Canad. Math. Bull. Vol. **17** (5), 1975

SEQUENCES REALIZABLE BY GRAPHS WITH HAMILTONIAN SQUARES

BY
V. CHUNGPHAISAN

ABSTRACT. Let $\mathbf{d}=(d_1, \ldots, d_n)$ be a sequence of positive integers. In this note we show that $\mathbf{d}$ is realizable by a graph whose square is hamiltonian if and only if (i) $\mathbf{d}$ is realizable by some graph, (ii) $n \geq 3$, and (iii) $d_1 + \cdots + d_n \geq 2(n-1)$. In fact, we prove that if $\mathbf{d}$ is realizable by a connected graph, then $\mathbf{d}$ is realizable by a graph with a spanning caterpillar. From this it follows that if $\mathbf{d}$ is realizable by a connected graph, it is realizable by a graph whose square is pancyclic. We also prove that $\mathbf{d}$ is realizable by a graph with a spanning wreath if and only if $\mathbf{d}$ is realizable by some graph and $d_1 + \cdots + d_n \geq 2n$. (A *wreath* is a connected graph that has exactly one cycle and all vertices not in the cycle monovalent.)

We consider finite undirected graphs without loops and multiple edges. If u, v, x, y are four distinct vertices of a graph G, by a (uv, xy)-*exchange* on G we mean an operation on G which removes the edges uv, xy in G and adjoins the edges ux, vy not already in G. A *unicyclic* graph is a connected graph with exactly one cycle. A *caterpillar* is a tree that has a path P and all vertices not in P monovalent. Similarly, a *wreath* is a unicyclic graph that has all vertices not in its cycle monovalent. Following J. A. Bondy [1], we call a graph G *pancyclic* if it is connected and has cycles of length t for every $t \in \{3, \ldots, |V(G)|\}$. Throughout this note, $\mathbf{d}$ denotes a sequence $(d_1, \ldots, d_n)$ of positive integers. We say that $\mathbf{d}$ is *realizable by*, or *a degree sequence of*, a graph G if $|V(G)|=n$ and $d_1, \ldots, d_n$ are the degrees of its vertices. Common definitions are omitted and can be found in [3].

The following Lemmas 1.1 and 1.2 are well-known.

LEMMA 1.1 [2]. *A sequence $\mathbf{d}=(d_1, \ldots, d_n)$ of positive integers is realizable by a connected graph if and only if $\mathbf{d}$ is realizable by some graph and $d_1 + \cdots + d_n \geq 2(n-1)$.*

LEMMA 1.2. *The square of a caterpillar R with at least 3 vertices is hamiltonian.*

Proof. Let $P=(v_0, \ldots, v_n)$ be a path in R such that all vertices not in P are monovalent. For $i=0, 1, \ldots, n$, let S_i be any ordering (or permutation) of the set of vertices of R that are incident with v_i and not in P. Let C denote the sequence of vertices

$$(v_0, S_1, v_2, S_3, \ldots, v_{n-1}, S_n, v_n, S_{n-1}, v_{n-2}, S_{n-3}, \ldots, v_3, S_2, v_1, S_0, v_0)$$

Received by the editors June 27, 1973 and, in revised form August 24, 1973.

https://doi.org/10.4153/CMB-1974-116-6 Published online by Cambridge University Press

if n is odd, or

$$(v_0, S_1, v_2, S_3, \ldots, S_{n-1}, v_n, S_n, v_{n-1}, S_{n-2}, v_{n-3}, \ldots, v_3, S_2, v_1, S_0, v_0)$$

if n is even. Then it can be verified that C is a hamiltonian cycle in R^2.

LEMMA 1.3 [1]. *The square of a caterpillar is pancyclic.*

Proof. We prove the lemma by induction on the number n of vertices of the caterpillar. The lemma holds trivially for $n=1$ or 2.

Assume that R is a caterpillar with at least 3 vertices and that the squares of all caterpillars with fewer vertices than R are pancyclic. Ly Lemma 1.2 R^2 has a cycle of length $|V(R)|$. Let v be a monovalent vertex of R. Then $R-v$, the graph obtained from R by removing v and its incident edges, is a caterpillar. Thus, by the induction hypothesis $(R-v)^2$, which is a subgraph of R^2, has cycles of length t for every $t \in \{3, \ldots, |V(R)|-1\}$. Hence, R^2 is pancyclic.

As an immediate consequence of Lemma 1.3, we have the following.

LEMMA 1.4. *If a graph has a spanning caterpillar, then its square is pancyclic.*

(REMARK. It can be shown that the cube of a connected graph is pancyclic and that the square of a tree is pancyclic if and only if the tree is a caterpillar.)

THEOREM 1. *For any sequence* $\mathbf{d}=(d_1, \ldots, d_n)$ *of positive integers, the following four statements are equivalent.*

(1) $\mathbf{d}$ *is realizable by some graph and* $d_1+\cdots+d_n \geq 2(n-1)$.

(2) $\mathbf{d}$ *is realizable by a connected graph.*

(3) $\mathbf{d}$ *is realizable by a graph with a spanning caterpillar.*

(4) $\mathbf{d}$ *is realizable by a graph whose square is pancyclic.*

Proof. By Lemma 1.1, (1) and (2) are equivalent. We have (4) implies (2), since a graph with a pancyclic square is connected. That (3) implies (4) follows immediately from Lemma 1.4. It thus remains to show that (2) implies (3).

Assume (2) holds. Let G be a connected graph with degree sequence $\mathbf{d}$ that has a longest possible path. Let $P=(v_0, \ldots, v_m)$ be a longest path in G. To show that G has a spanning caterpillar, it suffices to show that every vertex in G has distance at most 1 from P. Suppose there exists some vertex u at distance $t>1$ from P. Let $(u_0, \ldots, u_t)$ where $u_0=u$ and $u_t=v_i$ be a shortest path from u to P. Note that (i) $i\geq 1$ and v_{i-1} is not adjacent to u_1—for otherwise $P'=(v_0, \ldots, v_{i-1}, u_1, \ldots, u_t, v_{i+1}, \ldots, v_m)$ is a path in G longer than P, and that (ii) u is not adjacent to v_i, since $t>1$. Now the graph obtained from G by a (uu_1, v_iv_{i-1})-exchange is connected, has degree sequence $\mathbf{d}$, and contains a path (namely P') longer than P; this contradicts our choice of G.

COROLLARY 1.1. *A sequence* $\mathbf{d}=(d_1, \ldots, d_n)$ *of positive integers is realizable by a graph whose square is hamiltonian if and only if* (i) $\mathbf{d}$ *is realizable by some graph,* (ii) $n\geq 3$, *and* (iii) $d_1+\cdots+d_n\geq 2(n-1)$.

https://doi.org/10.4153/CMB-1974-116-6 Published online by Cambridge University Press

COROLLARY 1.2. *A sequence* $\mathbf{d}=(d_1, \ldots, d_n)$ *of positive integers is realizable by a caterpillar if and only if* $d_1+\cdots+d_n=2(n-1)$.

(NOTE. It is well-known that $d_1+\cdots+d_n=2(n-1)$ is also necessary and sufficient for $\mathbf{d}$ to be realizable by a tree.)

THEOREM 2. *A sequence* $\mathbf{d}=(d_1, \ldots, d_n)$ *of positive integers is realizable by a graph with a spanning wreath if and only if* (i) $\mathbf{d}$ *is realizable by some graph, and* (ii) $d_1+\cdots+d_n\geq 2n$.

Proof. (Necessity.) Let G be a graph with degree sequence $\mathbf{d}$ and a spanning wreath W. Then $d_1+\cdots+d_n=2\,|E(G)|\geq 2\,|E(W)|=2n$.

(Sufficiency.) Assume that (i) and (ii) hold. By Lemma 1.1 $\mathbf{d}$ is realizable by some connected graph which, since (ii) holds, must have a cycle. Let G be a connected graph with degree sequence $\mathbf{d}$ and a longest possible cycle $C=(v_0, \ldots, v_m)$. To prove that G has a spanning wreath, we show that every vertex u in G has distance at most 1 from C. Suppose there is a vertex u at distance $t>1$ from C. Let $(u_0, \ldots, u_t)$, where $u_0=u$, $u_t=v_i$ and $1\leq i\leq m$, be a path joining u to C. Then clearly u is not adjacent to v_i and u_1 is not adjacent to v_{i-1}. The graph obtained from G by a (uu_1, v_iv_{i-1})-exchange is connected, has degree sequence $\mathbf{d}$ and a longer cycle $(v_0, \ldots, v_{i-1}, u_1, \ldots, u_t, v_{i+1}, \ldots, v_m)$, contradicting the choice of G.

COROLLARY 2.1. *A sequence* $\mathbf{d}=(d_1, \ldots, d_n)$ *of positive integers is realizable by a wreath if and only if* (i) $\mathbf{d}$ *has at least three terms greater than* 1, *and* (ii) $d_1+\cdots+d_n=2n$.

(NOTE. It can be shown that conditions (i) and (ii) in Corollary 2.1 are also necessary and sufficient for $\mathbf{d}$ to be realizable by a unicyclic graph.)

ACKNOWLEDGEMENT. I am grateful to Professors J. A. Bondy and C. St. J. A. Nash-Williams for their helpful suggestions and comments.

REFERENCES

1. J. A. Bondy, *Pancyclic graphs*, Proc. Second Louisiana Conference on Combinatorics, Graph Theory and Computing, 167–172, Louisiana State University Baton Rouge, La., 1971.

2. J. Edmonds, *Existence of k-edge connected ordinary graphs with prescribed degrees*, J. Res. Nat. Bur. Standards Sect. B 68B (1964), 73–74.

3. F. Harary, *Graph Theory*. (Addison-Wesley, Reading, Mass. 1969).

DEPARTMENT OF COMBINATORICS AND OPTIMIZATION,
UNIVERSITY OF WATERLOO,
WATERLOO, ONTARIO, CANADA