mathrm{cs},l} \cap K^{\mathrm{ab}} = K^{\mathrm{cs},l}$. So, by Proposition 3.3, $N_G = \mathrm{Gal}(K^{\mathrm{cs},l}/K_m) \subset \mathrm{Gal}(K^{\mathrm{cs},l}/K)$ holds. So it is a free $\mathbb{Z}_l$-module. $\square$

By the above claim, we have $|M_{G,\mathrm{tor}}| \leqslant |M_{0,G,\mathrm{tor}}|$, so the inequality $c(L/K) \leqslant l^{n-m}|M_{0,G,\mathrm{tor}}|$ holds. Hence, to show the first inequality in Theorem 1.8 (3), it suffices to show the equality

$$|(M_0/(\sigma-1)M_0)_{\mathrm{tor}}| = l^{\sum_{j=m+1}^{n} a_j},$$

where σ is as in the proof of Theorem 1.8 (2). Let $\|\cdot\|$ be the norm on $\mathbb{Q}_l$ such that $\|l\| = l^{-1}$ holds. Then, since $\sigma - 1$ induces an automorphism on

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$M_0 \otimes_{\mathbb{Z}} \mathbb{Q} = \bigoplus_{j=m+1}^{n} V_j \,(=: W)$, we have

$$\begin{aligned}|(M_0/(\sigma-1)M_0)_{\mathrm{tor}}| &= |M_0/(\sigma-1)M_0| \\ &= \|\det(\sigma-1|W)\|^{-1} \\ &= \|\prod_{j=m+1}^{n} N_{A_j/\mathbb{Q}_l}(\zeta_{l^j}-1)^{a_j}\|^{-1} \\ &= l^{\sum_{j=m+1}^{n} a_j},\end{aligned}$$

as is desired.

Finally, we prove the last inequality in Theorem 1.8 (3). By Proposition 3.3, we have

$$M_G/M_{G,\mathrm{tor}} \cong \mathrm{Gal}(LK^{\mathrm{cs},l}/L)$$

and by the assumption $r(K)=0$, it is trivial. So we have

$$|M_{G,\mathrm{tor}}| = |M_G| \geqslant |M_G \otimes_{\mathbb{Z}_l} \mathbb{F}_l| = |(M \otimes_{\mathbb{Z}_l} \mathbb{F}_l)_G|.$$

If we fix a generator σ of G, we can regard $M \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ naturally as a finitely generated module over $B := \mathbb{F}_l[\sigma]/(\sigma-1)^{l^n}$. So there exists integers $b_j \in \mathbb{N}\,(1 \leqslant j \leqslant l^n)$ such that

$$M \otimes_{\mathbb{Z}_l} \mathbb{F}_l \cong \bigoplus_{j=1}^{l^n} (B/(\sigma-1)^j B)^{\oplus b_j}$$

holds (as B-modules). Since $M \otimes_{\mathbb{Z}_l} \mathbb{F}_l$ is an $r(L)$-dimensional vector space over $\mathbb{F}_l$, we have the equality $\sum_{j=1}^{l^n} jb_j = r(L)$. On the other hand, we have

$$(M \otimes_{\mathbb{Z}_l} \mathbb{F}_l)_G \cong (M \otimes \mathbb{F}_l)/(\sigma-1)(M \otimes \mathbb{F}_l) \cong \bigoplus_{j=1}^{l^n} \mathbb{F}_l^{\oplus b_j}.$$

So we have the inequality

$$|(M \otimes \mathbb{F}_l)_G| = l^{\sum_{j=1}^{l^n} b_j} \geqslant l^{\lceil \frac{1}{l^n} \sum_{j=1}^{l^n} jb_j \rceil} = l^{\lceil r(L)/l^n \rceil}.$$

So we have $|M_{G,\mathrm{tor}}| \geqslant l^{\lceil r(L)/l^n \rceil}$ and the desired inequality follows from this.

Finally we will give a proof of Corollary 1.9.

Proof of Corollary 1.9. If $d(L/K)$ is a prime, then the extension $K \subset L$ is cyclic. So the assertion is contained in Theorem 1.8.

Let us prove the general case by induction on $d(L/K)$. Let M be an intermediate field of K and L such that $[M:K]$ is a prime number. By definition, we have

$$r(L/K) = r(L/M) + r(M/K), \;\; d(L/M) \,|\, d(L/K), \;\; d(M/K) \,|\, d(L/K).$$

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By using this, the assertion (1) is easily deduced from the inductive hypothesis and Theorem 1.8 for the extension $K \subset M$.

Before proving (2), note the following claim:

CLAIM. *$d'(L/M)d'(M/K) \mid d'(L/K)$ holds. In particular, we have the inequality $d'(L/M)d'(M/K) \leqslant d'(L/K)$.*

Proof. If $K \subset M$ is a cs extension, $d'(M/K) = 1$ holds. So

$$d'(L/M)d'(M/K) = [L : M^{\mathrm{cs}} \cap L] \mid [L : K^{\mathrm{cs}} \cap L] = d'(L/K)$$

holds. If $K \subset M$ is not a cs extension, then M is not contained in $K^{\mathrm{cs}} \cap L$. Since $d(M/K)$ is prime, we have $[M(K^{\mathrm{cs}} \cap L) : K^{\mathrm{cs}} \cap L] = [M : K]$. Hence

$$\begin{aligned} &d'(L/M)d'(M/K) \\ &\quad = [L : M^{\mathrm{cs}} \cap L][M : K] \\ &\quad = [L : M^{\mathrm{cs}} \cap L][M(K^{\mathrm{cs}} \cap L) : K^{\mathrm{cs}} \cap L] \mid [L : K^{\mathrm{cs}} \cap L] = d'(L/K) \end{aligned}$$

holds. □

Now we prove the assertion (2). By definition,

$$c(L/K) = [C_K : NC_M][NC_M : NC_L] \leqslant c(L/M)c(M/K)$$

holds. Then, by using the above claim and the inductive hypothesis, we obtain

$$\begin{aligned} &c(L/M)c(M/K) \\ &\quad \leqslant d'(L/M)\Bigg(\prod_{l \mid d'(L/M)} l^{\frac{1}{l-1}}\Bigg)^{r(L/M)} d'(M/K)\Bigg(\prod_{l \mid d'(M/K)} l^{\frac{1}{l-1}}\Bigg)^{r(M/K)} \\ &\quad \leqslant d'(L/K)\Bigg(\prod_{l \mid d'(L/K)} l^{\frac{1}{l-1}}\Bigg)^{r(L/K)}, \end{aligned}$$

as is desired. □

4. Resolution of Two-Dimensional Abelian Covering Singularities

Throughout this section, let k be a finite field of characteristic p, let A be either $k[[x, y]]$ or $W(k)[[y]]$, and let K be the fraction field of A. In this section, we will describe how to calculate the rank $r(L)$ and the action of Galois group $\mathrm{Gal}(L/K)$ on $\mathrm{Gal}(L^{\mathrm{cs}}/L)$ for a Kummer extension $L \subset K$ satisfying some conditions. (If we can calculate them, one can calculate the quantities $c(L/M), r(L/M)$ for an intermediate field M such that $M \subset L$ is cyclic and $[L : M]$ is a power of a prime, by using Corollary 3.4. We will do it in two examples in the next section.)

To calculate the rank $r(L)$, it is important to resolve the singularities of the spectrum of the integral closure B of A in L. The singularities which occur in this way are sometimes called Abelian covering singularities, and in complex analytic case, the resolution of two-dimensional Abelian covering singularities is studied by Tsuchihashi ([T]). We study the case of positive characteristics or mixed characteristics, following his method. On the other hand, to see the action of $\mathrm{Gal}(L/K)$ on $\mathrm{Gal}(L^{\mathrm{cs}}/L)$, we slightly generalize an argument of Saito ([S1, II, Section 2]). This allows us to identify the group $\mathrm{Gal}(L^{\mathrm{cs}}/L)$ with the completion of the first homology group of a certain graph as $\mathrm{Gal}(L/K)$-modules.

Let s be a natural number and let $f_i \in A, r_i \in \mathbb{N}\,(1 \leqslant i \leqslant s)$. First we impose the following assumption on k:

(A1): k contains a primitive r_ith root of unity for any $1 \leqslant i \leqslant s$.

Let $\overline{A}$ be the unramified extension of A with residue field $\overline{k}$ (:= the algebraic closure of k), and let $\overline{K}$ be the fraction field of $\overline{A}$. Let $\overline{L}$ be $\overline{K}[z_i]_{1 \leqslant i \leqslant s}/(z_i^{r_i} - f_i)_{1 \leqslant i \leqslant s}$. Next we impose the following assumption:

(A2): $\overline{L}$ is a field.

Then, $L := K[z_i]_{1 \leqslant i \leqslant s}/(z_i^{r_i} - f_i)_{1 \leqslant i \leqslant s}$ is a finite Abelian extension of K, and the Galois group $G := \mathrm{Gal}(L/K)$ is naturally isomorphic to $\bigoplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}(1)$. Define $B, \overline{B}$ by

$$B := A[z_i]_{1 \leqslant i \leqslant s}/(z_i^{r_i} - f_i)_{1 \leqslant i \leqslant s}, \qquad \overline{B} := \overline{A}[z_i]_{1 \leqslant i \leqslant s}/(z_i^{r_i} - f_i)_{1 \leqslant i \leqslant s},$$

respectively. They are local integral domains.

Put $X = \operatorname{Spec} B$, $Y = \operatorname{Spec} A$ and let P, Q be the closed point of X, Y respectively. (Note that the Galois group G acts on X.) Let D_i be the divisor of Y defined by the ideal $(f_i) \subset A$, and put $D := \sum_i D_i$. Choose the minimal embedded resolution $\theta : Z \longrightarrow Y$ of (Y, D): That is, Z is regular, θ is a proper morphism which induces the isomorphism $Z - \theta^{-1}(D) \longrightarrow Y - D$ and $\theta^{-1}(D)_{\mathrm{red}}$ is a simple normal crossing divisor without (-1)-curves. Let I be the set of prime divisors with support in D and for $C \in I$, let E_C be the proper transform of C with respect to θ. Then $\theta^{-1}(D)_{\mathrm{red}}$ is expressed as the sum

$$\theta^{-1}(D)_{\mathrm{red}} = \sum_{C \in I} E_C + \sum_{j \in J} E_j,$$

where J is an index set and for $j \in J$, E_j is an irreducible divisor. Let E be $\sum_{j \in J} E_j$.

Let W be the normalization of $X \times_Y Z$. The action of the Galois group G on X induces the action of G on W. Let $\nu : W \longrightarrow Z$ (resp. $\lambda : W \longrightarrow X$) be the composite of the normalization $W \longrightarrow X \times_Y Z$ and the projection $X \times_Y Z \longrightarrow Z$ (resp. $X \times_Y Z \longrightarrow X$). Let us define $\widetilde{E}$ by $\widetilde{E} := \lambda^{-1}(P)_{\mathrm{red}} = \nu^{-1}(E)_{\mathrm{red}}$.

Let $\overline{Y}$ be $\operatorname{Spec} \overline{A}$, let $\overline{W}$ be $W \times_Y \overline{Y}$ and let $\overline{\widetilde{E}}$ be the pull-back of $\widetilde{E}$ to $\overline{W}$. Now we impose the following assumption:

(A3): The number of irreducible components of $\widetilde{E}$ is equal to that of $\overline{\widetilde{E}}$, and for any pair of distinct irreducible components $(\widetilde{E}_1, \widetilde{E}_2)$ of $\widetilde{E}$, the intersection $(\widetilde{E}_1 \cap \widetilde{E}_2)_{\mathrm{red}}$ is empty or a disjoint union of k-rational points.

Let us fix a topological generator of $\prod_{l \neq p} \mathbb{Z}_l(1)$ and identify $G = \bigoplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}(1)$ with $\bigoplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}$ by using it. Denote the element

$$(0, \cdots, 0, \overset{i}{\check{1}}, 0, \cdots, 0) \in G$$

simply by $e_i \, (1 \leqslant i \leqslant s)$. For $C \in I$, define the element $\tau_C \in G$ by $\tau_C := \sum_{i=1}^{s} m_{C,i} e_i$, where $m_{C,i} \in \mathbb{Z}$ is the multiplicity of C in D_i. For $j \in J$, define the element $\tau_j \in G$ by $\tau_j := \sum_{i=1}^{s} n_{j,i} e_i$, where $n_{j,i} \in \mathbb{Z}$ is the multiplicity of E_j in $\theta^* D_i$.

For $i \in I \cup J$, we define a subgroup $G_i \subset G$ as the subgroup generated by τ_j's $(j \in I \cup J)$ such that $E_i \cap E_j \neq \emptyset$, and for $i, j \in I \cup J$, we define a subgroup $G_{ij} \subset G$ as the subgroup generated by τ_i and τ_j.

Let Δ be the dual graph of $E = \sum_{j \in J} E_j$. For $\alpha \in \mathrm{vertex}(\Delta)$ corresponding to E_j, define $G_\alpha \subset G$ by $G_\alpha := G_j$, and for $\alpha \in \mathrm{edge}(\Delta)$ corresponding to $E_i \cap E_j$, define $G_\alpha \subset G$ by $G_\alpha := G_{ij}$. Now we define a graph $\widetilde{\Delta}$ which is endowed with an action of G (which we will call as a G-graph in the sequel) as follows: First, $\mathrm{vertex}(\widetilde{\Delta})$ and $\mathrm{edge}(\widetilde{\Delta})$ is defined as follows.

$$\mathrm{vertex}(\widetilde{\Delta}) := \{(\alpha, A) \mid \alpha \in \mathrm{vertex}(\Delta), A \in G/G_\alpha\},$$
$$\mathrm{edge}(\widetilde{\Delta}) := \{(\alpha, A) \mid \alpha \in \mathrm{edge}(\Delta), A \in G/G_\alpha\}.$$

For $(\alpha, A) \in \mathrm{vertex}(\widetilde{\Delta})$ and $(\beta, B) \in \mathrm{edge}(\widetilde{\Delta})$, we define that (α, A) is a face of (β, B) if and only if α is a face of β in the graph Δ and A contains B. The action of G on the graph $\widetilde{\Delta}$ is defined by $g \cdot (\alpha, A) := (\alpha, g + A)$. (Note that the above definition of the G-graph $\widetilde{\Delta}$ is independent of the choice of a topological generator of $\prod_{l \neq p} \mathbb{Z}_l(1)$, up to isomorphism.)

Then, our main theorem in this section is as follows (cf. Tsuchihashi [T, Section 3]):

THEOREM 4.1 *Under the assumptions* (A1), (A2) *and* (A3), *there exists an isomorphism of G-modules*

$$H_1(\widetilde{\Delta}, \mathbb{Z}) \otimes_{\mathbb{Z}} \hat{\mathbb{Z}} \cong \mathrm{Gal}(L^{\mathrm{cs}}/L).$$

(*In particular*, $r(L) = \mathrm{rk}\, H_1(\widetilde{\Delta}, \mathbb{Z})$.)

We first show an easy lemma which we need for the proof of the above theorem:

LEMMA 4.2. *Let R be a two-dimensional complete regular local ring with algebraically closed residue field of characteristic p, and x, y be a regular parameter of R. Put $U := \mathrm{Spec}\, R$. Let D_x, D_y be the divisor of U defined by x, y respectively and put $D := D_x \cup D_y$.*

Let $r_i \in \mathbb{N} \, (1 \leqslant i \leqslant s)$ be integers prime to p and let $a_i, b_i \in \mathbb{N} \, (1 \leqslant i \leqslant s)$ be integers. Let S' be $R[z_i]_{1 \leqslant i \leqslant s}/(z_i^{r_i} - x^{a_i} y^{b_i})_{1 \leqslant i \leqslant s}$ and let S be the normalization of S'. Put $V := \mathrm{Spec}\, S$. Fix a connected component V_0 of V and let $f : V_0 \longrightarrow U$ be the natural map.

Let

$$G := \mathrm{Aut}(V/U)\left(\cong \bigoplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}(1)\right) \textit{ and } H := \pi_1^{\mathrm{tame}}(U, D)\left(\cong \left(\prod_{l\neq p} \mathbb{Z}_l(1)\right)^2\right)$$

and let $\varphi : H \longrightarrow G$ be the natural map. Then:

(1) $\{g \in G \mid g(V_0) = V_0\} = \varphi(H)$ *holds.*

(2) *The singularities of V are all toric singularities* (*in the sense of* [Ka4]).

(3) $f^{-1}(D_x)_{\mathrm{red}}$ *and* $f^{-1}(D_y)_{\mathrm{red}}$ *are irreducible.*

Proof. The assertion (1) is obvious. Let us define the homomorphisms of monoids $\alpha\colon \mathbb{N}^s \longrightarrow \mathbb{N}^2$, $\beta\colon \mathbb{N}^s \longrightarrow \mathbb{N}^s$ by $\alpha(e_i) = (a_i, b_i)$, $\beta(e_i) = r_i e_i$, where $e_i\,(1 \leqslant i \leqslant s)$ is the natural basis of $\mathbb{N}^s$. Define the monoid Q by the push-out of the diagram

$$\mathbb{N}^2 \xleftarrow{\alpha} \mathbb{N}^s \xrightarrow{\beta} \mathbb{N}^s$$

in the category of monoids. Then $S' = R \otimes_{\mathbb{Z}[\mathbb{N}^2],\gamma} \mathbb{Z}[Q]$, where $\gamma\colon \mathbb{Z}[\mathbb{N}^2] \longrightarrow \mathbb{Z}[Q]$ is induced by the homomorphism $\mathbb{N}^2 \longrightarrow Q$ induced by the above diagram. Let Q^{sat} be the saturation of Q. Then one can check that $S = R \otimes_{\mathbb{Z}[\mathbb{N}^2],\gamma'} \mathbb{Z}[Q^{\mathrm{sat}}]$ holds, where $\gamma'\colon \mathbb{Z}[\mathbb{N}^2] \longrightarrow \mathbb{Z}[Q^{\mathrm{sat}}]$ be the composite

$$\mathbb{Z}[\mathbb{N}^2] \xrightarrow{\gamma} \mathbb{Z}[Q] \longrightarrow \mathbb{Z}[Q^{\mathrm{sat}}].$$

Then, since $(V, (Q^{\mathrm{sat}} \xrightarrow{\delta} S)^a)$ is log-étale over $(U, (\mathbb{N}^2 \xrightarrow{\varepsilon} R)^a)$ (where δ is the natural morphism and ε is defined by $\varepsilon((1,0)) = x$, $\varepsilon((0,1)) = y$), $(V, (Q^{\mathrm{sat}} \xrightarrow{\delta} S)^a)$ is log regular. So it has only toric singularities ([Ka4]).

Finally we will prove (3). Since $f\colon V_0 \longrightarrow U$ is a tame covering, there exists an integer $n > 0$ prime to p which satisfies the following: If we put $\widetilde{V} := \mathrm{Spec}\, R[z, w]/(z^n - x, w^n - y)$ and define $\widetilde{f}\colon \widetilde{V} \longrightarrow U$ as the natural homomorphism, there exists a morphism $g\colon \widetilde{V} \longrightarrow V$ such that $\widetilde{f} = f \circ g$ holds. Then it suffices to show that $\widetilde{f}^{-1}(D_x)_{\mathrm{red}}$ and $\widetilde{f}^{-1}(D_y)_{\mathrm{red}}$ are irreducible, and this is obvious. $\square$

Remark 4.3. If we fix a topological generator of $\prod_{l\neq p} \mathbb{Z}_l(1)$ and identify G (resp. H) with $\bigoplus_{i=1}^{s} \mathbb{Z}/r_i\mathbb{Z}$ (resp. $(\prod_{l\neq p} \mathbb{Z}_l)^2$) by using it (resp. it and the regular parameter x, y), the homomorphism $\varphi : H \longrightarrow G$ is expressed by $\varphi((1,0)) = (a_i)_{i=1}^{s}$, $\varphi((0,1)) = (b_i)_{i=1}^{s}$.

Proof of Theorem 4.1 Let us prepare the notation as follows:

$$\overline{Z} := Z \times_Y \overline{Y}, \qquad \overline{E}_i := E_i \times_Y \overline{Y}\,(i \in I \cup J), \quad \overline{E} := \sum_{j \in J} \overline{E}_j,$$

$$\overline{v} = v \times_Y \overline{Y} : \overline{W} \longrightarrow \overline{Z}.$$

First let us note the following claim:

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CLAIM 1. *With the above notations, we have the following*:

(1) *The singular points of* W *(resp.* $\overline{W}$*) is contained in* $v^{-1}(\bigcup_{i,j\in I\cup J}(E_i \cap E_j))$ *(resp.* $\overline{v}^{-1}(\bigcup_{i,j\in I\cup J}(\overline{E}_i \cap \overline{E}_j))$*), and they are all toric singularities.*
(2) *The intersections of irreducible components of* $\widetilde{E}$ *(resp.* $\widetilde{\overline{E}}$*) are double points.*

Proof. It is easy to see the former assertion of (1). As for the latter assertion of (1) and the assertion (2), it suffices to show for $\overline{W}$ and $\widetilde{\overline{E}}$. Let F be a double point of $\bigcup_{i\in I\cup J}(\overline{E}_i \cap \overline{E}_j)$, and let $\hat{\overline{Z}}, \hat{\overline{W}}$ be the completion of $\overline{Z}, \overline{W}$ at $F, \overline{v}^{-1}(F)$ respectively. Then, it suffices to show the assertions after pulling back to $\hat{\overline{Z}}, \hat{\overline{W}}$, since the assertions are local. Then they are reduced to (2) and (3). of Lemma 4.2 for $V = \hat{\overline{W}}, U = \hat{\overline{Z}}$.

By (2) of the above claim, we can define the dual graphs $\Gamma(\widetilde{E}), \Gamma(\widetilde{\overline{E}})$ of $\widetilde{E}, \widetilde{\overline{E}}$ respectively. Since the action of G on W induces the action of G on $\widetilde{E}$ and $\widetilde{\overline{E}}$, we can regard $\Gamma(\widetilde{E}), \Gamma(\widetilde{\overline{E}})$ as G-graphs, and by the assumption (A3), there exists a natural isomorphism of G-graphs $\Gamma(\widetilde{E}) \cong \Gamma(\widetilde{\overline{E}})$. By this observation, it suffices to show the following claims:

CLAIM 2. *There is an isomorphism of* G*-graphs* $\Gamma(\widetilde{\overline{E}}) \cong \widetilde{\Delta}$.

CLAIM 3. *There is an isomorphism of* G*-modules*

$$H_1(\Gamma(\widetilde{E}), \mathbb{Z}) \otimes \hat{\mathbb{Z}} \cong \mathrm{Gal}(L^{\mathrm{cs}}/L).$$

Proof of Claim 2. By using the assumption (A3), one can check that the dual graph of E and that of $\overline{E}$ are also isomorphic. Since they are trees, it suffices to show the following (cf. [T, Section 2]):

(1) For $i, j \in J$, we have the isomorphism of G-sets

$$\overline{v}^{-1}(\overline{E}_i \cap \overline{E}_j)_{\mathrm{red}} \cong G/G_{ij}.$$

(2) For $i \in J$, we have the isomorphism of G-sets

$$\left\{ \begin{array}{l} \text{the irreducible components} \\ \text{of } \ v^{-1}(\overline{E}_i) \end{array} \right\} \cong G/G_i.$$

First we prove the assertion (1). Let $\hat{\overline{W}}$ be the completion of $\overline{W}$ at $\overline{v}^{-1}(\overline{E}_i \cap \overline{E}_j)$. Then, by definition of G_{ij} and by Lemma 4.2 (1) (for $V = \hat{\overline{W}}$), the stabilizer of an element of $\pi_0(\hat{\overline{W}})$ is G_{ij}. Then, noting the isomorphism of G-sets

$$\overline{v}^{-1}(\overline{E}_i \cap \overline{E}_j)_{\mathrm{red}} \cong \pi_0(\hat{\overline{W}}),$$

we obtain the assertion.

Next we prove (2). By the argument of the previous paragraph and (3) of Lemma 4.2 (for $V = \hat{\overline{W}}$), one can see the following: For an irreducible component F in

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$\overline{v}^{-1}(\overline{E}_i)$ and for any $j \in I \cup J$ such that $\overline{E}_i \cap \overline{E}_j \neq \emptyset$, we have

$$\{g \in G \mid g(F) = F\} \supset G_{ij}.$$

So, by definition of G_i, we have $\{g \in G \mid g(F) = F\} \supset G_i$. On the other hand, one can check that the covering $F/G_i \longrightarrow \overline{E}_i$ is unramified. Since $\overline{E}_i$ is isomorphic to $\mathbb{P}^1_{\overline{k}}$ (where $\overline{k}$ is the algebraic closure of k), we have $F/G_i \cong \overline{E}_i$ and $\{g \in G \mid g(F) = F\} = G_i$. Hence, we obtain the assertion. □

Proof of Claim 3. Let $a\colon X' \longrightarrow X$ be the normalization of X and let P' be the unique point in $a^{-1}(P)$. Let $b\colon \mathcal{X} \longrightarrow W$ be a resolution of W defined by succesive toric blow-ups at singular points such that $b^{-1}(\widetilde{E})_{\mathrm{red}}$ is a simple normal crossing divisor. (Note that W has only toric singularities.) Then, there exists a morphism $c : \mathcal{X} \longrightarrow X'$ such that the following diagram is commutative:

$$\begin{tikzcd}
X' \arrow[d, "a"'] & \mathcal{X} \arrow[l, "c"'] \arrow[d, "b"] \\
X & W. \arrow[l, "\lambda"']
\end{tikzcd}$$

Let X_0 be $X - P(\cong X' - P')$. Then we have the following diagram:

$$\begin{tikzcd}
\pi_1^{\mathrm{ab}}(X_0) \arrow[r] \arrow[d, equal] & \pi_1^{\mathrm{ab}}(\mathcal{X}) \arrow[d] & \pi_1^{\mathrm{ab}}(c^{-1}(P')_{\mathrm{red}}) \arrow[l, "\sim"'] \arrow[d] \\
\pi_1^{\mathrm{ab}}(X_0) \arrow[r] & \pi_1^{\mathrm{ab}}(W) & \pi_1^{\mathrm{ab}}(\widetilde{E}). \arrow[l, "\sim"']
\end{tikzcd}$$

It induces the following diagram:

$$\begin{tikzcd}
\pi_1^{\mathrm{ab}}(X_0) \arrow[r, "\alpha"] \arrow[d, equal] & \pi_1^{\mathrm{cs}}(c^{-1}(P')_{\mathrm{red}}) \arrow[d] \\
\pi_1^{\mathrm{ab}}(X_0) \arrow[r, "\beta"] & \pi_1^{\mathrm{cs}}(\widetilde{E}),
\end{tikzcd}$$

where π_1^{cs} is the quotient of algebraic fundamental group which classifies complete splitting Abelian coverings ([S1, II,2]). Moreover, β is G-equivariant, since it is induced by λ, which is G-equivariant by definition.

By [S1, II, Proposition 2.2], α induces the isomorphism $\pi_1^{\mathrm{cs}}(X_0) \cong \pi_1^{\mathrm{cs}}(c^{-1}(P')_{\mathrm{red}})$. Now let us admit the following claim for a while, whose proof will be given later:

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CLAIM 4. *There exists a commutative diagram*

$$\begin{tikzcd}
\pi_1^{\mathrm{cs}}(c^{-1}(P')_{\mathrm{red}}) \arrow[r, "\gamma"] \arrow[d] & H_1(\Gamma(c^{-1}(P')_{\mathrm{red}}), \mathbb{Z}) \otimes \hat{\mathbb{Z}} \arrow[d, "\delta"'] \\
\pi_1^{\mathrm{cs}}(\tilde{E}) \arrow[r, "\epsilon"] & H_1(\Gamma(\widetilde{E}), \mathbb{Z}) \otimes \hat{\mathbb{Z}},
\end{tikzcd}$$

such that γ, δ, ϵ *are isomorphisms and* ϵ *is* G*-equivariant.*

Then, $\epsilon \circ \beta$ induces the G-equivariant isomorphism

$$\pi_1^{\mathrm{cs}}(X_0) \longrightarrow H_1(\Gamma(\widetilde{E}), \mathbb{Z}) \otimes \hat{\mathbb{Z}}.$$

Since we have $\mathrm{Gal}(L^{\mathrm{cs}}/L) = \pi_1^{\mathrm{cs}}(X_0)$ by definition ([S1], [S2]), we are done.

Hence Theorem 4.1 is reduced to Claim 4. Now let us prove this claim. Let $\mathcal{C}$ be the category of connected reduced schemes C of finite type over k which satisfy the following conditions:

(1) Each irreducible component $C_i\,(i \in I)$ is one-dimensional and the set S_C of singular points of C (regarded as the reduced closed subscheme of C) is equal to $(\bigcup_{i,j \in I}(C_i \cap C_j))_{\mathrm{red}}$.
(2) S_C is a disjoint union of k-rational points and for any $s \in S_C$, there exist exactly two irreducible components of C which contain s. (That is, any point $s \in S_C$ is a double point.)

Now we introduce the notion of modification in the category $\mathcal{C}$: Let $\varphi : C' \longrightarrow C$ be a morphism in the category $\mathcal{C}$ and let $s \in S_C$. Then φ is called a modification at s if $\varphi|_{C' - \varphi^{-1}(s)} : C' - \varphi^{-1}(s) \longrightarrow C - \{s\}$ is an isomorphism and $\varphi^{-1}(s) \subset C'$ is an irreducible component of C' which is smooth over k. For a modification $\varphi : C' \longrightarrow C$ at $s \in S_C$ and an irreducible component C_i of C, we define the proper transform of C_i as the closure of $\varphi^{-1}(C_i - \{s\})$.

Then $c^{-1}(P)_{\mathrm{red}}$ and $\widetilde{E}$ are in the category $\mathcal{C}$ and the morphism $b|_{c^{-1}(P)_{\mathrm{red}}} : c^{-1}(P)_{\mathrm{red}} \longrightarrow \widetilde{E}$ is a composite of modifications, since b is a composite of toric blow-ups at k-rational points. So, to prove Claim 4, it suffices to show the following claim:

CLAIM 5. *For* $C \in \mathcal{C}$, *denote its dual graph by* $\Gamma(C)$. *Then we have a system of isomorphisms* $\{f_C : \pi_1^{\mathrm{cs}}(C) \xrightarrow{\sim} H_1(\Gamma(C), \mathbb{Z}) \otimes \hat{\mathbb{Z}}\}_{C \in \mathcal{C}}$ *which satisfies the following conditions*:

(1) *If* C *admits an action of a group* G, *then* f_C *is* G*-equivariant.*
(2) *For any modification* $\varphi : C' \longrightarrow C$, *there exists an isomorphism* $g : H_1(\Gamma(C'), \mathbb{Z}) \otimes \hat{\mathbb{Z}} \longrightarrow H_1(\Gamma(C), \mathbb{Z}) \otimes \hat{\mathbb{Z}}$ *which makes the following diagram com-*

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mutative:

$$\begin{array}{ccc} \pi_1^{\mathrm{cs}}(C') & \xrightarrow{f_{C'}} & H_1(\Gamma(C'), \mathbb{Z}) \otimes \hat{\mathbb{Z}} \\ {\scriptstyle\varphi_*}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \pi_1^{\mathrm{cs}}(C) & \xrightarrow{f_C} & H_1(\Gamma(C), \mathbb{Z}) \otimes \hat{\mathbb{Z}}. \end{array}$$

Now we prove Claim 5. For $C \in \mathcal{C}$ and a closed subscheme $D \subset C$, denote the Henselization of C at D by C_D. Since the argument in the proof of [S1, II, Theorem 2.4] is valid for any $C \in \mathcal{C}$, we have the exact sequence

$$H^0_{\mathrm{et}}(C - S_C, \mathbb{Q}/\mathbb{Z}) \xrightarrow{h_C} \bigoplus_{x \in S_C} \frac{H^0_{\mathrm{et}}(C_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \longrightarrow \pi_1^{\mathrm{cs}}(C)^* \longrightarrow 0.$$

One can check easily the isomorphisms

$$H^0_{\mathrm{et}}(C - S_C, \mathbb{Q}/\mathbb{Z}) \cong \bigoplus_{i \in I} \mathbb{Q}/\mathbb{Z}, \quad \frac{H^0_{\mathrm{et}}(C_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \cong \mathbb{Q}/\mathbb{Z},$$

and via these isomorphisms, h_C is nothing but the cochain homomorphism of the dual graph $\Gamma(C)$. Hence, we can identify $\mathrm{Coker}(h_C)$ naturally with the dual of $H_1(\Gamma(C), \mathbb{Z}) \otimes \hat{\mathbb{Z}}$. We define the homomorphism f_C as the dual of the natural isomorphism $\mathrm{Coker}(h_C) \xrightarrow{\sim} \pi_1^{\mathrm{cs}}(C)^*$. Then one can easily check that the homomorphism f_C satisfies the condition (1). of the claim.

Now let $C \in \mathcal{C}$, $s \in S_C$ and let $\varphi : C' \longrightarrow C$ be a modification at s. Put $S := \varphi^{-1}(S_C)$. Then, the diagram in [S1, p.60] is functorial with respect to the morphism of pairs $(C', S) \longrightarrow (C, S_C)$, since it is induced by the localization sequence of etale cohomology. Moreover, since $\varphi^{-1}(s)$ is smooth over k, we have (cf. [S1, p.61])

$$\begin{aligned} \pi_1^{\mathrm{cs}}(C')^* = \mathrm{Ker}(&H^1_{\mathrm{et}}(C', \mathbb{Q}/\mathbb{Z}) \longrightarrow H^1_{\mathrm{et}}(C' - S, \mathbb{Q}/\mathbb{Z}) \oplus H^1_{\mathrm{et}}(\varphi^{-1}(s), \mathbb{Q}/\mathbb{Z}) \oplus \\ &\bigoplus_{x \in \varphi^{-1}(S_C - \{s\})} H^1_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})). \end{aligned}$$

Hence we obtain the following commutative diagram:

$$\begin{array}{ccccccc} H^0_{\mathrm{et}}(C - S_C, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{h_C} & \displaystyle\bigoplus_{x \in S_C - \{s\}} \frac{H^0_{\mathrm{et}}(C_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \oplus \frac{H^0_{\mathrm{et}}(C_s - \{s\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(s, \mathbb{Q}/\mathbb{Z})} & \longrightarrow & \pi_1^{\mathrm{cs}}(C)^* & \longrightarrow & 0 \\ {\scriptstyle a}\big\downarrow & & \big\downarrow{\scriptstyle b} & & \big\downarrow{\scriptstyle \varphi^*} & & \\ H^0_{\mathrm{et}}(C' - S, \mathbb{Q}/\mathbb{Z}) & \xrightarrow{j} & \displaystyle\bigoplus_{x \in \varphi^{-1}(S_C - \{s\})} \frac{H^0_{\mathrm{et}}(C'_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \oplus \frac{H^0_{\mathrm{et}}(C'_{\varphi^{-1}(s)} - \varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(\varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})} & \longrightarrow & \pi_1^{\mathrm{cs}}(C')^* & \longrightarrow & 0, \end{array}$$

where φ^* is the dual of φ_*. Since the morphism $C' - S \longrightarrow C - S_C$ is an isomorphism, a is an isomorphism, and the first factor of the homomorphism b is also an isomorphism. Moreover, if we set $s \in C_{i_1} \cap C_{i_2}$, $\varphi^{-1}(s)$ meets with the proper transforms of C_{i_1} and C_{i_2}, and does not meet with other irreducible components of

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C'. Hence, we have

$$\frac{H^0_{\mathrm{et}}(C'_{\varphi^{-1}(s)} - \varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(\varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})} \cong \mathbb{Q}/\mathbb{Z}$$

and one can check easily that the second factor of b is also an isomorphism. Therefore, we have the commutative diagram

$$\begin{tikzcd}
\mathrm{Coker}\,(h_C) \arrow[r, "f_C^*"] \arrow[d, "g_1"'] & \pi_1^{\mathrm{cs}}(C)^* \arrow[d, "\varphi^*"'] \\
\mathrm{Coker}\,(j) \arrow[r, "\sim"] & \pi_1^{\mathrm{cs}}(C')^*,
\end{tikzcd}$$

where all the homomorphisms are isomorphisms.

Next, let us consider the functoriality of the diagram in [S1, p.60] with respect to the inclusion $S_{C'} \subset S$. Then we obtain the following diagram:

$$\begin{tikzcd}
H^0_{\mathrm{et}}(C' - S, \mathbb{Q}/\mathbb{Z}) \arrow[r, "j"] & \displaystyle\bigoplus_{x \in \varphi^{-1}(S_C - \{s\})} \frac{H^0_{\mathrm{et}}(C'_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \oplus \frac{H^0_{\mathrm{et}}(C'_{\varphi^{-1}(s)} - \varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(\varphi^{-1}(s), \mathbb{Q}/\mathbb{Z})} \arrow[r] & \pi_1^{\mathrm{cs}}(C')^* \arrow[r] & 0 \\
H^0_{\mathrm{et}}(C' - S_{C'}, \mathbb{Q}/\mathbb{Z}) \arrow[r, "h_{C'}"] \arrow[u] & \displaystyle\bigoplus_{x \in S_{C'}} \frac{H^0_{\mathrm{et}}(C_x - \{x\}, \mathbb{Q}/\mathbb{Z})}{H^0_{\mathrm{et}}(x, \mathbb{Q}/\mathbb{Z})} \arrow[r] \arrow[u] & \pi_1^{\mathrm{cs}}(C)^* \arrow[r] \arrow[u] & 0,
\end{tikzcd}$$

So the following diagram is induced:

$$\begin{tikzcd}
\mathrm{Coker}\,(j) \arrow[r, "\sim"] & \pi_1^{\mathrm{cs}}(C')^* \arrow[d, equal] \\
\mathrm{Coker}\,(h_{C'}) \arrow[r, "f_{C'}^*"] \arrow[u, "g_2"] & \pi_1^{\mathrm{cs}}(C')^*.
\end{tikzcd}$$

Since $f_{C'}^*$ is an isomorphism, g_2 is also an isomorphism. If we define the homomorphism $g : H_1(\Gamma(C'), \mathbb{Z}) \otimes \hat{\mathbb{Z}} \longrightarrow H_1(\Gamma(C), \mathbb{Z}) \otimes \hat{\mathbb{Z}}$ as the dual of the composite $g_2^{-1} \circ g_1$, we have the diagram in the condition (2). in Claim 5. Hence Claim 5 is proved, and so the proof of Theorem 4.1 is now finished. $\square$

Remark 4.4. Let the notations be as in this section and assume that the conditions (A1) and (A2) are satisfied. Then, the condition (A3) is also satisfied if we replace A by some unramified extension of A.

5. Proofs (II)

In this section, first we give two examples of Abelian extensions $K \subset L$ such that $c(L/K)$ is computable by using the result of previous sections. Then we give a proof of Proposition 1.10 and Theorem 1.12.

EXAMPLE 5.1. Let $n, a \in \mathbb{N}$ and let p and l be distinct primes. Let k be a finite field of characterictic p such that $|k| > a$ holds and k contains a primitive l^n-th root of unity. Let A be $k[[x, y]]$ (resp. $W(k)[[y]]$) and let K be the fraction field of A. For $\alpha \in k$ (resp. $\alpha \in W(k)$), define g_α by $g_\alpha := (y + \alpha x)^l + x^{l+1}$ (resp. $g_\alpha := (y + \alpha p)^l + p^{l+1}$.) Fix a subset I of k (resp. $W(k)$) such that $|I| = a + 1$ (resp. $|I| = a + 1$ and the map $I \hookrightarrow W(k) \longrightarrow k$ is injective). Let $f_1 \in A$ be $\prod_{\alpha \in I} g_\alpha$ and let $L := K[z_1]/(z_1^{l^n} - f_1)$. Then,

PROPOSITION 5.2. *The conditions* (A1), (A2) *are satisfied for the extension* $K \subset L$.

The condition (A1) is trivial. To show the condition (A2), first note the following lemma:

LEMMA 5.3. *Let the notations be as above and let* $\overline{A}$ *be as in the previous section. Then,*

(1) g_α is a prime element of $\overline{A}$ for any $\alpha \in \overline{A}$.
(2) g_α and g_β are coprime in $\overline{A}$ for $\alpha, \beta \in I, \alpha \neq \beta$.

Proof. In this proof, we denote the element $p \in A$ by x in the case $A = W(k)[[y]]$ to simplify the description of the proof.

First we prove (1). Let $\varphi_\alpha : \overline{A} \longrightarrow \overline{A}$ be the automorphism defined by

$$\varphi_\alpha\left(\sum_i c_i y^i\right) = \sum_i c_i (y - \alpha x)^i,$$

where $c_i \in k[[x]]$ (resp. $W(k)$) in the case $A = k[[x, y]]$ (resp. $A = W(k)[[y]]$). Then we have $\varphi_\alpha(g_\alpha) = g_0$. So it suffices to show that g_0 is a prime element of $\overline{A}$.

Proof. Let us define the order $\succ$ on $\mathbb{N}^2$ by

$$(i, j) \succ (i', j') \iff i + j > i' + j' \quad \text{or} \quad i + j = i' + j' \text{ and } j \geqslant j'.$$

For $\alpha \in \mathbb{N}^2$, let $\alpha' \in \mathbb{N}^2$ be the unique element which satisfies the following two conditions:

(1) $\alpha' \neq \alpha$ and $\alpha' \succ \alpha$ hold.
(2) If $\beta \in \mathbb{N}^2$ satisfies $\beta \neq \alpha$ and $\beta \succ \alpha$, then $\beta \succ \alpha'$ holds.

Let $\mathfrak{m}$ be the maximal ideal of $\overline{A}$. For $(i, j) \in \mathbb{N}^2$, let $I_{(i,j)}$ be the ideal of $\overline{A}$ which is generated by $\mathfrak{m}^{i+j+1}$ and the element of the form $x^a y^b$ $((a, b) \in \mathbb{N}^2, (a, b) \succ (i, j))$. Take a multiplicatively closed subset $\Lambda \subset \overline{A}$ of $\overline{A}$ containing $0, 1$ such that the map $\Lambda \hookrightarrow \overline{A} \longrightarrow \overline{A}/\mathfrak{m} =: \overline{k}$ is bijective. Then, since $I_{(i,j)}/I_{(i,j)'} \cong \overline{k}x^i y^j$ holds for any $(i,j) \in \mathbb{N}^2$, we have the following:

(*) For any nonzero element f of $\overline{A}$, there exists a unique pair $((i,j), \lambda) \in \mathbb{N}^2 \times (\Lambda - \{0\})$ such that $f \in \lambda x^i y^j + I_{(i,j)'}$ holds.

Now let us assume that $g_0 = h_1 h_2$ holds. Then there exist $((i_1, j_1), \lambda_1), (i_2, j_2), \lambda_2) \in \mathbb{N}^2 \times (\Lambda - \{0\})$ satisfying

$$h_1 \in \lambda_1 x^{i_1} y^{j_1} + I_{(i_1,j_1)'}, \qquad h_2 \in \lambda_2 x^{i_2} y^{j_2} + I_{(i_2,j_2)'}.$$

Then, we have

$$g_0 = h_1 h_2 \in \lambda_1 \lambda_2 x^{i_1+i_2} y^{j_1+j_2} + I_{(i_1+i_2,j_1+j_2)'}.$$

On the other hand, we have $g_0 = y^l + x^{l+1} \in y^l + I_{(l+1,0)}$. Hence, by the uniqueness of the expression in $(*)$, we have $\lambda_1\lambda_2 = 1, i_1 = i_2 = 0$ and $j_1 + j_2 = l$.

Then, by using $(*)$ repeatedly, one can write

$$h_1 \in \lambda_1 y^{j_1} + \sum_{\nu=0}^{j_1+1} \mu_{1,\nu} x^\nu y^{j_1+1-\nu} + I_{(j_1+2,0)}, \ (\mu_{1,\nu} \in \Lambda),$$
$$h_2 \in \lambda_2 y^{j_2} + \sum_{\nu=0}^{j_2+1} \mu_{2,\nu} x^\nu y^{j_2+1-\nu} + I_{(j_2+2,0)}, \ (\mu_{2,\nu} \in \Lambda).$$

For $\nu > j_1 + 1$ (resp. $\nu > j_2 + 1$), put $\mu_{1,\nu} = 0$ (resp. $\mu_{2,\nu} = 0$). Then, we have

$$g_0 - y^l = h_1 h_2 - y^l \in \sum_{\nu=0}^{\max(j_1,j_2)+1} (\lambda_1 \mu_{2,\nu} + \lambda_2 \mu_{1,\nu}) x^\nu y^{l+1-\nu} + I_{(l+2,0)}.$$

If we assume $j_1 j_2 > 0$, then $l + 1 - \nu$ is always positive on the right-hand side. Hence we have $g_0 - y^l \in I_{(l,1)}$. On the other hand, we have $g_0 - y^l = x^{l+1} \in x_{l+1} + I_{(l,1)}$. It is a contradiction. Hence, we have $j_1 j_2 = 0$. Then, either h_1 or h_2 is invertible. So g is a prime element in $\overline{A}$, which is a UFD.

Next we prove (2). By using the automorphism φ_α, we may assume that $\alpha = 0$ and β is invertible (but not necessarily in I). Assume g_0 and g_β are not coprime. Then there exists an element $u \in \overline{A}^\times$ such that $g_\beta = u g_0$ holds.

Let $\overline{\beta}$ and $\overline{u}$ be the image of β and u in $\overline{A}/(y, x^{l+1})$ respectively. Then, from the equality $g_\beta = u g_0$, we obtain

$$\overline{\beta}^l x^l = 0 \ \text{in} \ \overline{A}/(y, x^{l+1}).$$

But one can check that it is impossible, since $\overline{\beta}$ is invertible. Hence we obtain the assertion.

Proof of Proposition 5.2 (A2). By Lemma 5.3, $f_1 \in \overline{A}$ is a product of distinct primes. Since $\overline{A}$ is a UFD, one can see that the order of f_1 in $\overline{K}^\times/(\overline{K}^\times)^{l^n}$ (where $\overline{K} := \mathrm{Frac}\overline{A}$) is equal to l^n. Then, by Kummer theory, $\overline{L} = \overline{K}[z_1]/(z^{l^n} - f_1)$ is a field. $\square$

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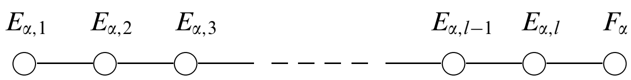
Now let us replace k by a finite extension so that the assumption (A3) is also satisfied. Let $D, \theta, \Delta, \widetilde{\Delta}$ be as in the previous section and let $C_\alpha\,(\alpha \in I)$ be the prime divisor defined by the ideal $(g_\alpha) \subset A$. Then one can check, by direct calculation, the following descriptions of $D, \theta^{-1}(D)_{\mathrm{red}}, \theta^* D$ and Δ:

(1) $D = \sum_{\alpha \in I} C_\alpha$.
(2) $\theta^{-1}(D)_{\mathrm{red}} := \sum_{\alpha\in I} E_\alpha + \sum_{1 \leqslant i \leqslant l, \alpha \in I} E_{\alpha,i} + E_0$, where $E_\alpha, E_{\alpha,i}$ and E_0 are all irreducible and E_α is the proper transform of C_α. For any $\alpha \in I$, E_α meets with $E_{\alpha,1}$ at one point and does not meet with the other components.
(3) $\theta^* D := \sum_{\alpha\in I} E_\alpha + \sum_{1 \leqslant i \leqslant l, \alpha\in I} i(al+1)E_{\alpha,i} + alE_0$.
(4) Let $\Gamma_\alpha\,(\alpha \in I)$ be the following graph:



Then, Δ is defined by $\Delta := \coprod_{\alpha\in I} \Gamma_\alpha / \sim$, where $\sim$ is the equivalent relation generated by $F_\alpha \sim F_\beta$ for $\alpha, \beta \in I$, and E_0 is defined to be the vertex $\mathrm{Im}(F_\alpha \mid \Gamma_\alpha \to \Delta)$.

From these descriptions, the structure of the G-graph $\widetilde{\Delta}$ is described as follows. First, let $\Gamma'_\alpha\,(\alpha \in I)$ be the graph in Figure 1. Then, as a graph, $\widetilde{\Delta}$ is expressed as $\widetilde{\Delta} = \coprod_{\alpha\in I} \Gamma'_\alpha / \sim$, where $\sim$ is the equivalent relation generated by $F^i_\alpha \sim F^i_\beta\,(\alpha, \beta \in I)$. Denote the vertex $\mathrm{Im}(F^i_\alpha \mid \Gamma'_\alpha \to \widetilde{\Delta})$ by $E^i_0\,(0 \leqslant i \leqslant l-1)$. Then the action of $G \cong \mathbb{Z}/l\mathbb{Z}$ on $\widetilde{\Delta}$ is described as follows: Fix a generator σ of G. Then the action of σ on $\widetilde{\Delta}$ is defined by

$$\sigma(E_{\alpha,i}) = E_{\alpha,i}, \qquad \sigma(E^i_0) = E^{i+1}_0 \;(0 \leqslant i \leqslant l-2), \qquad \sigma(E^{l-1}_0) = E^0_0.$$

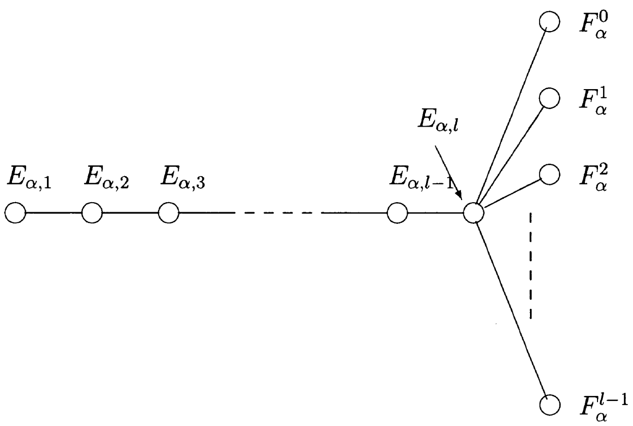


Figure 1. The graph Γ'_α.

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Then, by Corollary 3.4, Theorem 4.1 and direct calculation, we have

$$d(L/K) = l^n, \;\; r(L) = a(l-1), \;\; r(K) = 0, \;\; c(L/K) = l^{a+n}.$$

Let L' be $L^{\mathrm{cs}} \cap K^{\mathrm{ab}}$. Then we have

$$d(L'/K) = c(L/K) > d(L/K) \quad \text{and} \quad d'(L'/L) = 1.$$

On the other hand, we have

$$d'(L'/K) = d(L'/K), d'(L/K) = d(L/K),$$

since $r(K)$ is equal to zero. Hence, we have

$$d'(L'/L)d'(L/K) = d(L/K) < d(L'/K) = d'(L'/K).$$

In particular, the 'multiplicativity' does not hold for $d'(-/-)$.

EXAMPLE 5.4. Let k be a finite field of characteristic at least 5 which contains a primitive third root of unity. Let A be $k[[x,y]]$ and let K be the fraction field of A. Define $f_1, f_2 \in A$ by $f_1 := y^2 + x^3, f_2 := (y+x)^3 + x^4$. Let L be $K[z_1, z_2]/(z_1^2 - f_1, z_2^3 - f_2)$. Then, one can check, by using Lemma 5.3, that the conditions (A1), (A2) are satisfied for $K \subset L$.

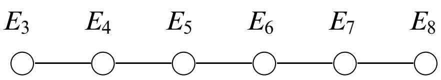
Now let us replace k by a finite extension so that the assumption (A3) is also satisfied. Let $D = D_1 + D_2, \theta, \Delta, \widetilde{\Delta}$ be as in the previous section. Then D_i is a prime divisor for $i = 1, 2$ by Lemma 5.3 and one can check, by direct calculation, the following descriptions of $\theta^{-1}(D)_{\mathrm{red}}$, $\theta^* D_1$, $\theta^* D_2$ and Δ:

(1) $\theta^{-1}(D)_{\mathrm{red}} = \sum_{i=1}^{8} E_i$, where $E_i \, (1 \leqslant i \leqslant 8)$ are irreducible and E_i is the proper transform of D_i for $i = 1, 2$. E_1 meets only with E_3 at one point and E_2 meets only with E_8 at one point.

(2)

$$\theta^* D_1 = E_1 + 3E_3 + 6E_4 + 2E_5 + 6E_6 + 4E_7 + 2E_8,$$
$$\theta^* D_2 = E_2 + 3E_3 + 6E_4 + 3E_5 + 12E_6 + 8E_7 + 4E_8.$$

(3) The graph Δ is as follows:



From these descriptions, the structure of the G-graph $\widetilde{\Delta}$ (here $G \cong \mathbb{Z}/6\mathbb{Z}$) is described as follows. As a graph, $\widetilde{\Delta}$ is as in Figure 2. The action of G on $\widetilde{\Delta}$ is described as follows: let us fix a generator σ of G. Then the action of σ is defined by

$$\sigma(E_{i,j}) = \begin{cases} E_{i,0} & \text{if } (i,j) = (3,2), (4,2), (5,5), (6,1), (7,1), (8,1), \\ E_{i,j+1} & \text{otherwise.} \end{cases}$$

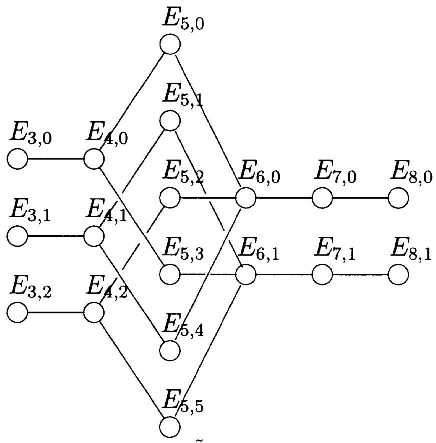


Figure 2. The graph $\tilde{\Delta}$.

Let M_1 (resp. M_2) be the intermediate field between K and L such that $[L : M_1] = 2$ (resp. $[L : M_2] = 3$) holds. Then, by using the above description of G-graph $\widetilde{\Delta}$ and Corollary 3.4, Theorem 4.1, One can check the equalities $c(L/M_1) = 2^3$, $c(L/M_2) = 3^2$. One can also check the equality $r(M_1) = r(M_2) = 0$. So one has $c(M_1/K) = 3, c(M_2/K) = 2$. By definition, both $c(L/M_1)c(M_1/K)$ and $c(L/M_2)c(M_2/K)$ are divisible by $c(L/K)$. Hence 6 is divisible by $c(L/K)$. On the other hand, we have $c(L/K) \geqslant d'(L/K) = 6$. So we have $c(L/K) = 6$. In particular, we have

$$c(L/M_i)c(M_i/K) \neq c(L/K) \ (i = 1, 2).$$

Hence the 'multiplicativity' does not hold for $c(-/-)$.

In this example, $K \subset L$ is a cyclic extension of degree 6 and we have $r(K) = 0, r(L) = 2, d'(L/K) = c(L/K) = 6$. So it gives an example which we mentioned in Remark 1.13(2).

Now we give a proof of Proposition 1.10 and Theorem 1.12.

Proof of Proposition 1.10. By Example 5.1, it suffices to show the following:

CLAIM. *In the situation of Example 5.1, let us assume the following: k contains a primitive l-th root of unity, $|k| > a$ holds and $n = 1$. Then, the condition* (A3) *is automatically satisfied for $K \subset L$. (That is, we do not have to enlarge k in the case $n = 1$.)*

Now we prove this claim. Let $v\colon W \longrightarrow Z$ as in the previous section and let $E_0 \subset Z$ be as in Example 5.1. Then it suffices to show that $v^{-1}(E_0)$ has l connected components. By the proof of Claim 2 in the proof of Theorem 4.1, $v^{-1}(E_0)$ is etale over $E_0 \cong \mathbb{P}^1_k$. So, it suffices to show that $v^{-1}(R)$ has l connected components for a k-rational point R in E_0.

Take a k-rational point R of $E_0 - \bigcup_{\alpha \in I} E_{\alpha,l}$, where the notations are as in Example 5.1. (It is possible since $E_0 \cong \mathbb{P}^1_k$ and $|k| > a$ holds.) Then, by easy calculation, one can check the following: There exists a regular parameter $x, y \in \hat{\mathcal{O}}_{Z,R}$ and $b \in \hat{\mathcal{O}}^{\times}_{Z,R}$ such that

$$\hat{\mathcal{O}}_{Z \times_Y X, p^{-1}(R)} \cong \hat{\mathcal{O}}_{Z,R}[z]/(z^l - b^l x^{(a+1)l})$$

holds. (Here, $\hat{}$ denotes the completion with respect to the Jacobson radical and $p : Z \times_Y X \longrightarrow Z$ is the projection.) Then we have

$$\begin{aligned}\hat{\mathcal{O}}_{W, v^{-1}(R)} &\cong \text{normalization of } \hat{\mathcal{O}}_{Z \times_Y X, p^{-1}(R)} \\ &\cong \prod_{\zeta^l = 1} \hat{\mathcal{O}}_{Z,R}[z]/(z - \zeta b x^{a+1}).\end{aligned}$$

So $v^{-1}(R)$ has l connected components, as is desired. □

Proof of Theorem 1.12. First we prove the assertion (2). If $r(K) > 0$ holds, then it is easy to see that Ψ_K is not injective. So we may assume $r(K)$ is equal to 0 to prove the assertion (2). Let A_0 be $k[[x, y]]$ if the characterictic of K is positive and $W(k)[[y]]$ if the characteristic of K is equal to zero. Fix a prime number l which divides $|k| - 1$. (Here we used the assumption $k \neq \mathbb{F}_2$.) Then, by Proposition 1.10, there exists an Abelian extension $K_0 \subset L_0$ such that $d(L_0/K_0) = l$ and $r(L_0) = l - 1 > 0$ hold. Fix an inclusion $A_0 \subset A$ such that A is finite over A_0, and let $K_0 \subset K$ be the induced inclusion. Let L be the composite field KL_0. (It is Abelian over K.) Then we have the following:

CLAIM. *$r(L) > 0$ holds.*

Proof. Put $m := [L : L_0]$. Let p be a prime number which does not divide m, and let L_0' be a cs extension of L_0 such that $[L_0' : L_0] = p$ holds. Then, since L_0' and L are linearly disjoint over L_0, we have $[LL_0' : L] = p$. On the other hand, one can see that $L \subset LL_0'$ is also a cs extension. Hence L has a nontrivial cs extension. So $r(L)$ is positive. □

By the claim, we have $r(L) > 0, r(K) = 0$ and $[L : K] = l$. Then, by the last inequality in Theorem 1.8(3), $c(L/K) > l = d(L/K)$ holds. So, by Proposition 2.7, Ψ_K is not surjective.

Next we prove the assertion (1). Let A_0, K_0 be as above and fix an inclusion $A_0 \subset A$ as above. Let $K_0 \subset K$ be the induced inclusion and put $d := [K_0 : K]$. For a finite extension $\mathfrak{k}$ of k, Let $K_0(\mathfrak{k}), K(\mathfrak{k})$ be the unramified extension of K_0, K with residue field $\mathfrak{k}$, respectively. Then one can see that $[K(\mathfrak{k}) : K_0(\mathfrak{k})] = d$ holds for any $\mathfrak{k}$. First, let us note the following claim:

CLAIM. *$r := \sup\{r(K(\mathfrak{k})) \mid k \subset \mathfrak{k}$ finite extension$\} < \infty$.*

Proof. Let P be the closed point of Spec A, let $\pi : X \longrightarrow \text{Spec}\, A$ be a resolution and let Y be $\pi^{-1}(P)_{\text{red}}$. For a finite extension $\mathfrak{k}$ of k, let $\Gamma_{\mathfrak{k}}$ be the dual graph of

$Y_{\mathfrak{k}} := Y \otimes_k \mathfrak{k}$. Then we have $r(K(\mathfrak{k})) = \mathrm{rk}\, H_1(\Gamma_{\mathfrak{k}}, \mathbb{Z})$. Let $\mathfrak{k}_0$ be a finite extension of k such that every irreducible component of $Y_{\mathfrak{k}_0}$ is geometrically connected and every double point of $Y_{\mathfrak{k}_0}$ is $\mathfrak{k}_0$-rational. Then we have

$$\begin{aligned} r(K(\mathfrak{k})) &\leqslant r(K(\mathfrak{k}\mathfrak{k}_0)) \ \ \text{(by Corollary 1.9)} \\ &= \mathrm{rk}\ H_1(\Gamma_{\mathfrak{k}\mathfrak{k}_0}, \mathbb{Z}) \\ &= \mathrm{rk}\ H_1(\Gamma_{k_0}, \mathbb{Z}) \ \ \text{(by definition of } \mathfrak{k}_0) \\ &= r(K(\mathfrak{k}_0)). \end{aligned}$$

So we have the assertion.

Now fix a prime number l which is prime to pd and fix a finite extension $k \subset \mathfrak{k}_0$ which satisfies the following:

(1) $\mathfrak{k}_0$ contains a primitive l-th root of unity.
(2) $|\mathfrak{k}_0| > r + 1$ holds.

Let $\mathfrak{k}$ be a finite extension of $\mathfrak{k}_0$. Then, by Proposition 1.10, there exists a finite Abelian extension $K_0(\mathfrak{k}) \subset L_0$ such that $[L_0 : K_0(\mathfrak{k})] = l$ and $c(L_0/K_0(\mathfrak{k})) \geqslant l^{r+2}$ holds. Let $\widetilde{L}_0$ be $L_0^{\mathrm{cs}} \cap K_0(\mathfrak{k})^{\mathrm{ab}}$. Then, by Lemma 3.1, there exists an integer $b \geqslant r + 1$ such that $[\widetilde{L}_0 : L_0] = l^b$ holds. As for the structure of the Galois group $\mathrm{Gal}(\widetilde{L}_0/K_0(\mathfrak{k}))$, we have the following claim.

CLAIM. $\mathrm{Gal}(\widetilde{L}_0/K_0(\mathfrak{k})) \cong (\mathbb{Z}/l\mathbb{Z})^{\oplus(b+1)}$ *holds.* (*Hence we have* $\mathrm{Gal}(\widetilde{L}_0/L_0) \cong (\mathbb{Z}/l\mathbb{Z})^{\oplus b}$.)

Proof. Assume the claim is false. Then there exists a surjection

$$\varphi : \mathrm{Gal}(\widetilde{L}_0/K_0(\mathfrak{k})) \longrightarrow \mathbb{Z}/l^2\mathbb{Z}.$$

Let M be the intermediate field between $K_0(\mathfrak{k})$ and $\widetilde{L}_0$ which corresponds to $\mathrm{Ker}(\varphi)$. Then, by Lemma 3.2, there exists an element $x \in P_{K_0(\mathfrak{k})}$ and an element $y \in P_M$ lying above x such that $[M_y : K_0(\mathfrak{k})_x] = l^2$ holds. Let $z \in P_{\widetilde{L}_0}$ be an element lying above y and let $w \in P_{L_0}$ be the element lying under z. Then, since $L_0 \subset \widetilde{L}_0$ is a cs extension, we have

$$[M_y : K_0(\mathfrak{k})_x] \leqslant [\widetilde{L}_{0,z} : K_0(\mathfrak{k})_x] = [L_{0,w} : K_0(\mathfrak{k})_x] \leqslant l.$$

This is a contradiction. So we have the assertion. □

Now let L, $\widetilde{L}$ be the composite field $L_0K(\mathfrak{k})$, $\widetilde{L}_0K(\mathfrak{k})$, respectively. Then, since l is prime to d, $\widetilde{L}$ is an Abelian extension of $K(\mathfrak{k})$ and we have $\mathrm{Gal}(\widetilde{L}/K(\mathfrak{k})) \cong (\mathbb{Z}/l\mathbb{Z})^{\oplus(b+1)}$. (Hence we have $\mathrm{Gal}(\widetilde{L}/L) \cong (\mathbb{Z}/l\mathbb{Z})^{\oplus b}$.) Moreover, one can see that $L \subset \widetilde{L}$ is a cs extension. Using these, we can show the following claim:

CLAIM. $[L^{\mathrm{cs}} \cap K(\mathfrak{k})^{\mathrm{ab}} : LK(\mathfrak{k})^{\mathrm{cs}}] > 1$ *holds.*

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Proof. It suffices to show the inequality $L^{\mathrm{cs},l} \cap K(\mathfrak{k})^{\mathrm{ab}} \neq LK(\mathfrak{k})^{\mathrm{cs},l}$, where $^{\mathrm{cs},l}$ denotes the maximal pro-l cs extension. Assume the contrary. Then $\widetilde{L}$ is contained in $LK(\mathfrak{k})^{\mathrm{cs},l}$. So there exists the natural surjection

$$\pi : \mathrm{Gal}(LK(\mathfrak{k})^{\mathrm{cs},l}/L) \longrightarrow \mathrm{Gal}(\widetilde{L}/L) \cong (\mathbb{Z}/l\mathbb{Z})^{\oplus b}.$$

On the other hand, since we have the inclusion $\mathrm{Gal}(LK(\mathfrak{k})^{\mathrm{cs},l}/L) \subset \mathrm{Gal}(K(\mathfrak{k})^{\mathrm{cs},l}/K(\mathfrak{k}))$ (in fact they are equal), we have the isomorphism $\mathrm{Gal}(LK(\mathfrak{k})^{\mathrm{cs},l}/L) \cong \mathbb{Z}_l^{r(K(\mathfrak{k}))}$. Since we have the inequalities $r(K(\mathfrak{k})) \leqslant r < b$, it contradicts to the existence of the surjection π. So the claim is proved. $\square$

Note that, by Lemma 3.1, we have the equality

$$c(L/K(\mathfrak{k})) = d'(L/K(\mathfrak{k}))[L^{\mathrm{cs}} \cap K(\mathfrak{k})^{\mathrm{ab}} : LK(\mathfrak{k})^{\mathrm{cs}}].$$

So, by the above claim, we obtain the inequality $c(L/K(\mathfrak{k})) > d'(L/K(\mathfrak{k}))$. Hence, $\Psi_{K(\mathfrak{k})}$ is not surjective by Proposition 2.7. So the proof is finished. $\square$

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