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TRANSCENDENTAL MEROMORPHIC SOLUTIONS OF SOME ALGEBRAIC DIFFERENTIAL EQUATIONS

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Abstract

In this paper we treat transcendental meromorphic solutions of some algebraic differential equations. We consider the number of distinct transcendental meromorphic solutions. Algebraic relations between meromorphic solutions and comparisons of the growth of transcendental meromorphic solutions are also discussed.

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1. Introduction

The binomial differential equation

$$(y')^n = R(z, y),$$

where n is a positive integer and $R(z, y)$ is a rational function in z and y, has been studied under the assumption that it has a transcendental meromorphic solution y in the complex plane (for example, Yosida [17], Laine [10]). The result due to Steinmetz [13], Bank and Kaufman [2] states that by a suitable Möbius transformation $v = (\alpha y + \beta)/(\gamma y + \delta)$, where $\alpha\delta - \beta\gamma \neq 0$, the binomial equation is classified into

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the following six simple differential equations:

$$v' = a_2(z)v^2 + a_1(z)v + a_0(z) \tag{1.1}$$

$$(v')^2 = a(z)(v - b(z))^2(v - \tau_1)(v - \tau_2) \tag{1.2}$$

$$(v')^2 = a(z)(v - \tau_1)(v - \tau_2)(v - \tau_3)(v - \tau_4) \tag{1.3}$$

$$(v')^3 = a(z)(v - \tau_1)^2(v - \tau_2)^2(v - \tau_3)^2 \tag{1.4}$$

$$(v')^4 = a(z)(v - \tau_1)^2(v - \tau_2)^3(v - \tau_3)^3 \tag{1.5}$$

$$(v')^6 = a(z)(v - \tau_1)^3(v - \tau_2)^4(v - \tau_3)^5 \tag{1.6}$$

where $\tau_1, \ldots, \tau_4$ are distinct constants and $a_j(z)(\not\equiv 0)$, $j = 0, 1, 2$, $a(z)$, $b(z)$ are rational functions. The result of Steinmetz cited above [13, Theorem 2], was generalized by v. Rieth [16] and He-Laine [7] to cover the case when $R(z, y)$ is rational in y with meromorphic coefficients.

Throughout this paper 'meromorphic' means 'meromorphic in the complex plane' and we use the standard notation of the Nevanlinna theory of meromorphic functions (see for example, [6, 11, 12]).

We consider the following three problems for the equations (1.2) and (1.3). For equation (1.2) we especially consider the case when $b(z)$ is a constant and we refer to this case as (1.2*), The equations (1.2*) and (1.3) are treated in Section 2 and in Section 3 respectively.

The first problem is to classify the equations by the number of transcendental meromorphic solutions. The differential equations (1.1)–(1.6) do not always admit transcendental meromorphic solutions. It depends on the coefficients of the equations. We investigate how many transcendental meromorphic solutions the differential equations have and under what conditions they have an infinite number of transcendental meromorphic solutions. Some results are already known about the number of meromorphic solutions to the Ricatti equation (1.1) (see, for example, [1], or [11, Chapter 9]). Answers to this problem for (1.2*) are given in Corollary 2.4 and for (1.3) are given in parts (a) and (b) of Corollary 3.2.

The second problem is to find algebraic relations between meromorphic solutions. For the case of the Riccati equation (1.1), four distinct solutions f_1, f_2, f_3, f_4 of (1.1) satisfy $\mathscr{R}(f_1, f_2, f_3, f_4) = c$, for a constant c, where $\mathscr{R}$ is a cross ratio of four elements (see, for example, [9, Section 4.2]). We shall give an answer for (1.2*) by proving Theorem 2.1, and give an answer for (1.3) by proving Theorem 3.1 (iii).

The third problem is to compare the growth of transcendental meromorphic solutions. There are many results on the growth of transcendental meromorphic solutions of these six differential equations (see, for example, [2, 13, 14]). The fact proved in [2] and in [14] is that for the transcendental meromorphic solutions f of (1.2*) or (1.3), the order of f is a positive integral multiple of 1/2, which is dependent on the coefficients of the equation. For example (*cf.* [14, Satz 1]), for any solution of $(f')^2 = A(z)(f^2 - 1)$ the order of f is equal to $1 + d/2$ where $d \geq -1$ and

$$A(z) = c_1 z^d + c_2 z^{d-1} + \cdots \quad \text{for } z \to \infty,\ c_1 \neq 0.$$

This says that for given (fixed) coefficients, all transcendental meromorphic solutions f and g of the equation have the same order of growth.

We shall give more detailed estimates of growth for transcendental meromorphic solutions of (1.2*) in Theorem 2.1 and of (1.3) in part (c) of Corollary 3.2.

2. Results for the equation (1.2)

This section is devoted to answering the question, which we posed in Section 1 for the equation (1.2*) (that is, equation (1.2) in the case where $b(z)$ is a constant).

The linear transformation

$$f = \frac{2(\tau_2 - b)(v - \tau_1)}{(\tau_2 - \tau_1)(v - b)} - 1$$

can be used to transform equation (1.2*) into

$$(f')^2 = A(z)(f^2 - 1), \tag{2.1}$$

where $A(z) = (b - \tau_1)(b - \tau_2)a(z)$. This form of the equation is more suitable for investigation of solutions. We denote by $\mathfrak{S}(A)$ the set of transcendental meromorphic solutions of (2.1) for a given rational function A, and denote by $\#\mathfrak{S}(A)$ the number of functions in $\mathfrak{S}(A)$.

In this section we prove the following theorems and corollaries.

THEOREM 2.1. *Suppose that the differential equation* (2.1) *possesses distinct transcendental meromorphic solutions f and g. Then there is a constant c such that*

$$f^2 + 2cfg + g^2 = 1 - c^2. \tag{2.2}$$

Conversely, if there are two nonconstant meromorphic functions f and g satisfying (2.2), *then the following relation holds:*

$$\frac{(f')^2}{f^2 - 1} = \frac{(g')^2}{g^2 - 1}, \tag{2.3}$$

so that if f is a solution of (2.1) *then so is g.*

COROLLARY 2.2. *Suppose that the differential equation* (2.1) *possesses transcendental meromorphic solutions f and g. Then we have*

$$T(r, g) = T(r, f) + O(1). \tag{2.4}$$

THEOREM 2.3. *Suppose that the differential equation* (2.1) *admits at least three transcendental meromorphic solutions. Then the following statements are true.*

(i) *There is a rational function $\alpha(z)$ such that $A(z) = \alpha(z)^2$.*
(ii) *We can write $\alpha(z)$ in* (i) *as a decomposition of partial fractions*

$$\alpha(z) = p(z) + \sum_{j=1}^{n} k_j (z - \tau_j)^{-1}, \tag{2.5}$$

where $p(z)$ is a polynomial not identically equal to 0, k_j $(j = 1, \cdots, n)$ are integers and τ_j $(j = 1, \ldots, n)$ are distinct constants. Moreover, for any transcendental meromorphic solution f there exists a constant $C \in \mathbb{C}$ such that

$$f(z) = \cosh\left(\int_0^z p(z)dz + \sum_{j=1}^{n} \log(z - \tau_j)^{k_j} + C\right). \tag{2.6}$$

COROLLARY 2.4. (a) *If the differential equation* (2.1) *admits at least three transcendental meromorphic solutions then $\#\mathfrak{S}(A) = \infty$.*
(b) *For a rational function A, the number of transcendental meromorphic solutions of* (2.1) *is 0, 2 or ∞.*

We note that any nonconstant meromorphic solution f of (2.1) satisfies the second order linear differential equation

$$f'' - \left(\frac{A'}{2A}\right) f' - Af = 0. \tag{2.7}$$

In fact, differentiating (2.1), we have $2f'f'' = A'(f^2 - 1) + 2Aff'$. Combining this with (2.1), we obtain (2.7) since $f' \not\equiv 0$.

For the proofs of Theorems 2.1 and 2.3 we need some lemmas given below.

LEMMA 2.5. [5, Theorem 1] *Let F and G be meromorphic functions. F and G satisfy $F^2 + G^2 = 1$ if and only if there is a meromorphic function $\beta(z)$ such that*

$$F = \frac{2\beta}{1 + \beta^2} \quad \text{and} \quad G = \frac{1 - \beta^2}{1 + \beta^2}.$$

LEMMA 2.6. *Let f be a nonconstant meromorphic function and put*

$$R(z) = \frac{(f')^2}{f^2 - 1}. \tag{2.8}$$

If $R(z)$ has poles then any pole of $R(z)$ is of order at most 2.

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PROOF. Any pole z_0 of $R(z)$ is either a pole of f, a zero of $f(z)-1$ or a zero of $f(z)+1$. If z_0 is a pole of f then a standard pole order comparison of (2.8) implies that $R(z)$ has a double pole at z_0. By similar reasoning, if $f(z)=\pm 1+\sum_{j=k}^{\infty}\alpha_j(z-z_0)^j$ around z_0 then $R(z)$ is regular at z_0 when $k\geq 2$, while $R(z)$ has a simple pole at z_0 when $k=1$. □

LEMMA 2.7. *Suppose that a meromorphic function α can be expressed in a neighbourhood of a_0 as*

$$\alpha(z)=\frac{k}{z-a_0}+h(z),\quad (k\neq 0), \tag{2.9}$$

where $h(z)$ is regular at a_0. Then, the differential equation

$$w''-\left(\frac{\alpha'(z)}{\alpha(z)}\right)w'-\alpha^2(z)w=0 \tag{2.10}$$

has a single-valued meromorphic solution in a neighbourhood of a_0 if and only if k is an integer.

PROOF. From (2.9), it is easy to see that a_0 is a regular-singular point for (2.10) (see [8, Satz 3.2]). The corresponding indicial equation at a_0 is

$$\rho(\rho-1)+\rho-k^2=\rho^2-k^2=0$$

and its solutions are $\rho=k$ and $\rho=-k$. Therefore it is easy to see that (2.10) has a nonconstant meromorphic solution in a neighbourhood of a_0 if and only if k is an integer. □

PROOF OF THEOREM 2.1. Assume that f and g are transcendental meromorphic solutions to (2.1), so that

$$(f')^2=A(f^2-1)\quad\text{and}\quad (g')^2=A(g^2-1). \tag{2.11}$$

Then it follows from (2.7) that

$$f''-\left(\frac{A'}{2A}\right)f'-Af=0\quad\text{and}\quad g''-\left(\frac{A'}{2A}\right)g'-Ag=0. \tag{2.12}$$

We add the two equations in (2.12) and then multiply the obtained equality by $2(f'+g')/A$ to obtain

$$\frac{2(f'+g')(f''+g'')}{A}-\frac{A'}{A^2}(f'+g')^2=2(f+g)(f'+g'),$$

from which we have $((f'+g')^2/A)'=((f+g)^2)'$ and hence

$$\frac{(f'+g')^2}{A}=(f+g)^2+c', \tag{2.13}$$

where c' is a constant. From (2.11) and (2.13) we eliminate A, f' and g' to obtain (2.2), where $c = 1 + c'/2$.

To prove the converse statement in the theorem, we suppose that two nonconstant meromorphic functions f and g satisfy (2.2). When $c^2 = 1$, we have $f = \pm g$ and so the relation (2.3) holds. We consider the case $c^2 \neq 1$. Write (2.2) as

$$(f + cg)^2 + (1 - c^2)g^2 = 1 - c^2. \tag{2.14}$$

Differentiating both sides of (2.14), we have

$$(f' + cg')(f + cg) + (1 - c^2)g'g = 0. \tag{2.15}$$

Combining (2.14) and (2.15), we obtain

$$\frac{(g')^2}{1 - g^2} = \frac{(f' + cg')^2}{(1 - c^2)g^2}. \tag{2.16}$$

Similarly we obtain by symmetry

$$\frac{(f')^2}{1 - f^2} = \frac{(g' + cf')^2}{(1 - c^2)f^2}. \tag{2.17}$$

We can write (2.15) as $f(f' + cg') = -g(g' + cf')$, so that the right-hand sides of (2.16) and (2.17) are equal, which proves that f and g satisfy (2.3). □

PROOF OF COROLLARY 2.2. If $c^2 = 1$, then $f = \pm g$ and we have $T(r, f) = T(r, g)$. Hence we only treat the case $c^2 \neq 1$. From (2.2), we have

$$\left(\frac{f}{g}\right)^2 + \frac{2cf}{g} + 1 = \frac{1 - c^2}{g^2},$$

from which we have, by Nevanlinna's first fundamental theorem,

$$2T(r, g) = 2T(r, f/g) + O(1).$$

Changing the roles of f and g and using Nevanlinna's first fundamental theorem, we obtain the relation $2T(r, f) = 2T(r, g/f) + O(1) = 2T(r, f/g) + O(1)$. Combining the two relations above, we obtain (2.4). □

PROOF OF THEOREM 2.3(i). By the hypothesis of this theorem and by Theorem 2.1, there are transcendental meromorphic functions f and g satisfying

$$f^2 + 2cfg + g^2 = 1 - c^2 \quad (c^2 \neq 1),$$

from which we have $f^2 + \left((cf + g)/(\sqrt{1 - c^2})\right)^2 = 1$. By Lemma 2.5, there is a meromorphic function $\beta(z)$ such that $f = 2\beta/(1 + \beta^2)$. We see that because f is transcendental, so is β. Hence

$$A(z) = \frac{(f')^2}{f^2 - 1} = -\left(\frac{2\beta'}{1 + \beta^2}\right)^2 = \left(\frac{2i\beta'}{1 + \beta^2}\right)^2.$$

That is to say, $A(z) = \alpha(z)^2$ where $\alpha(z) = (2i\beta')/(1+\beta^2)$. Since $A(z)$ is a rational function, $\alpha(z)$ must be a rational function. $\square$

PROOF OF THEOREM 2.3(ii). By Theorem 2.3 (i), we can write $A(z) = \alpha(z)^2$ for a rational function $\alpha(z)$. If $\alpha(z)$ has a pole, then the pole is simple by Lemma 2.6 and the residue at the pole must be an integer by Lemma 2.7. Hence we can write $\alpha(z)$ in the form

$$\alpha(z) = p(z) + \sum_{j=1}^{n} k_j (z-\tau_j)^{-1},$$

where $p(z)$ is a polynomial, n is the number of poles of $\alpha(z)$, k_j $(j = 1, \dots, n)$ are integers, and τ_j $(j = 1, \dots, n)$ are distinct constants. If we put $\zeta(z) = \int_0^z p(t)dt$ then the meromorphic functions

$$f_1(z) = e^{\zeta(z)} \prod_{j=1}^{n} (z-\tau_j)^{k_j} \quad \text{and} \quad f_2(z) = e^{-\zeta(z)} \prod_{j=1}^{n} (z-\tau_j)^{-k_j}, \tag{2.18}$$

which are linearly independent, satisfy the linear differential equation (2.10). Since any solution $f(z)$ of (2.1) satisfies (2.10), $f(z)$ can be expressed as a linear combination of f_1 and f_2, say

$$f(z) = C_1 f_1(z) + C_2 f_2(z), \tag{2.19}$$

where C_1 and C_2 are constants. As $f_1'(z) = \alpha(z) f_1(z)$, $f_2'(z) = -\alpha(z) f_2(z)$ and $f_1 f_2 = 1$ from (2.18), by substituting (2.19) into (2.1), we obtain that $C_1 C_2 = 1/4$. Therefore we see that for some $C \in \mathbb{C}$, $f(z)$ is represented in the form

$$f(z) = \cosh\left(\zeta(z) + \sum_{j=1}^{n} \log(z-\tau_j)^{k_j} + C\right).$$

It is immediately concluded that if $p(z) \equiv 0$ then meromorphic solutions to (2.1) are rational functions, which is a contradiction. Hence $p(z) \not\equiv 0$ and the assertion follows. $\square$

PROOF OF COROLLARY 2.4. (a) It follows from the proof of Theorem 2.3 (ii) that if $A = \alpha^2$, where α satisfies (2.5), then a meromorphic function of the form (2.6) is a solution of (2.1). This implies that $\mathfrak{S}(A)$ is an uncountable set when $p(z) \not\equiv 0$. This implies that if (2.1) possesses at least three distinct transcendental meromorphic solutions then $\#\mathfrak{S}(A) = \infty$.

(b) It is clear that if f is a transcendental meromorphic solution of (2.1) then $-f$ is also a transcendental meromorphic solution of (2.1). By part (a), $\#\mathfrak{S}(A) \geq 3$ implies $\#\mathfrak{S}(A) = \infty$. Therefore we have proved (b). $\square$

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REMARK 1. We mention a condition which implies $\mathfrak{S}(A)$ is an empty set: *if A has at least one pole of order not less than 3 then* $\#\mathfrak{S}(A) = 0$. This is a direct consequence of Lemma 2.6.

PROOF. We proved part (a) of Corollary 2.4 by means of Theorem 2.3. We mention here that we can get the same result by only using the algebraic relation (2.2) and the relation (2.4). In fact, by the hypothesis of this Corollary, there are two meromorphic functions f and g in $\mathfrak{S}(A)$ satisfying $f^2 + 2cfg + g^2 = 1 - c^2$ ($c^2 \neq 1$), from which we have

$$f^2 + \left(\frac{cf + g}{\sqrt{1 - c^2}}\right)^2 = 1.$$

This shows that $(cf + g)/\sqrt{1 - c^2} \in \mathfrak{S}(A)$ by (2.3) in Theorem 2.1 and (2.4) since $f \in \mathfrak{S}(A)$. Put

$$h = \frac{cf + g}{\sqrt{1 - c^2}} \quad \text{and} \quad F = \gamma f + \delta h,$$

where γ and δ are constants satisfying $\gamma^2 + \delta^2 = 1$. Then

$$f^2 + h^2 = 1 \quad \text{and} \quad ff' = -hh'. \tag{2.20}$$

Now we are going to prove that $F \in \mathfrak{S}(A)$. In fact, by (2.20),

$$\begin{aligned}(F')^2 &= \gamma^2 (f')^2 + 2\gamma\delta f'h' + \delta^2 (h')^2 \\ &= \gamma^2 A(f^2 - 1) + \delta^2 A(h^2 - 1) - \frac{2\gamma\delta f (f')^2}{h} \\ &= A(\gamma^2 f^2 + \delta^2 h^2 - 1) + 2\gamma\delta A f h = A\left((\gamma f + \delta h)^2 - 1\right)\end{aligned} \tag{2.21}$$

since $(f')^2/h^2 = (h')^2/f^2 = -A$ by (2.20) and $f, h \in \mathfrak{S}(A)$. It follows from (2.21) that $F = \gamma f + \delta h$ is a meromorphic solution of (2.1) and by (2.4) that $\gamma f + \delta h \in \mathfrak{S}(A)$. This proves the assertion. □

3. Results for the equation (1.3)

In this section we are concerned with the differential equation of the type (1.3) in Section 1. It will be seen below that solutions of the equation (1.3) are closely connected with the Weierstrass $\wp$-function. We choose and fix a $\wp$-function satisfying

$$(\wp')^2 = 4\wp^3 - \tilde{g}_2\wp - \tilde{g}_3, \tag{3.1}$$

where $\tilde{g}_2$, $\tilde{g}_3$, are constants satisfying $27\tilde{g}_3^2 - \tilde{g}_2^3 \neq 0$. For the sake of brevity we put $G(x) = 4x^3 - \tilde{g}_2 x - \tilde{g}_3$, and we denote by e_1, e_2, e_3 the distinct roots of $G(x) = 0$. For any solution v of (1.3), we set

$$f(z) = \frac{\alpha}{v(z) - \tau_4} - \beta \quad \text{with} \quad \alpha = -\frac{(\tau_1 - \tau_4)(\tau_2 - \tau_4)(\tau_3 - \tau_4)}{4},$$
$$\beta = \frac{1}{12}\left(2\tau_4(\tau_1 + \tau_2 + \tau_3) - (\tau_1\tau_2 + \tau_2\tau_3 + \tau_3\tau_1) - 3\tau_4^3\right).$$

Then the equation of type (1.3) can be translated into the form

$$(f')^2 = A(z)\left(4f^3 - \tilde{g}_2 f - \tilde{g}_3\right) = A(z)G(f), \tag{3.2}$$

where $A(z) \not\equiv 0$ is a rational function. We denote by $\mathfrak{T}(A)$ the set of transcendental meromorphic solutions of (3.2) for a given rational function A, and denote by $\#\mathfrak{T}(A)$ the number of functions in $\mathfrak{T}(A)$.

The purpose of this section is to prove the following theorem and corollary.

THEOREM 3.1. *Suppose that the equation* (3.2) *admits two transcendental meromorphic solutions f and g such that $f \neq L(g)$ for some Möbius transformation L such that $L(z) \not\equiv z$. Then the following statements are true.*

(i) *There exists a polynomial $a(z)$ such that $A(z) = a'(z)^2$.*

(ii) *Any $f(z) \in \mathfrak{T}(A)$ can be expressed as*

$$f(z) = \wp(a(z) + c), \quad c \in \mathbb{C}, \tag{3.3}$$

where $\wp$ is the Weierstrass $\wp$ function given in (3.1).

(iii) *Let $u(z)$ and $v(z)$ denote arbitrary distinct transcendental meromorphic solutions of* (3.2). *Then there exists a constant $d_0 \in \mathbb{C}$, such that $U = u - d_0$ and $V = v - d_0$ satisfy an algebraic relation*

$$U^2V^2 - G_2UV - G_1(U + V) - G_0 = 0, \tag{3.4}$$

where G_0, G_1 and G_2 are constants.

Conversely, if transcendental meromorphic functions U and V satisfy (3.4) *then we have*

$$\frac{(U')^2}{K(U)} = \frac{(V')^2}{K(V)}, \tag{3.5}$$

where $K(x)$ is a polynomial of degree 3 expressed as

$$K(x) = 4x^3 + \left(\frac{G_0 + G_2^2}{G_1}\right)x^2 + 2G_2x + G_1. \tag{3.6}$$

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COROLLARY 3.2. (a) *If the equation (3.2) admits two transcendental meromorphic solutions f and g such that $f \neq L(g)$ for some Möbius transformation L which is not the identity, then $\#\mathfrak{T}(A) = \infty$.*

(b) *For a rational function A, the number of transcendental meromorphic solutions of (3.2) is 0, 4 or ∞.*

(c) *For any transcendental meromorphic solutions f and g of (3.2) we have*

$$T(r, g) = T(r, f) + S(r), \tag{3.7}$$

where $S(r)$ is small with respect to $T(r, f)$ and $T(r, g)$.

We need the following results due to Bank and Kaufman [2, Lemma 5], and Valiron [15].

LEMMA 3.3. *Let $H(w)$ be a polynomial having constant coefficients, and let $w(z)$ be a nonconstant elliptic function of elliptic order q which is a solution of the differential equation $(w')^q = H(w)$. Then the following statements are true.*

(a) *If c_0 and c_1 are complex numbers satisfying $c_1^q = H(c_0)$ then there exists a complex number ζ such that $w(\zeta) = c_0$ and $w'(\zeta) = c_1$.*

(b) *Any solution of the differential equation $(w')^q = H(w)$ which is meromorphic and nonconstant in a region of the plane must be of the form $w(z + C)$ where C is a constant.*

The lemma given below is also needed for the proof of Theorem 3.1.

LEMMA 3.4. *Suppose that* (3.2) *has distinct transcendental meromorphic solutions f and g. If f and g have a common pole z_0 and we let $\varphi = f - g$ then φ does not have a zero at z_0.*

PROOF. We write A in a neighbourhood of z_0 as

$$A(z) = R_A(z - z_0)^\lambda + O(z - z_0)^{\lambda+1}, \quad R_A \neq 0, \tag{3.8}$$

where λ is an integer. Let μ_f and μ_g denote the orders of the poles of f and g at z_0. From (3.2), we have $-2(\mu_f + 1) = \lambda - 3\mu_f$, so $\mu_f = 2 + \lambda$. Similarly we have $\mu_g = 2 + \lambda$. For the sake of brevity we write $\mu_f = \mu_g = \mu$.

Write f and g in a neighbourhood of z_0 as

$$f(z) = \frac{R_f}{(z - z_0)^\mu} + O(z - z_0)^{-(\mu-1)}, \quad R_f \neq 0, \tag{3.9}$$

$$g(z) = \frac{R_g}{(z - z_0)^\mu} + O(z - z_0)^{-(\mu-1)}, \quad R_g \neq 0. \tag{3.10}$$

Substituting these representations into (3.2) and comparing the coefficients of terms $(z-z_0)^{-2(\mu+1)}$, we obtain

$$R_f = R_g = \frac{\mu^2}{4R_A}. \tag{3.11}$$

It follows from (3.2) that

$$\left(\frac{\varphi'}{\varphi}\right)(f'+g') = A\left(4\left(f^2+fg+g^2\right)-\tilde{g}_2\right). \tag{3.12}$$

Assume that φ has a zero at z_0 of order $\sigma > 0$. We compare the coefficients of $(z-z_0)^{-(\mu+2)}$ in the Laurent expansions in both sides of (3.12). Using (3.11), we obtain

$$\sigma\left(-\frac{\mu^3}{4R_A}-\frac{\mu^3}{4R_A}\right) = R_A\left(4\left(\frac{\mu^4}{16R_A^2}+\frac{\mu^4}{16R_A^2}+\frac{\mu^4}{16R_A^2}\right)\right),$$

which implies $-\sigma = 3\mu/2$, which is absurd. We have thus proved Lemma 3.4. □

The following remark states some basic properties of solutions of (3.2).

REMARK 2. (A) Every solution f of (3.2) satisfies

$$f'' = \frac{A'(z)}{2A(z)}f' + \frac{A(z)}{2}\left(12f^2-\tilde{g}_2\right). \tag{3.13}$$

Moreover, if f and g are distinct solutions of (3.2), then we have

$$\varphi'' - \frac{A'(z)}{2A(z)}\varphi' - 6A(z)(f+g)\varphi = 0, \quad \text{or} \tag{3.14}$$

$$\frac{\varphi''}{\varphi} - \frac{A'(z)}{2A(z)}\frac{\varphi'}{\varphi} = 6A(z)(f+g), \tag{3.15}$$

where $\varphi = f-g$.

(B) Let f be a transcendental meromorphic solution of (3.2). We introduce here the following four Möbius transformations:

$$L_0(x) = x, \qquad L_1(x) = \frac{e_1x+e_1^2-e_2^2-e_1e_2}{x-e_1},$$

$$L_2(x) = \frac{e_2x+e_2^2-e_3^2-e_2e_3}{x-e_2}, \quad L_3(x) = \frac{e_3x+e_3^2-e_1^2-e_3e_1}{x-e_3}.$$

We see that $L_j(f)$ $(j=0,1,2,3)$ are also solutions of (3.2), which is verified by direct computations. Moreover, we assert that for any other Möbius transformation $L(x) = (ax+b)/(cx+d)$, with $\Delta = ad-bc \neq 0$, the equation (3.2) is not satisfied by $L(f)$. To show this, we assume that $L(f)$ satisfies (3.2), that is,

$$\Delta^2\frac{(f')^2}{(cf+d)^4} = A(z)\left(4\left(\frac{af+b}{cf+d}\right)^3 - \tilde{g}_2\left(\frac{af+b}{cf+d}\right)-\tilde{g}_3\right). \tag{3.16}$$

First we treat the case $c = 0$. In this case we may assume that $d = 1$ and $a \neq 0$. Using (3.2) and (3.16), we eliminate f' and obtain a polynomial in f which must vanish. Then we have that $a = 1$ and $b = 0$ since f is a transcendental function. This implies that L must be L_0 in this case.

Next we consider the case $c \neq 0$. We may assume that $c = 1$ in this case. Using the same argument as above, we obtain a polynomial in f of degree 4 which must vanish. Since f is transcendental, all coefficients must vanish. From the coefficients of f^4, f^3 and f^2, we obtain the following relations

$$4a^3 - \tilde{g}_2 a - \tilde{g}_3 = 0, \tag{3.17}$$

$$12a^2 b - 4b^2 + 4a^3 d + 8abd - 4a^2 d^2 - b\tilde{g}_2 - 3ad\tilde{g}_2 - 4d\tilde{g}_3 = 0, \tag{3.18}$$

and

$$4ab^2 + 4a^2 bd - bd\tilde{g}_2 - ad^2\tilde{g}_2 - 2d^2\tilde{g}_3 = 0. \tag{3.19}$$

From (3.17) and (3.19), we eliminate $\tilde{g}_3$. Then, noting that $ad - b \neq 0$, we have

$$4ab + d(8a^2 - \tilde{g}_2) = 0. \tag{3.20}$$

(i) When $a \neq 0$, substituting $b = -d\left(8a^2 - \tilde{g}_2\right)/(4a)$ from (3.20) and $\tilde{g}_3 = 4a^3 - \tilde{g}_2 a$ from (3.17) into (3.18), we obtain $(a + d)d\left(12a^2 - \tilde{g}_2\right)^2 = 0$. We note that $d(12a^2 - \tilde{g}_2) \neq 0$. In fact, if $d(12a^2 - \tilde{g}_2) = 0$, by (3.20) we obtain that $a = 0$ since $ad - b \neq 0$, which is a contradiction. We have

$$d = -a \tag{3.21}$$

and from (3.20) we have

$$b = \frac{8a^2 - \tilde{g}_2}{4}. \tag{3.22}$$

(ii) When $a = 0$, we have $\tilde{g}_3 = 0$ by (3.17) and $d\tilde{g}_2 = 0$ by (3.20). If $d \neq 0$, $\tilde{g}_2 = 0$. This implies that $27\tilde{g}_3^2 - \tilde{g}_2^3 = 0$, which is a contradiction. We have $d = 0$. Substituting $a = 0$, $d = 0$ into (3.18), we obtain the equality $b(4b + \tilde{g}_2) = 0$. As $b \neq 0$ in this case $(ad - b \neq 0)$, we have $b = -\tilde{g}_2/4$.

(i) and (ii) imply that (3.21) and (3.22) hold in any case.

By (3.17), we see that a coincides with one of the roots of $G(x) = 0$, say e_1, e_2 or e_3. We note that $\tilde{g}_2 = -4(e_1e_2 + e_2e_3 + e_3e_1)$ and $\tilde{g}_3 = 4e_1e_2e_3$. In view of (3.21) and (3.22), if $a = e_1$ then $b = e_1^2 - e_2^2 - e_1e_2$ and $d = -e_1$. This implies that L coincides with L_1. Similarly we see that $L = L_2$ when $a = e_2$ and $L = L_3$ when $a = e_3$.

PROOF OF THEOREM 3.1(i). Let f and g be two transcendental meromorphic solutions of (3.2) satisfying the hypothesis of this theorem. First we will show that $A(z)$ in (3.2) has no poles. From (3.2),

$$A(z) = \frac{(f')^2}{G(f)} = \frac{(g')^2}{G(g)} \quad \text{and} \quad \frac{G(f)}{G(g)} = \left(\frac{f'}{g'}\right)^2. \tag{3.23}$$

Suppose that A has a pole z_0. From (3.23), there are four possibilities:

(i.1) z_0 is a pole of f and a pole of g,
(i.2) z_0 is a pole of f and a zero of $G(g)$,
(i.3) z_0 is a pole of g and a zero of $G(f)$,
(i.4) z_0 is a zero of $G(f)$ and a zero of $G(g)$.

Here we make a remark. In the cases (i.2)–(i.4) we consider the zeros of $G(f)$ and $G(g)$. Assume that z_0 is a zero of $G(f)$. It follows that f has one of the e_j ($j = 1, 2, 3$) points at z_0. Without loss of generality we may assume that it is an e_1 point. We set $f_1 = L_1(f)$, where L_1 is given in Remark 2 (B), that is,

$$f_1 = \frac{e_1 f + e_1^2 - e_2^2 - e_1 e_2}{f - e_1}. \tag{3.24}$$

Then we see by a simple computation that f_1 also satisfies (3.2) and z_0 is a pole of f_1. Hence the cases (i.2)–(i.4) reduce to the case (i.1), by using a suitable Möbius transformation which can be defined in a similar way to (3.24). Thus we have only to consider the case (i.1). Denote by μ_A, μ_f and μ_g the orders of the poles at z_0 for A, f and g.

From (3.2), we have $2(\mu_f + 1) = 3\mu_f + \mu_A$, that is, $(1 \le)\mu_f = 2 - \mu_A$. Hence $\mu_f = \mu_A = 1$ and similarly $\mu_g = 1$. Here we consider the Laurent expansions of A, f and g in a neighbourhood of z_0 as follows:

$$\begin{aligned}
A(z) &= \frac{R_A}{z - z_0} + \alpha_A + O(z - z_0), \quad R_A \neq 0, \\
f(z) &= \frac{R_f}{z - z_0} + \alpha_f + O(z - z_0), \quad R_f \neq 0, \\
g(z) &= \frac{R_g}{z - z_0} + \alpha_g + O(z - z_0), \quad R_g \neq 0.
\end{aligned}$$

From (3.11), $R_f = R_g = 1/4R_A$. Further, substituting these representations into (3.2) and comparing the coefficients of terms $(z - z_0)^{-3}$, we have

$$\alpha_f = \alpha_g = \frac{-R_f \alpha_A}{3R_A} = \frac{-\alpha_A}{12R_A^2}. \tag{3.25}$$

By the assumption of this lemma, the function $\varphi = f - g$ does not vanish identically, and by (3.25) φ has a zero at z_0. However, by Lemma 3.4 it is impossible that φ has a zero at z_0, a contradiction.

Secondly, we will show that all zeros of A are of even order. Let z_1 be a zero of A. From (3.23), if z_1 is a zero of f' (respectively g') and if z_1 is not a zero of $G(f)$, (respectively $G(g)$), then the order of the zero of A at z_1 is an even integer. Hence we shall consider the following four possibilities.

(i.5) z_1 is a pole of f and a pole of g,
(i.6) z_1 is a pole of f, a zero of g' and a zero of $G(g)$,
(i.7) z_1 is a pole of g, a zero of f' and a zero of $G(f)$,
(i.8) z_1 a zero of f', a zero of $G(f)$, a zero of g' and a zero of $G(g)$.

We only have to treat the case (i.5). In fact, as in the cases (i.2)–(i.4) above, the cases (i.6)–(i.8) can be reduced to the case (i.5) by using suitable Möbius transformations. We denote by λ the order of the zero of A at z_1, and denote by μ_f and μ_g the orders of the poles of f and g at z_1.

In a manner similar to the proof of Lemma 3.4, we obtain $(1 \leq)\lambda = \mu_f - 2 = \mu_g - 2$, which implies $\mu_f \geq 3$. As before, we write $\mu_f = \mu_g = \mu$.

Consider the Laurent expansions of A, f and g in a neighbourhood of z_1. Denote by R_A the coefficient of $(z - z_1)^{\mu-2}$ in the expansion of A, and denote by R_f, R_g the coefficients of $(z - z_1)^{-\mu}$ in the expansions of f, g respectively. From (3.2), similarly to (3.11), we have

$$R_f = R_g = \frac{\mu^2}{4R_A}. \tag{3.26}$$

We see that the coefficient of the term $(z - z_1)^{-2}$ in the right-hand side of (3.15) is $6R_A(R_f + R_g) = 3\mu^2$ by (3.26).

We divide the behaviour of φ at $z = z_1$ into three cases, namely, φ has a pole at z_1, φ has a zero at z_1, or φ does not have a pole nor a zero at z_1.

We first assume that φ has a pole at z_1 of order ν. Note that by (3.26) ν is at most $\mu - 1$. In the left-hand side of (3.15), the coefficient of double pole z_1 is $\nu(\nu + 1) + (\mu - 2)\nu/2 = \nu^2 + \mu\nu/2$. Hence we have $2\nu^2 + \mu\nu - 6\mu^2 = 0$, so $\nu = -2\mu$ or $2\nu = 3\mu$. Since μ and ν are positive, $\nu = -2\mu$ is absurd. If $2\nu = 3\mu$ then since $\nu \leq \mu - 1$, we have $\mu \leq -2$ which is also absurd.

Next we treat the case where φ has a zero at z_1. By the assumption, the function $\varphi = f - g$ does not vanish. Hence, in view of Lemma 3.4, this case does not occur.

Finally we consider the case where φ does not have a pole nor a zero at z_1. In this case z_1 is a simple pole or a regular point of the left-hand side of (3.15). However the right-hand side has a double pole, a contradiction.

Therefore A must be a polynomial whose zeros are of even order, which implies that there exists a polynomial a such that $A = (a')^2$. □

PROOF OF THEOREM 3.1(ii). We follow the idea in the proofs of Lemma 3.3 parts (a) and (b), (see Bank and Kaufman [2]). Let f be a transcendental meromorphic solution of (3.2). We fix $z_0 \in \mathbb{C}$ which is not a pole of f satisfying the conditions $a'(z_0) \neq 0$, $\wp'(z_0) \neq 0$ and $f'(z_0) \neq 0$ (or $G(f(z_0)) \neq 0$). Denote by D_0 a fundamental parallelogram of $\wp$ that contains z_0. Further we set $f(z_0) = b_0$ and $f'(z_0)/a'(z_0) = b_1$. Then from (3.2), $b_1^2 = G(b_0)$. In view of Lemma 3.2, there exists $z_1 \in D_0$ such that $\wp(z_1) = b_0$ and $\wp'(z_1) = b_1$. We set $\alpha(z) = a(z) + z_1 - a(z_0)$ and

$$f_1 = f_1(z) = \wp(\alpha(z)) = \wp(a(z) + z_1 - a(z_0)).$$

Then $f_1'(z) = \wp'(\alpha(z))\alpha'(z) = \wp'(\alpha(z))a'(z)$ and hence

$$(f_1')^2 = (\wp'(\alpha))^2(a')^2 = AG(\wp(\alpha)) = AG(f_1)$$

which implies that f_1 is a meromorphic solution of (3.2). We have that

$$f_1(z_0) = \wp(a(z_0) + z_1 - a(z_0)) = \wp(z_1) = b_0 = f(z_0), \tag{3.27}$$

$$f_1'(z_0) = \wp'(a(z_0) + z_1 - a(z_0))\,a'(z_0) = \wp'(z_1)a'(z_0) = b_1a'(z_0) = f'(z_0). \tag{3.28}$$

Set $\psi = f - f_1$. Then from (3.27) and (3.28) we have that $\psi(z_0) = \psi'(z_0) = 0$. We see that $A'/2A$ and A are analytic at z_0 from our assumption. Regarding g as f_1 and φ as ψ in (3.14), we conclude that $\psi = 0$, so f and f_1 must coincide. This proves (ii). □

PROOF OF THEOREM 3.1(iii). Let u and v denote meromorphic solutions of (3.2) and let $a(z)$ be a polynomial given in (i). We may assume that $u = u(z) = \wp(a(z))$ and we can write $v = v(z) = \wp(a(z) + c)$ for a constant $c \in \mathbb{C}$ by (ii). Put $\wp(c) = d_0$ and $\wp'(c) = d_1$. Then by the addition formula for the $\wp$-function,

$$\wp(a(z) + c) = \frac{1}{4}\left(\frac{\wp'(a(z)) - \wp'(c)}{\wp(a(z)) - \wp(c)}\right)^2 - \wp(a(z)) - \wp(c),$$

that is,

$$v = \frac{1}{4}\left(\frac{\wp'(a(z)) - d_1}{u - d_0}\right)^2 - u - d_0. \tag{3.29}$$

Since $d_1^2 = G(d_0)$ and $(a'(z))^2 = A(z)$, from (3.2) and (3.29) we obtain

$$\left(4(v + u + d_0)(u - d_0)^2 - G(u) - G(d_0)\right)^2 = 4G(d_0)G(u). \tag{3.30}$$

Put $U = U(z) = u(z) - d_0$ and $V = V(z) = v(z) - d_0$. Then since

$$G(d_0) = 4d_0^3 - \tilde{g}_2 d_0 - \tilde{g}_3 \quad \text{and} \quad G'(d_0) = 12d_0^2 - \tilde{g}_2,$$

we can write (3.30) as

$$U^2V^2 - \frac{1}{2}G'(d_0)UV - G(d_0)(U + V) + \frac{1}{16}(G'(d_0)^2 - 48d_0G(d_0)) = 0,$$

which confirms that U and V satisfy a relation of the form (3.4).

Conversely, we suppose that the relation (3.4) holds for meromorphic functions U and V. We differentiate (3.4) to obtain

$$U'(2UV^2 - G_2V - G_1) = -V'(2VU^2 - G_2U - G_1). \tag{3.31}$$

Using (3.4) we have

$$\begin{aligned}(2VU^2 - G_2U - G_1)^2 &\\ = 4U^2(G_2UV + G_1(U+V) + G_0) &+ G_2^2U^2 + G_1^2 - 4U^3VG_2 - 4U^2VG_1 \\ +2G_2G_1U = 4G_1U^3 &+ (4G_0 + G_2^2)U^2 + 2G_1G_2U + G_1^2 = G_1K(U).\end{aligned} \tag{3.32}$$

Similarly we obtain

$$(2UV^2 - G_2V - G_1)^2 = G_1K(V). \tag{3.33}$$

Combining (3.31)–(3.33), we obtain the assertion (3.5) with (3.6). □

PROOF OF COROLLARY 3.2. (a) This assertion follows from (ii) of Theorem 3.1.

(b) Suppose that (3.2) has a transcendental meromorphic solution f. If there exists a transcendental meromorphic solution g of (3.2) such that $g \neq L(f)$ for some Möbius transformation then we have $\#\mathfrak{T}(A) = \infty$ by (a). For the proof of (b), it remains to find the number of Möbius transformations L_j such that $L_j(f)$ satisfies the equation (3.2) if $\#\mathfrak{T}(A) \neq 0, \infty$. By means of Remark 2(B), the number of such Möbius transformations is equal to four, so $\#\mathfrak{T}(A) = 4$.

(c) In the case $f = L(g)$ for a Möbius transformation L, we have, by means of Nevanlinna's first fundamental theorem, $T(r, f) = T(r, g) + O(1)$. We may suppose that $f \neq L(g)$ for any Möbius transformation L. Then in view of Theorem 3.1(iii), there is a $d_0 \in \mathbb{C}$ such that the functions $f_0 = f - d_0$ and $g_0 = g - d_0$ satisfy an algebraic relation (3.4). Since $T(r, f_0) = T(r, f) + O(1)$ and $T(r, g_0) = T(r, g) + O(1)$, we need only show that f_0 and g_0 satisfy the assertion of part (c), namely that $T(r, f_0) = T(r, g_0) + O(1)$. If $G_1 = 0$ in (3.4) then f_0g_0 is a constant, from which we obtain that $T(r, f_0) = T(r, g_0) + O(1)$. In what follows, we assume that $G_1 \neq 0$. Define meromorphic functions

$$f_1 = -\frac{G_1g_0 + G_0}{f_0g_0^2} \quad \text{and} \quad g_1 = -\frac{G_1f_0 + G_0}{g_0f_0^2}. \tag{3.34}$$

From (3.3) for $U = f_0$, $V = g_0$, we have

$$f_0 - \frac{G_2g_0 + G_1}{g_0^2} = \frac{G_1g_0 + G_0}{f_0g_0^2}.$$

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Eliminating f_0 from this equation by using the first one of (3.33), we see that f_1 and g_0 satisfy (3.4). Similarly we see that f_0 and g_1 satisfy (3.4), so

$$f_1^2 g_0^2 - G_2 f_1 g_0 - G_1(f_1 + g_0) - G_0 = 0, \tag{3.35}$$

$$f_0^2 g_1^2 - G_2 f_0 g_1 - G_1(f_0 + g_1) - G_0 = 0. \tag{3.36}$$

Thus f_0, g_0, f_1 and g_1 are transcendental meromorphic solutions of

$$(w')^2 = A(z)K(w), \tag{3.37}$$

where $A(z)$ is given in (3.2) and $K(w)$ is given in (3.6). We also have

$$f_0 + f_1 = \frac{G_2 g_0 + G_1}{g_0^2} \quad \text{and} \quad g_0 + g_1 = \frac{G_2 f_0 + G_1}{f_0^2}. \tag{3.38}$$

It follows from (3.38) and $G_1 \neq 0$ that

$$2T(r, g_0) \leq T(r, f_0) + T(r, f_1) + O(1). \tag{3.39}$$

Using (3.34) and (3.38), we obtain

$$\frac{1}{f_0} + \frac{1}{f_1} = -\frac{G_2 g_0 + G_1}{G_1 g_0 + G_0}. \tag{3.40}$$

By means of Nevanlinna's first fundamental theorem of and (3.40),

$$T(r, f_1) \leq T(r, g_0) + T(r, f_0) + O(1). \tag{3.41}$$

Combining (3.39) and (3.41), we have $T(r, g_0) \leq 2T(r, f_0) + O(1)$. Changing the roles of $f_0(z)$ and $g_0(z)$, we obtain $T(r, f_0) \leq 2T(r, g_0) + O(1)$. This implies that if $\varphi(r) = S(r, f_0)$, then $\varphi(r) = S(r, g_0)$, and if $\varphi(r) = S(r, g_0)$, then $\varphi(r) = S(r, f_0)$. Hence for two meromorphic functions f and g, we can write $S(r, f) = S(r)$ and $S(r, g) = S(r)$.

We recall some properties of a transcendental meromorphic solution $w(z)$ of (3.37). Let $w(z)$ be a transcendental meromorphic solution of (3.37). Then, by means of Gol'dberg's theorem [4], we see that $w(z)$ is of finite order. We also have that all poles of $w(z)$, except for a finite number, are double, and $m(r, w) = O(\log r)$. Also all zeros of $w(z)$, except for a finite number, are simple, and $m(r, 1/w) = O(\log r)$ since we assume $G_1 \neq 0$. Hence,

$$N(r, w) = 2\overline{N}(r, w) + O(\log r) = T(r, w) + O(\log r), \quad \text{and} \tag{3.42}$$

$$N(r, 1/w) = \overline{N}(r, 1/w) + O(\log r) = T(r, w) + O(\log r). \tag{3.43}$$

Let z_0 be a pole of $f_0(z)$ and let z_1 be a pole of $f_1(z)$. Then we see from (3.4) and (3.35) (or (3.34)) that z_0 is a zero of g_0 and z_1 is also a zero of g_0. If both $f_0(z)$

and $f_1(z)$ have a common double pole z_2 then z_2 is a zero of $g_0(z)$ of multiplicity at least two. From (3.43), the counting function of such common poles is $O(\log r)$. Thus it follows that

$$\overline{N}(r, f_0) + \overline{N}(r, f_1) \le \overline{N}(r, 1/g_0) + O(\log r). \tag{3.44}$$

From (3.42)–(3.44),

$$T(r, f_0) + T(r, f_1) \le 2T(r, g_0) + O(\log r).$$

Combining this with (3.39), we obtain

$$T(r, f_0) + T(r, f_1) = 2T(r, g_0) + O(\log r). \tag{3.45}$$

Further we define

$$g_2 = -\frac{G_1 f_1 + G_0}{g_0 f_1^2} \quad \text{and} \quad f_2 = -\frac{G_1 g_1 + G_0}{f_0 g_1^2}.$$

Repeating this process, we define sequences of meromorphic functions $f_0, g_1, f_2, g_3, \dots$, and $g_0, f_1, g_2, f_3, \dots$. Namely, for $k = 0, 1, 2, \dots$, we set

$$f_{2k+3} = -\frac{G_1 g_{2k+2} + G_0}{f_{2k+1} g_{2k+2}^2}, \quad g_{2k+2} = -\frac{G_1 f_{2k+1} + G_0}{g_{2k} f_{2k+1}^2},$$
$$g_{2k+3} = -\frac{G_1 f_{2k+2} + G_0}{g_{2k+1} f_{2k+2}^2}, \quad f_{2k+2} = -\frac{G_1 g_{2k+1} + G_0}{f_{2k} g_{2k+1}^2}.$$

Then we see that all the functions

$$\{f_j(z)\}\ (j = 0, 1, \dots) \quad \text{and} \quad \{g_k(z)\} \quad (k = 0, 1, \dots)$$

are transcendental and satisfy the differential equation (3.37), and all pairs

$$(f_j(z), g_{j+1}(z)) \quad \text{and} \quad (g_j(z), f_{j+1}(z)) \quad (j = 0, 1, \dots)$$

satisfy (3.4) and all triples

$$(f_{j-1}(z), f_{j+1}(z), g_j(z)) \quad \text{and} \quad (g_{j-1}(z), g_{j+1}(z), f_j(z)) \quad (j = 1, 2, \dots)$$

satisfy (3.45). For $j = 0, 1, 2, \dots$, we write

$$h_j(z) = \begin{cases} f_j(z), & \text{if } j \text{ is odd} \\ g_j(z), & \text{if } j \text{ is even.} \end{cases}$$

Let a_0, b_0, a_1 and b_1 be positive constants. We assume that there exists a sequence $\{r_n\}$ such that $r_n \to \infty$ as $n \to \infty$ and

$$\begin{aligned} T(r_n, h_0) &\le a_0 T(r_n, f_0) + O(\log r_n) \\ T(r_n, h_0) &\ge b_0 T(r_n, f_0) + O(\log r_n) \end{aligned} \tag{3.46}$$

and

$$
\text{(3.47)} \qquad \begin{aligned} T(r_n, h_1) &\le a_1 T(r_n, f_0) + O(\log r) \\ T(r_n, h_1) &\ge b_1 T(r_n, f_0) + O(\log r). \end{aligned}
$$

We assert that there exist sequences $\{a_j\}$ and $\{b_j\}$, $j = 0, 1, 2, \dots$, such that

$$
\text{(3.48)} \qquad T(r_n, h_j) \le a_j T(r_n, f_0) + O(\log r_n) \quad \text{and}
$$

$$
\text{(3.49)} \qquad T(r_n, h_j) \ge b_j T(r_n, f_0) + O(\log r_n).
$$

In view of (3.45) and the comment that we made after the definitions of $\{f_j(z)\}$ and $\{g_j(z)\}$, we have, for $j = 1, 2, \dots$,

$$
\text{(3.50)} \qquad T(r_n, h_{j-1}) + T(r_n, h_{j+1}) = 2T(r_n, h_j) + O(\log r_n).
$$

Assume that (3.48) and (3.49) hold for $j = 0, 1, 2, \dots, k$. Then from (3.50),

$$
\begin{aligned} T(r_n, h_{k+1}) &= 2T(r_n, h_k) - T(r_n, h_{k-1}) + O(\log r_n) \\ &\le 2a_k T(r_n, f_0) - b_{k-1} T(r_n, f_0) + O(\log r_n), \end{aligned}
$$

which gives

$$
\text{(3.51)} \qquad a_{k+1} = 2a_k - b_{k-1}.
$$

Similarly, we obtain

$$
\text{(3.52)} \qquad b_{k+1} = 2b_k - a_{k-1}.
$$

Therefore, using the assumptions (3.46) and (3.47), we can use (3.51) and (3.52) recursively to obtain a sequence $\{a_n\}$ which satisfies (3.48) and a sequence $\{b_n\}$ which satisfies (3.49).

We now compute a_k and b_k concretely. Put $c_k = a_k + b_k$. Then we have $c_{k+1} - 2c_k + c_{k-1} = 0$ and hence $c_k = (c_1 - c_0)k + c_0$, $k = 0, 1, 2, \dots$. Thus we obtain

$$
\text{(3.53)} \qquad a_{k+1} - 2a_k - a_{k-1} = \mu k + \nu,
$$

where $\mu = c_0 - c_1$ and $\nu = c_1 - 2c_0$. In (3.53), we set $d_k = a_{k+1} - a_k$. Then we have $d_k - 2d_{k-1} - d_{k-2} = \mu$. Further, we put $e_k = d_k + \mu/2$. Then

$$
\text{(3.54)} \qquad e_k - 2e_{k-1} - e_{k-2} = 0.
$$

Thus we can write e_k with some constants γ_1 and γ_2 as

$$
\text{(3.55)} \qquad e_k = \gamma_1 \lambda_1^k + \gamma_2 \lambda_2^k,
$$

where $\lambda_1 = 1 + \sqrt{2}$ and $\lambda_2 = 1 - \sqrt{2}$, (which are the roots of $t^2 - 2t - 1 = 0$), see, for example, [3]. Thus $d_k = e_k - \mu/2$ and hence for $k = 1, 2, \dots,$ we have

(3.56)
$$\begin{aligned}
a_k &= \sum_{j=0}^{k-1} d_j + a_0 = \sum_{j=0}^{k-1}\left(e_j - \frac{\mu}{2}\right) + a_0 = \sum_{j=0}^{k-1}\left(\gamma_1\lambda_1^j + \gamma_2\lambda_2^j - \frac{\mu}{2}\right) + a_0 \\
&= \gamma_1\frac{1-\lambda_1^k}{1-\lambda_1} + \gamma_2\frac{1-\lambda_2^k}{1-\lambda_2} - \frac{\mu}{2}k + a_0, \quad \text{and} \\
b_k &= 2a_k - a_{k+1} \\
&= \frac{\gamma_1}{1-\lambda_1}\left(1 - 2\lambda_1^k + \lambda_1^{k+1}\right) + \frac{\gamma_2}{1-\lambda_2}\left(1 - 2\lambda_2^k + \lambda_2^{k+1}\right) - \frac{\mu}{2}(k-1) + a_0.
\end{aligned}$$

We assert that

$$\liminf_{r\to\infty} \frac{T(r, f_1)}{T(r, f_0)} \geq 1 \quad \text{and} \quad \liminf_{r\to\infty} \frac{T(r, g_1)}{T(r, g_0)} \geq 1. \tag{3.57}$$

To show this, we assume that

$$\liminf_{r\to\infty} \frac{T(r, f_1)}{T(r, f_0)} = \alpha < 1. \tag{3.58}$$

For any $\epsilon > 0$ such that $\alpha + \epsilon < 1$, there exists a sequence $\{r_n\} = \{r_n(\epsilon)\}$ satisfying

$$T(r_n, f_1) \leq (\alpha + \epsilon)T(r_n, f_0) \quad \text{and} \quad T(r_n, f_1) \geq (\alpha - \epsilon)T(r_n, f_0), \tag{3.59}$$

for $n \geq n_0(\epsilon)$. Later we choose a suitable ϵ. From (3.45),

$$\begin{aligned}
T(r_n, h_0) = T(r_n, g_0) &= \frac{T(r_n, f_0) + T(r_n, f_1)}{2} + O(\log r_n) \\
&\leq \frac{T(r_n, f_0) + (\alpha+\epsilon)T(r_n, f_0)}{2} + O(\log r_n) \\
&= \frac{(1+\alpha+\epsilon)T(r_n, f_0)}{2} + O(\log r_n).
\end{aligned}$$

Similarly, we have

$$T(r_n, h_0) \geq \frac{(1+\alpha-\epsilon)T(r_n, f_0)}{2} + O(\log r_n).$$

We now set

$$a_0 = \frac{1+\alpha+\epsilon}{2}, \quad b_0 = \frac{1+\alpha-\epsilon}{2}, \quad a_1 = \alpha + \epsilon, \quad \text{and} \quad b_1 = \alpha - \epsilon.$$

We compute μ, ν, γ_1 and γ_2 concretely under our assumptions. We have

$$\mu = c_0 - c_1 = (a_0 + b_0) - (a_1 + b_1) = 1 - \alpha \quad \text{and} \quad \nu = c_1 - 2c_0 = -2.$$

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From (3.53),

$$a_2 = 2a_1 + a_0 + \mu + \nu = \frac{3}{2}\alpha + \frac{5}{2}\epsilon - 1/2.$$

On the other hand, from (3.56),

$$a_1 = \gamma_1 + \gamma_2 + \alpha + \frac{\epsilon}{2}$$

$$a_2 = (1+\lambda_1)\gamma_1 + (1+\lambda_2)\gamma_2 + \frac{3}{2}\alpha + \frac{1}{2}\epsilon - \frac{1}{2}.$$

Hence we have

$$\gamma_1 + \gamma_2 = \frac{\epsilon}{2} \quad \text{and} \quad (1+\lambda_1)\gamma_1 + (1+\lambda_2)\gamma_2 = 2\epsilon.$$

Since $\lambda_1 = 1+\sqrt{2}$ and $\lambda_2 = 1-\sqrt{2}$, we obtain

$$\gamma_1 = \frac{(1+\sqrt{2})}{4}\epsilon, \quad \gamma_2 = \frac{(1-\sqrt{2})}{4}\epsilon.$$

Hence we can write

$$\begin{aligned} \text{(3.60)} \quad a_k =& (\alpha-1)k + \frac{1+\alpha+\epsilon}{2} \\ &+ \epsilon\left(\left(\frac{1+\sqrt{2}}{4}\right)\frac{(1+\sqrt{2})^k - 1}{\sqrt{2}} + \left(\frac{1-\sqrt{2}}{4}\right)\frac{1-(1-\sqrt{2})^k}{\sqrt{2}}\right). \end{aligned}$$

Since we assume that $\alpha < 1$, we can take $k = k(\alpha)$ so large that $(\alpha - 1)k + 1 < 0$. Once we find such a k, we fix it. Then we choose ϵ so small that $a_k < 0$. For this ϵ, there exists $\{r_n\} = \{r_n(\epsilon)\}$ satisfying (3.59), in particular,

$$\text{(3.61)} \qquad T(r_n, h_j) \le a_k T(r_n, f_0) + O(\log r_n).$$

We observe the term $O(\log r_n)$ in (3.61). Write this term as $\psi(\log r_n)$. Then the function $\psi(x)$ in x depends on k. However, it is independent of ϵ. Since h_0 is transcendental and $a_k < 0$, the right hand side of (3.61) is negative for sufficiently large n, a contradiction. This gives the first inequality in (3.57). On the other hand, we consider a sequence of functions

$$h_j^*(z) = \begin{cases} f_j(z), & \text{if } j \text{ is even} \\ g_j(z), & \text{if } j \text{ is odd} \end{cases}$$

instead of $h_j(z)$ above. Then we obtain the second inequality in (3.57) by similar arguments. Hence the assertion (3.57) follows. It follows from (3.45) and the first inequality in (3.57) that

$$\text{(3.62)} \qquad \liminf_{r\to\infty} \frac{T(r, g_0)}{T(r, f_0)} \ge 1.$$

We recall the remark that we posed after the definitions of $\{f_j(z)\}$ and $\{g_j(z)\}$, in particular,

$$T(r, g_0) + T(r, g_1) = 2T(r, f_0) + O(\log r).$$

From this and the second inequality in (3.57), we have

$$\liminf_{r\to\infty} \frac{T(r, f_0)}{T(r, g_0)} \geq 1,$$

and hence

$$\limsup_{r\to\infty} \frac{T(r, g_0)}{T(r, f_0)} \leq 1. \tag{3.63}$$

Hence we see that $\lim_{r\to\infty} T(r, f_0)/T(r, g_0) = 1$. This implies that as $r \to \infty$, $T(r, f_0) = (1 + o(1))T(r, g_0)$ which gives (3.7). Finally, we comment that the case $h_j(z) = h_i(z)$ for some $j \neq i$ is included in our arguments. We have thus proved (c). □

4. Examples

Finally, we state some examples in this section. As mentioned in the statement in Theorem 2.3, a condition that gives $\#\mathfrak{S}(A) \geq 3$ is obtained and in Remark 1, a condition that gives $\#\mathfrak{S}(A) = 0$ is obtained.

A natural question arises: Under what conditions does $\#\mathfrak{S}(A) = 2$ occur?

We shall give some examples of A in (2.1) for which $\#\mathfrak{S}(A) = 2$ and an example for (3.2) having the property $\#\mathfrak{T}(A) = 4$.

EXAMPLE 1. $\mathfrak{S}\left(\frac{1}{4z}\right) = \left\{\cosh\sqrt{z}, -\cosh\sqrt{z}\right\}$.

In fact, it is easy to see that $\cosh\sqrt{z}$ and $-\cosh\sqrt{z}$ are transcendental entire solutions of the differential equation

$$(f')^2 = \frac{1}{4z}(f^2 - 1).$$

It follows from Theorem 2.3 (i) that there is no other solution to the equation above.

Similarly let $p(z)$ be a polynomial with simple zeros only. Then,

$$\mathfrak{S}\left(p\,(p')^2\right) = \left\{\pm\cosh\left(\frac{2}{3}p^{3/2}\right)\right\},$$

$$\mathfrak{S}\left(\frac{(p')^2}{p}\right) = \left\{\pm\cosh 2p^{1/2}\right\},$$

$$\mathfrak{S}\left(4\left(z^2 - 1\right)\right) = \left\{\pm\cosh\left(z\sqrt{z^2 - 1} - \log\left(z + \sqrt{z^2 - 1}\right)\right)\right\}.$$

EXAMPLE 2. The equation

$$(f')^2 = \frac{1}{4z}\left(4f^3 - \tilde{g}_2 f - \tilde{g}_3\right)$$

possesses a solution $\wp(\sqrt{z})$, where $\wp$ is Weierstrass' elliptic function satisfying (3.1). Clearly $\wp(\sqrt{z} + c)$, $c \neq 0 \in \mathbb{C}$, is not meromorphic and hence $\#\mathfrak{T}(1/4z) = 4$.

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